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IDENTITIES WITH GENERALIZED DERIVATIONS IN SEMIPRIME RINGS

Abstract. Let R be a semiprime ring. An additive mapping $F : R \to R$ is called a generalized derivation of R if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds, for all $x, y \in R$. The objective of the present paper is to study the following situations: (1) $[d(x), F(y)] = \pm[x, y]$; (2) $[d(x), F(y)] = \pm x \circ y$; (3) $[d(x), F(y)] = 0$; (4) $d(x) \circ F(y) = \pm x \circ y$; (5) $d(x) \circ F(y) = \pm[x, y]$; (6) $d(x) \circ F(y) = 0$; (7) $d(x)F(y) \pm xy \in Z(R)$, for all x, y in some appropriate subset of R.

1. Introduction

Let R be an associative ring with center $Z(R)$. A ring R is said to be n-torsion free, where n is an integer, if $nx = 0$, $x \in R$ implies $x = 0$. For $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ stand for the anticommutator $xy + yx$. We shall use basic commutator identities: $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = 0$ implies $a = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds, for all $x, y \in R$. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds, for all $x, y \in R$. Obviously, every derivation is a generalized derivation of R, but the converse is not true in general. If the associated derivation d is zero, then the generalized derivation F is said to be left multiplier of R.

Over the last some decades, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of derivations of R. The first result in this direction is due to E. C. Posner [15] who proved that if a prime ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$, for all $x \in R$, then R is commutative. This result was subsequently refined and extended by a number of algebraists; we refer to

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[7], [6], [8] for a state-of-art account and a comprehensive bibliography. Recently, some authors have obtained commutativity of prime and semiprime rings with derivations satisfying certain polynomial constraints (viz. [1], [2], [3] and [5] where further references can be found). In [12], Herstein proved that if R is a 2-torsion free prime ring with a nonzero derivation d of R such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. In [9], Daif showed that if R is a 2-torsion free semiprime ring with a nonzero ideal I of R and d is a derivation of R such that $d(I) \neq 0$ and $[d(x), d(y)] = 0$, for all $x, y \in I$, then R contains a nonzero central ideal. Moreover, Bell and Daif [6] proved that if R is a semiprime ring with U a nonzero right ideal and if R admits a nonzero derivation d such that $[d(x), d(y)] = [x, y]$, for all $x, y \in U$, then $U \subseteq Z(R)$. Recently, Ashraf et al. [4] investigated the commutativity of a prime ring R admitting a generalized derivation F associated with a nonzero derivation d satisfying any one of the following conditions: (1) $d(x) \circ F(y) = 0$, (2) $[d(x), F(y)] = 0$, (3) $d(x) \circ F(y) = x \circ y$, (4) $d(x) \circ F(y) + x \circ y = 0$, (5) $[d(x), F(y)] = [x, y]$, (6) $[d(x), F(y)] + [x, y] = 0$, (7) $d(x)F(y) \pm xy \in Z(R)$, for all $x, y \in I$, where I is a nonzero ideal of R. In [13], Shuliang has studied these situations for a Lie ideal U of a prime ring R such that $u^2 \in U$, for all $u \in U$ and obtained that either $d = 0$ or $U \subseteq Z(R)$.

In the present paper, we shall study all these cases in the setting of semiprime ring.

2. Main results

We begin with our first main result:

Theorem 2.1. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R and F a generalized derivation of R associated with a derivation d of R such that $d(I) \neq 0$. If $[d(x), F(y)] = \pm[x, y]$ holds, for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. By our assumption, we have

$$[d(x), F(y)] = \pm[x, y], \tag{1}$$

for all $x, y \in I$. Putting $x = xz$, where $z \in I$, in (1) we get

$$[d(x)z + xd(z), F(y)] = \pm([x, y]z + x[z, y]), \tag{2}$$

for all $x, y, z \in I$. This implies

$$\begin{aligned} d(x)[z, F(y)] + [d(x), F(y)]z + x[d(z), F(y)] + [x, F(y)]d(z) \\ = \pm([x, y]z + x[z, y]), \end{aligned} \tag{3}$$

