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Liangpeng Xiong, Lin Tian, Shaoxi Luo, Xiaoli Liu

EXTREME POINTS AND SUPPORT POINTS OF A CLASS OF ANALYTIC FUNCTIONS WITH MISSING COEFFICIENTS

Abstract. Let $\mathcal{M}_n(a,b,c)$ denote a class of functions of the form $f(z) = z - a_{n+1}z^{n+1} - \cdots - a_k z^k - \cdots$ which are analytic in open unit disk $U = \{z : |z| < 1\}$ and satisfy the condition

$$\sum_{k=n+1}^{\infty} \frac{k(k+a)}{k+b} a_k < c, \quad a_k \geq 0, \quad a \geq b \geq 0, \quad 0 < c \leq 1, \quad n \in N = \{1, 2, \dots\}.$$

In this paper, we obtain the extreme points and support points of the class $\mathcal{M}_n(a,b,c)$ of functions.

1. Introduction

Let $\mathcal{A}$ denote the space of functions which are analytic in the unit disk $U = \{z : |z| < 1\}$. If a function $f(z) \in \mathcal{A}$ then $f(z)$ has the general form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad z \in U. \tag{1.1}$$

The topology of $\mathcal{A}$ is defined to be the topology of uniform convergence on compact subsets of the unit disk U. Suppose that K be a subset of the space $\mathcal{A}$, then $f \in K$ is called an extreme point of K if and only if f can not be expressed as a proper convex combination of two distinct elements of K. The set of all extreme points of K is denoted by EK.

Furthermore, a function f is called a support point of a compact $\mathcal{F}$ of $\mathcal{A}$ if $f \in \mathcal{F}$ and if there is a continuous linear functional J on $\mathcal{A}$ such that ReJ is non-constant on $\mathcal{F}$ and

$$ReJ(f) = \max\{ReJ(g) : g \in \mathcal{F}\}.$$

We shall denote the set of all support points of $\mathcal{F}$ by $\operatorname{supp} \mathcal{F}$.

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Let T be the subclass of $\mathcal{A}$ consisting functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{1.2}$$

Indeed, some authors studied various subclasses of functions related to T, see [1, 10, 11]. Also, H. Silverman [8] studied the class $F(\{b_k\})$ given by

$$F(\{b_k\}) = \left\{ f(z) : f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ \sum_{k=2}^{\infty} a_k b_k \leq 1,\ a_k \geq 0,\ z \in U \right\},$$

where $\{b_k\}$ is a positive sequence. Furthermore, H. Silverman [9] obtained the extreme points of class $F(\{k\})$ and W. Deeb [2] obtained the support points of $F(\{k\})$, respectively, where

$$F(\{k\}) = \left\{ f(z) : f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ \sum_{k=2}^{\infty} k a_k \leq 1,\ a_k \geq 0,\ z \in U \right\}.$$

Recently, Z. G. Peng [6], [7] have extended their results by considering a more general subclass concerning $F(\{b_n\})$.

Denote by T_n the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad a_k \geq 0,\ z \in U,\ n \in N = \{1, 2, \dots\} \tag{1.3}$$

that are analytic in U. Here, we want to introduce and study the subclass $\mathcal{M}_n(a, b, c)$ of T_n. A function $f(z) \in \mathcal{M}_n(a, b, c)$ if and only if

$$\mathcal{M}_n(a, b, c) = \Bigg\{ f(z) : f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k,\ \sum_{k=n+1}^{\infty} \frac{k(k+a)}{k+b} a_k \leq c,\ a_k \geq 0,$$
$$a \geq b \geq 0, 0 < c \leq 1,\ n \in N = \{1, 2, \dots\} \Bigg\}.$$

In particular, we have $\mathcal{M}_1(0, 0, 1) \equiv F(\{k\})$ when $n = 1$, $a = b = 0$, $c = 1$, therefore, the class $\mathcal{M}_n(a, b, c)$ is the generalization of the class $F(\{k\})$ studied by H. Silverman [9].

Goodman [4] proved that a sufficient condition for functions of the form (1.1) to be univalent in U is that

$$\sum_{k=2}^{\infty} k|a_k| \leq 1, \tag{1.4}$$

moreover, condition (1.4) implies that such functions must be in $S^*(0)$. Obviously, it is clear that all the functions belonging to $\mathcal{M}_n(a, b, c)$ are univalent and are in $S^*(0)$.

We note that the class $\mathcal{M}_n(a,b,c)$ is non-empty as it contains the functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} \frac{c(k+b)}{k(k+a)} \varphi_k z^k, \tag{1.5}$$

where $\varphi_k \geq 0$, $\sum_{k=n+1}^{\infty} \varphi_k \leq 1, a \geq b \geq 0$, $0 < c \leq 1$, $n \in N$.

In this paper, we obtain the extreme points and support points of the subclass $\mathcal{M}_n(a,b,c)$. Our results are the generalizations of the corresponding results due to H. Silverman [9] and W. Deeb [2].

2. The extreme points of $\mathcal{M}_n(a,b,c)$

Lemma 2.1. (See [5, P_{44}]) *Let $\mathcal{A}$ be a locally convex linear topological space and let $\mathcal{F}$ be a compact subset of $\mathcal{A}$.*
(1) *If $\mathcal{F}$ is non-empty then $E\mathcal{F}$ is non-empty.*
(2) *$HE\mathcal{F} = H\mathcal{F}$.*
(3) *If $H\mathcal{F}$ is compact then $EH\mathcal{F} \subset \mathcal{F}$.*

Lemma 2.2. *Let $f_1(z) = z$ and $f_k(z) = z - \frac{c(k+b)}{k(k+a)} z^k$, $(k \geq n+1)$, then $f(z) \in \mathcal{M}_n(a,b,c)$ if and only if $f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z)$, where $\lambda_k \geq 0$ and $\lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1$.*

Proof. Firstly, if $f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z)$, we have

$$f(z) = z - \sum_{k=n+1}^{\infty} \frac{c(k+b)}{k(k+a)} \lambda_k z^k \in T_n, \tag{2.1}$$

where

$$a_k = \frac{c(k+b)}{k(k+a)} \lambda_k.$$

So,

$$\sum_{k=n+1}^{\infty} \frac{k(k+a)}{c(k+b)} a_k = \sum_{k=n+1}^{\infty} \frac{k(k+a)}{c(k+b)} \frac{c(k+b)}{k(k+a)} \lambda_k = \sum_{k=n+1}^{\infty} \lambda_k = 1 - \lambda_1 \leq 1,$$

hence, we know $f(z) \in \mathcal{M}_n(a,b,c)$.

Conversely, suppose that $f(z) \in \mathcal{M}_n(a,b,c)$, then it is easy to know that

$$a_k \leq \frac{c(k+b)}{k(k+a)}, \quad (k \geq n+1). \tag{2.2}$$

Now, suppose that

$$\lambda_k = \frac{k(k+a)}{c(k+b)} a_k, \quad (k \geq n+1), \lambda_1 = 1 - \sum_{k=n+1}^{\infty} \lambda_k,$$

then $f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z)$. ∎

Theorem 2.3. *The extreme points of the class $\mathcal{M}_n(a,b,c)$ are given by*

$$E\mathcal{M}_n(a,b,c) = V = \left\{ z, z - \frac{c(n+1+b)}{(n+1)(n+1+a)} z^{n+1}, \right.$$
$$\left. z - \frac{c(n+2+b)}{(n+2)(n+2+a)} z^{n+2}, \dots, z - \frac{c(k+b)}{k(k+a)} z^k, \dots \right\},$$

where $a \geq b \geq 0$, $0 < c \leq 1$, $k \geq n+1$, $n \in N$.