for all $x, y, z \in I$. Applying (1), (3) yields that

$$d(x)[z, F(y)] + [x, F(y)]d(z) = 0, \tag{4}$$

for all $x, y, z \in I$. Substituting zx for x in (4), we get

$$(d(z)x + zd(x))[z, F(y)] + z[x, F(y)]d(z) + [z, F(y)]xd(z) = 0, \tag{5}$$

for all $x, y, z \in I$. Left multiplying (4) by z, we obtain

$$zd(x)[z, F(y)] + z[x, F(y)]d(z) = 0, \tag{6}$$

for all $x, y, z \in I$. Subtracting (6) from (5), we have

$$d(z)x[z, F(y)] + [z, F(y)]xd(z) = 0, \tag{7}$$

for all $x, y, z \in I$. In (7), replacing x with $x[z, F(y)]u$, we obtain

$$d(z)x[z, F(y)]u[z, F(y)] + [z, F(y)]x[z, F(y)]ud(z) = 0, \tag{8}$$

for all $x, y, z, u \in I$. Right multiplying by $u[z, F(y)]$ in (7), we have

$$d(z)x[z, F(y)]u[z, F(y)] + [z, F(y)]xd(z)u[z, F(y)] = 0, \tag{9}$$

for all $x, y, z, u \in I$. On subtracting (9) from (8), we get

$$[z, F(y)]x([z, F(y)]ud(z) - d(z)u[z, F(y)]) = 0, \tag{10}$$

for all $x, y, z, u \in I$. Substituting $ud(z)x$ for x in (10), we find

$$[z, F(y)]ud(z)x([z, F(y)]ud(z) - d(z)u[z, F(y)]) = 0. \tag{11}$$

Left multiplying by $d(z)u$, (10) gives

$$d(z)u[z, F(y)]x([z, F(y)]ud(z) - d(z)u[z, F(y)]) = 0. \tag{12}$$

Subtracting (12) from (11), we get

$$([z, F(y)]ud(z) - d(z)u[z, F(y)])x([z, F(y)]ud(z) - d(z)u[z, F(y)]) = 0, \tag{13}$$

for all $x, y, u, z \in I$. The last expression forces that

$$(([z, F(y)]ud(z) - d(z)u[z, F(y)])I)^2 = 0, \tag{14}$$

for all $y, u, z \in I$. Since R is semiprime, we conclude that

$$([z, F(y)]ud(z) - d(z)u[z, F(y)])I = 0, \tag{15}$$

for all $y, u, z \in I$. Hence, $[z, F(y)]ud(z) - d(z)u[z, F(y)] \in I \cap Ann(I) = 0$, for all $y, u, z \in I$ that gives

$$[z, F(y)]xd(z) - d(z)x[z, F(y)] = 0, \tag{16}$$

for all $x, y, z \in I$. Subtracting (16) from (7), we have $2d(z)x[z, F(y)] = 0$, for all $x, y, z \in I$. Since R is 2-torsion free ring, $d(z)x[z, F(y)] = 0$, for all $x, y, z \in I$. Putting $y = yz$, we obtain $0 = d(z)x[z, F(yz)] = d(z)x[z, F(y)z + yd(z)] = d(z)x[z, yd(z)]$ and hence $[z, yd(z)]x[z, yd(z)] = 0$, for all $x, y, z \in I$. Since R is semiprime, it follows that $[z, yd(z)] = 0$. We put $y = d(z)y$ and then obtain $0 = [z, d(z)yd(z)] = d(z)[z, yd(z)] + [z, d(z)]yd(z) = [z, d(z)]yd(z)$, for all $y, z \in I$. Now, it follows from $[z, d(z)]yd(z) = 0$ that $[z, d(z)]y[z, d(z)] = 0$, for all $y, z \in I$, implying

$[d(z), z] = 0$, for all $z \in I$. Hence, by [7, Theorem 3], R contains a nonzero central ideal. This completes the proof of the theorem. ■

Theorem 2.2. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R and F a generalized derivation of R associated with a derivation d of R such that $d(I) \neq 0$. If $[d(x), F(y)] = \pm x \circ y$ holds, for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. By the hypothesis, we have

$$[d(x), F(y)] = \pm x \circ y, \tag{17}$$

for all $x, y \in I$. Replacing x by xz in (17), we obtain

$$[d(x)z + xd(z), F(y)] = \pm xz \circ y, \tag{18}$$

which implies

$$\begin{aligned} d(x)[z, F(y)] + [d(x), F(y)]z + x[d(z), F(y)] + [x, F(y)]d(z) \\ = \pm\{(x \circ y)z + x[z, y]\}, \end{aligned} \tag{19}$$

for all $x, y, z \in I$. In view of (17), the expression reduces to

$$d(x)[z, F(y)] + x[d(z), F(y)] + [x, F(y)]d(z) = \pm x[z, y], \tag{20}$$

for all $x, y, z \in I$. Putting $x = zx$, we have

$$\begin{aligned} d(z)x[z, F(y)] + zd(x)[z, F(y)] + zx[d(z), F(y)] \\ + z[x, F(y)]d(z) + [z, F(y)]xd(z) = \pm zx[z, y]. \end{aligned} \tag{21}$$