Proof. Suppose that $z = tf_1(z) + (1-t)f_2(z)$ and $0 < t < 1$, where

$$f_i(z) = z - \sum_{k=n+1}^{\infty} a_{k,i} z^k \in \mathcal{M}_n(a,b,c), \quad i = 1, 2.$$

Then

$$0 = ta_{k,1} + (1-t)a_{k,2}, \quad k \geq n+1,\, n \in N.$$

Because $a_{k,1} \geq 0$, $a_{k,2} \geq 0$, it follows that $a_{k,1} = a_{k,2} = 0$, for $k \geq n+1$, $n \in N$. Hence $f_1(z) = f_2(z) = z$. This shows that $z \in E\mathcal{M}_n(a,b,c)$. If

$$z - \frac{c(k+b)}{k(k+a)} z^k = tg_1(z) + (1-t)g_2(z), \quad 0 < t < 1,\, k \geq n+1,\, n \in N,$$

where

$$g_i(z) = z - \sum_{k=n+1}^{\infty} a_{k,i} z^k \in \mathcal{M}_n(a,b,c), \quad i = 1, 2,$$

then we have

$$\frac{c(k+b)}{k(k+a)} = ta_{k,1} + (1-t)a_{k,2}, \quad k \geq n+1,\, n \in N. \tag{2.3}$$

Since $g_i(z) \in \mathcal{M}_n(a,b,c)$, definition gives us

$$a_{k,i} \leq \frac{c(k+b)}{k(k+a)}, \quad i = 1, 2. \tag{2.4}$$

This implies that

$$a_{k,1} = a_{k,2} = \frac{c(k+b)}{k(k+a)}, \quad k \geq n+1,\, n \in N.$$

So $g_1(z) = g_2(z)$. It gives us

$$z - \frac{c(k+b)}{k(k+a)} z^k \in E\mathcal{M}_n(a,b,c), \quad k \geq n+1,\, n \in N.$$

So $V \subset E\mathcal{M}_n(a,b,c)$.

Conversely, from Lemma 2.2, we know $\mathcal{M}_n(a,b,c) = HV$. In fact, it is easy to prove that V is a compact set. Following the Lemma 2.1, it gives $E\mathcal{M}_n(a,b,c) = EHV \subset V$. So $E\mathcal{M}_n(a,b,c) = V$. ■

Putting $n=1$, $a=b=0$, $c=1$ in Theorem 2.1, we have the Corollary 2.1 proved by H. Silverman [9].

Corollary 2.4. *The extreme points of the class $F(\{k\})$ are given by*

$$EF(\{k\}) = V = \left\{z, z-\frac{1}{2}z^2, z-\frac{1}{3}z^3, \dots, z-\frac{1}{k}z^k, \dots, (k\geq 2)\right\}.$$

3. The support points of $\mathcal{M}_n(a,b,c)$

Lemma 3.1. (See [5, P_{42}]) *J is a complex-valued continuous linear functional on $\mathcal{A}$ if and only if there is a sequence $\{b_n\}$ of complex numbers satisfying $\overline{\lim}_{n\to\infty}(|b_n|)^{\frac{1}{n}} < 1$ and such that $J(f) = \sum_{n=0}^{\infty} b_n a_n$, where $f \in \mathcal{A}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $(|z|<1)$.*

Theorem 3.2. *The support points of the class $\mathcal{M}_n(a,b,c)$ are given by*

$$\begin{aligned} Supp\,\mathcal{M}_n(a,b,c) = \Bigg\{ & f(z) \in \mathcal{M}_n(a,b,c) : f(z) \\ & = z - \frac{c(n+1+b)}{(n+1)(n+1+a)}\phi_{n+1}z^{n+1} \\ & \quad - \frac{c(n+2+b)}{(n+2)(n+2+a)}\phi_{n+2}z^{n+2} - \cdots - \frac{c(k+b)}{k(k+a)}\phi_k z^k - \dots \Bigg\}, \end{aligned}$$

where $a \geq b \geq 0$, $0 < c \leq 1$, $\phi_k \geq 0$, $\sum_{k=n+1}^{\infty}\phi_k \leq 1$, $n \in N$ and $\phi_i = 0$, for some $i \geq n+1$.

Proof. Firstly, let a function $f_0(z) \in \mathcal{M}_n(a,b,c)$ and put

$$\begin{aligned} f_0(z) &= z - \frac{c(n+1+b)}{(n+1)(n+1+a)}\phi_{n+1}z^{n+1} - \frac{c(n+2+b)}{(n+2)(n+2+a)}\phi_{n+2}z^{n+2} - \dots \\ &\quad - \frac{c(k+b)}{k(k+a)}\phi_k z^k - \cdots = z - \sum_{k=n+1}^{\infty}\frac{c(k+b)}{k(k+a)}\phi_k z^k, \end{aligned}$$

where $\sum_{k=n+1}^{\infty}\phi_k \leq 1$, $\phi_k \geq 0$, $\phi_i = 0$, for some $i \geq n+1$. Now, we need to take

$$b_k = \begin{cases} 0, & k>1, k\neq i, \\ 1, & k=1, k=i. \end{cases}$$

Obviously, we have $\overline{\lim}_{n\to\infty}(|b_k|)^{1/k} < 1$. Furthermore, we define a functional J on $\mathcal{A}$ by

$$J(f) = \sum_{n=0}^{\infty} a_k b_k, \quad \text{where} \quad f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \in T_n \subset \mathcal{A}.$$

It is clear that J is a continuous linear functional on T_n by Lemma 3.1. Moreover, we note that $J(f_0) = a_1 b_1 + a_i b_i = 1 - 0 = 1$. However, for any function

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \in \mathcal{M}_n(a,b,c), \tag{3.1}$$

we can note that $J(f) = a_1 b_1 + a_i b_i = 1 - a_i \leq 1$, $(a_i \geq 0)$. So we have

$$ReJ(f_0) = \max\{ReJ(f) : f \in \mathcal{M}_n(a,b,c)\}$$

and $ReJ(f)$ are not constant on $\mathcal{M}_n(a,b,c)$. Hence, f_0 is a support point of $\mathcal{M}_n(a,b,c)$.

Conversely, suppose that f_0 is a support point of $\mathcal{M}_n(a,b,c)$, and J is a continuous linear functional on $\mathcal{M}_n(a,b,c)$. Note that ReJ is also a continuous linear and is non-constant on $\mathcal{M}_n(a,b,c)$. Consequently, we have

$$ReJ(f_0) = \max\{ReJ(f) : f \in \mathcal{M}_n(a,b,c)\}.$$

Let

$$M = ReJ(f_0)$$

and

$$G_J = \{f(z) : f \in \mathcal{M}_n(a,b,c) : \ ReJ(f) = M\}.$$

On one hand, suppose that $ReJ(f_1) = ReJ(f_2) = M$, where $f_1 \in G_J, f_2 \in G_J, 0 < t < 1$. Then $ReJ[tf_1 + (1-t)f_2] = tReJ(f_1) + (1-t)ReJ(f_2) = tM + (1-t)M = M$ and so, $tf_1 + (1-t)f_2 \in G_J$, which gives the convexity of G_J.

On the other hand, suppose that $ReJ(f_n) = M$ and $f_n \to f$, where $f_n \in G_J$. Then $ReJ(f_n) \to ReJ(f)$ and so, $ReJ(f) = M$, which implies that the G_J is closed. Furthermore, because S^* is compact, we can claim that G_J is compact due to the relation $G_J \subset \mathcal{M}_n(a,b,c) \subset S^*$.

So, the G_J is a convex compact subset of $\mathcal{M}_n(a,b,c)$. Thus, EG_J is not empty (see Lemma 2.1). Now, suppose that $g_0 \in EG_J$ and $g_0 = tg_1(z) + (1-t)g_2(z)$, where $0 < t < 1, g_1(z) \in \mathcal{M}_n(a,b,c), g_2(z) \in \mathcal{M}_n(a,b,c)$. Then, since

$$ReJ(g_1) \leq M,\ ReJ(g_2) \leq M,\ tReJ(g_1) + (1-t)ReJ(g_2) = ReJ(g_0) = M,$$