Left multiplication of (20) by z yields

$$zd(x)[z, F(y)] + zx[d(z), F(y)] + z[x, F(y)]d(z) = \pm zx[z, y], \tag{22}$$

for all $x, y, z \in I$. Subtracting (22) from (21), we have

$$d(z)x[z, F(y)] + [z, F(y)]xd(z) = 0, \tag{23}$$

for all $x, y, z \in I$. The last expression is the same as the relation (7) and hence, using similar aurgument as used in the proof of Theorem 2.1, we get the required result. ■

Similarly, we can prove the following.

Theorem 2.3. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R and F a generalized derivation of R associated with a derivation d of R such that $d(I) \neq 0$. If $[d(x), F(y)] = 0$ holds, for all $x, y \in I$, then R contains a nonzero central ideal.*

Theorem 2.4. *Let R be a 2-torsion free semiprime ring, and F a generalized derivation of R associated with a nonzero derivation d of R. If $d(x) \circ F(y) = \pm x \circ y$ holds, for all $x, y \in R$, then R contains a nonzero central ideal.*

Proof. For any $x, y \in R$, we have

$$d(x)F(y) + F(y)d(x) = \pm(xy + yx). \tag{24}$$

Putting $y = yx$ in (24), we get

$$d(x)\{F(y)x + yd(x)\} + \{F(y)x + yd(x)\}d(x) = \pm(xy + yx)x, \tag{25}$$

for all $x, y \in R$. Right multiplying (24) by x, we obtain

$$d(x)F(y)x + F(y)d(x)x = \pm(xy + yx)x, \tag{26}$$

for all $x, y \in R$. Subtracting (26) from (25), we get

$$d(x)yd(x) + F(y)[x, d(x)] + yd(x)^2 = 0, \tag{27}$$

for all $x, y \in R$. Replacing y with $y[x, d(x)]$ in (27) we find that

$$d(x)y[x, d(x)]d(x) + \{F(y)[x, d(x)] + y[x, d^2(x)]\}[x, d(x)] + y[x, d(x)]d(x)^2 = 0, \tag{28}$$

for all $x, y \in R$. Right multiplication of (27) by $[x, d(x)]$ gives

$$d(x)yd(x)[x, d(x)] + F(y)[x, d(x)]^2 + yd(x)^2[x, d(x)] = 0, \tag{29}$$

for all $x, y \in R$. Subtracting (29) from (28), we obtain

$$d(x)y[[x, d(x)], d(x)] + y[x, d^2(x)][x, d(x)] + y[[x, d(x)], d(x)^2] = 0, \tag{30}$$

for all $x, y \in R$. Putting $y = xy$ in (30), we get

$$d(x)xy[[x, d(x)], d(x)] + xy[x, d^2(x)][x, d(x)] + xy[[x, d(x)], d(x)^2] = 0, \tag{31}$$

for all $x, y \in R$. Left multiplying (30) by x and then subtracting from (31), we get that

$$[d(x), x]y[[x, d(x)], d(x)] = 0, \tag{32}$$

for all $x, y \in R$. This implies

$$[[x, d(x)], d(x)]y[[x, d(x)], d(x)] = 0, \tag{33}$$

for all $x, y \in R$. Since R is semiprime, the last expression yields that $[[x, d(x)], d(x)] = 0$, for all $x \in R$. By [11], we conclude that $d(R) \subseteq Z(R)$. In view of [7, Theorem 3], the proof of theorem is completed. ∎

Theorem 2.5. *Let R be a 2-torsion free semiprime ring, and F a generalized derivation of R associated with a nonzero derivation d of R. If $d(x) \circ F(y) = \pm[x, y]$ holds for all $x, y \in R$, then R contains a nonzero central ideal.*