it follows that

$$ReJ(g_1) = ReJ(g_2) = M,$$

which implies $g_1 \in G_J, g_2 \in G_J$. Again, because $g_0 \in EG_J$, so $g_1 = g_2 = g_0$. Thus $g_0 \in E\mathcal{M}_n(a,b,c)$. This shows that $EG_J \subset E\mathcal{M}_n(a,b,c)$. Suppose

$$EG_J - \{z\} = \left\{ z - \frac{c(k+b)}{k(k+a)} z^k : k \in Z_1 \right\},$$

where Z_1 is a subset of $Z_0 = \{n+1, n+2, \dots, n \in N\}$. We assert that Z_1 is a proper subset of Z_0. In fact, if it is not the case, then

$$EG_J - \{z\} = \left\{ z - \frac{c(k+b)}{k(k+a)} z^k : k = n+1, n+2, \dots, n \in N \right\}.$$

Since $EG_J \subset G_J$, it follows that

$$ReJ(z - \frac{c(k+b)}{k(k+a)} z^k) = M, \tag{3.2}$$

for all $k \geq n+1, n \in N$. Hence,

$$ReJ(z) - \frac{c(k+b)}{k(k+a)} ReJ(z^k) = M, \tag{3.3}$$

for all $k \geq n+1, n \in N$. Let $k \to \infty$. Since $z^k \to 0$ in the metric of $\mathcal{A}$ and J is a continuous linear functional on $\mathcal{A}$, it follows that $ReJ(z^k) \to 0$. Thus, by (3.2) and (3.3), we have $ReJ(z) = M$ and we also find that $ReJ(z^k) = 0$, for all $k \geq n+1, n \in N$. Furthermore, for any $f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \in \mathcal{M}_n(a, b, c)$, since J is continuous on $\mathcal{A}$ and $ReJ(z^k) = 0$, for $k \geq n+1$, it follows that

$$ReJ(f) = ReJ(z) - \sum_{k=n+1}^{\infty} a_k ReJ(z^k) = ReJ(z) = M,$$

which contradicts the fact that ReJ is not constant on $\mathcal{M}_n(a, b, c)$. This shows that there is an integer $i \geq n+1$ not belonging to Z_1. In other words, $z - \frac{c(i+b)}{i(i+a)} z^i$ does not belong to EG_J. Because G_J is a convex compact set, so $G_J = HEG_J$ (see Lemma 2.1). Since $f_0 \in G_J$, it follows that

$$f_0(z) = \phi_1 z + \sum_{k=n+1}^{\infty} \phi_k f_k(z), \tag{3.4}$$

where $\phi_1 \geq 0, \phi_k \geq 0$ and $\phi_1 + \sum_{k=n+1}^{\infty} \phi_k = 1, f_k(z) \in EG_j$. Because $z - \frac{c(i+b)}{i(i+a)} z^i$ does not belong to EG_J. So

$$f_0(z) = z - \sum_{k=n+1, k \neq i}^{\infty} \phi_k \frac{c(k+b)}{k(k+a)} z^k. \blacksquare$$

Putting $n = 1, a = b = 0, c = 1$ in Theorem 3.1, we have the Corollary 3.1 proved by W. Deeb [2].

Corollary 3.3. *The support points of the class $F(\{k\})$ are given by*

$$SuppF(\{k\}) = \left\{ f(z) \in F(\{k\}) : f(z) = z - \frac{1}{2}\phi_2 z^2 - \frac{1}{3}\phi_3 z^3 - \dots - \frac{1}{k}\phi_k z^k - \dots \right\},$$

where $\phi_k \geq 0, \sum_{k=2}^{\infty} \phi_k \leq 1$ and $\phi_i = 0$, for some $i \geq 2$.

Remark. For specific choices of parameters n, a, b, c, we can obtain the extreme points and support points of the classes of functions concerning $\mathcal{M}_n(a, b, c)$.

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THE COLLEGE OF ENGINEERING AND TECHNICAL
CHENGDU UNIVERSITY OF TECHNOLOGY
LESHAN, SICHUAN, 614000, P.R. CHINA
E-mail: xlpwxf@163.com (L.P. Xiong)

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