Proof. For any $x, y \in R$, we have

$$d(x)F(y) + F(y)d(x) = \pm[x, y], \tag{34}$$

for all $x, y \in R$. Putting $y = yx$ in (34), we get

$$d(x)\{F(y)x + yd(x)\} + \{F(y)x + yd(x)\}d(x) = \pm[x, y]x, \tag{35}$$

for all $x, y \in R$. Right multiplying (34) by x, we obtain

$$d(x)F(y)x + F(y)d(x)x = \pm[x, y]x, \tag{36}$$

for all $x, y \in R$. Subtracting (36) from (35), we find that

$$d(x)yd(x) + F(y)[x, d(x)] + yd(x)^2 = 0, \tag{37}$$

for all $x, y \in R$. This relation is the same as (27) in Theorem 2.4. Then, by same argument as we used in Theorem 2.4, the conclusion is obtained. ∎

Theorem 2.6 below can be proved by using the same techniques as used in Theorem 2.5.

Theorem 2.6. *Let R be a 2-torsion free semiprime ring, and F a generalized derivation of R associated with a nonzero derivation d of R. If $d(x) \circ F(y) = 0$ holds, for all $x, y \in R$, then R contains a nonzero central ideal.*

Theorem 2.7. *Let R be a semiprime ring with center $Z(R)$, I a nonzero ideal of R and F a generalized derivation of R associated with a nonzero derivation d of R. If $d(x)F(y) \pm xy \in Z(R)$ holds, for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. We notice that $d(x)F(y) \pm xy \in Z(R)$, for all $x, y \in I$. Replacing y with yx, we obtain

$$d(x)\{F(y)x + yd(x)\} \pm xyx \in Z(R), \tag{38}$$

for all $x, y \in I$. Since $d(x)F(y) \pm xy \in Z(R)$, commuting both sides with x, above relation yields that

$$[d(x)yd(x), x] = 0, \tag{39}$$

that is

$$d(x)yd(x)x - xd(x)yd(x) = 0, \tag{40}$$

for all $x, y \in I$.

Replacing $y = yd(x)z$, $z \in I$, we get

$$d(x)yd(x)zd(x)x - xd(x)yd(x)zd(x) = 0. \tag{41}$$

By (40), this can be written as

$$d(x)yxd(x)zd(x) - d(x)yd(x)xzd(x) = 0 \tag{42}$$

which is

$$d(x)y[d(x), x]zd(x) = 0, \tag{43}$$

for all $x, y, z \in I$. This implies

$$[d(x), x]y[d(x), x]z[d(x), x] = 0. \tag{44}$$

That is $([d(x),x]I)^3 = 0$. Since R is semiprime ring, $[d(x),x]I = 0$. Thus $[d(x),x] \subseteq I \cap ann(I) = 0$. Hence, by [7, Theorem 3], if d is nonzero on I, then R contains a nonzero central ideal.

If $d(I) = 0$, then by our hypothesis, we have $xy \in Z(R)$, for all $x, y \in I$. It yields $[xy, x] = 0$, that is $x[x,y] = 0$, for all $x, y \in I$. Replacing y with yz, it gives $0 = x[x,yz] = x[x,y]z + xy[x,z] = xy[x,z]$, for all $x,y,z \in I$. This implies that $[x,z]I[x,z] = \{0\}$, for all $x, z \in I$, and hence $([x,z]I)^2 = \{0\}$. Since R is semiprime ring, $[x,z]I = 0$. Thus $[x,z] \subseteq I \cap ann(I) = 0$, for all $x, z \in I$. Therefore, I is commutative. Since in a semiprime ring, any commutative ideal is contained in the center of R (see [10, Lemma 2]), $I \subseteq Z(R)$. Hence the theorem is proved. ∎

We conclude our paper with the following example which shows that the above theorems do not hold for arbitrary rings.

Example. Consider S be any ring. Next, let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in S \right\}$. We define maps $F, d : R \to R$ by $F\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $d\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$. Then F is a generalized derivation of R associated with the derivation d of R. Consider $I = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} | x \in S \right\}$, as an both sided ideal of R. Then we find that the following conditions holds: (1) $[d(x), F(y)] = \pm[x,y]$, (2) $[d(x), F(y)] = \pm x \circ y$, (3) $[d(x), F(y)] = 0$ and (4) $d(x)F(y) \pm xy \in Z(R)$, for all $x, y \in I$. Note that R is not semiprime for $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$. Since $d(I) \neq 0$ and R contains no nonzero central ideal for $Z(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$, the semiprimeness hypothesis in Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.7 is not superfluous.

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