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Seyit Temir

CONVERGENCE OF AN IMPLICIT ITERATION PROCESS WITH ERRORS FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

Abstract. The purpose of this paper is to introduce an implicit iterative process with errors for approximating common fixed point of two finite families of asymptotically nonexpansive mappings in the framework of Banach space. The results presented in this paper extend and generalize the corresponding results of Qin et al. [Convergence analysis of implicit iterative algorithms for asymptotically nonexpansive mappings, Appl. Math. Comp. 210 (2009), 542–550], Thakur [Weak and strong convergence of composite implicit iteration process, Appl. Math. Comp. 190 (2007), 965–973] and some others.

1. Introduction

Let K be a nonempty subset of a real Banach space X and $T : K \to K$ be a mapping. Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of T.

A mapping $T : K \to K$ is called *nonexpansive* provided

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in K$ and $n \geq 1$.

An important generalization of the class of nonexpansive mappings and the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4]. They proved that every asymptotically nonexpansive self-mapping of a nonempty closed convex subset of a uniformly convex Banach space has a fixed point.

T is called *asymptotically nonexpansive* mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in K$ and $n \geq 1$. A mapping $T : K \to K$ is called *uniformly*

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Lipschitzian with a Lipschitzian constant $L \geq 1$, if there exists Lipschitzian constant $L \geq 1$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in K$ and $n \geq 1$. It is obvious that an asymptotically nonexpansive mapping is also uniformly L-Lipschitzian with $L = \sup\{k_n : n \geq 1\}$.

In 2001, Xu and Ori [10] introduced an implicit iteration process for a finite family of nonexpansive mappings. Xu and Ori [10] proved the weak convergence of an implicit iterative process to a common fixed point of the finite family of nonexpansive mappings defined in Hilbert space. Zhou and Chang [11], in 2002, studied the weak and strong convergence of implicit iteration process to a common fixed point for a finite family of nonexpansive mappings in Banach spaces. Chidume–Shahzad [2], in 2005, proved the strong convergence of an implicit iteration process to a common fixed point for a finite family of nonexpansive mappings in Banach spaces. Sun [7], in 2003, extended an implicit iteration process for a finite family of nonexpansive mappings due to Xu and Ori [10] to the case of asymptotically quasi-nonexpansive mappings in a setting of Banach spaces. Chang et al. [1], in 2003, studied the weak and strong convergence of a Mann implicit iteration process with errors to a common fixed point for a finite family of asymptotically nonexpansive mappings in Banach spaces. Qin et al. [5], in 2009, proved the weak and strong convergence theorems of an implicit iterative process with errors for two finite families of asymptotically nonexpansive mappings in the framework of Banach spaces. Thakur [9], studied weak and strong convergence of composite implicit iteration process for a finite family of asymptotically nonexpansive mappings in Banach spaces.

The purpose of this paper is to establish weak and strong convergence theorems of the following implicit iterative process for two finite families of asymptotically nonexpansive mapping in Banach space. In this paper, motivated and inspired by [10], [5], [9], we introduce the following implicit iteration process for two finite families of asymptotically nonexpansive mappings. Let K be a nonempty subset of Banach space X. Let $T_1, T_2, \ldots, T_N$ and $S_1, S_2, \ldots, S_N$ be asymptotically nonexpansive self-mappings on K. Let $\{u_n\}$, $\{v_n\}$ be two bounded sequences in K and $\{\alpha_n\}, \{\beta_n\}$ be two real sequences in [0,1]. For arbitrary $x_0 \in K$, the sequence $\{x_n\}$ is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1[\beta_1 x_0 + (1 - \beta_1) S_1 x_1 + v_1] + u_1, \\
x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2[\beta_2 x_1 + (1 - \beta_2) S_2 x_2 + v_2] + u_2, \\
&\vdots
\end{aligned}$$

$$
\begin{aligned}
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N[\beta_N x_{N-1} + (1-\beta_N)S_N x_N + v_N] + u_N,\\
x_{N+1} &= \alpha_{N+1}x_N + (1-\alpha_{N+1}T_1^2[\beta_{N+1}x_{N+1}\\
&\quad + (1-\beta_{N+1})S_1^2 x_{N+1} + v_{N+1}] + u_{N+1},\\
&\ \vdots\\
x_{2N} &= \alpha_{2N}x_{2N-1} + (1-\alpha_{2N})T_N^2[\beta_{2N}x_{2N}\\
&\quad + (1-\beta_{2N})S_N^2 x_{2N} + v_{2N}] + u_{2N},\\
x_{2N+1} &= \alpha_{2N+1}x_{2N} + (1-\alpha_{2N+1})T_1^3[\beta_{2N+1}x_{2N+1}\\
&\quad + (1-\beta_{2N+1})S_1^3 x_{2N+1} + v_{2N+1}] + u_{2N+1}\\
&\ \vdots
\end{aligned}
$$

which can be written in the following compact form:

$$
\begin{cases}
y_n = \beta_n x_{n-1} + (1-\beta_n)S_{i(n)}^{k(n)} x_n + v_n;\\
x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} y_n + u_n,
\end{cases}
\tag{1.1}
$$

$\forall n \geq 1$, where $n = (k(n)-1)N + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$, $k(n) \geq 1$.

2. Preliminaries and notations

Throughout this paper, we assume that X is a real Banach space, K is a nonempty closed convex subset of X and $F(T)$ is the set of fixed points of T. A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$
\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\right\} > 0,
$$

for all $0 < \varepsilon \leq 2$ (i.e., $\delta : (0,2] \to [0,1]$).

A Banach space X is said to satisfy *Opial's condition* if for each sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in X$ with $y \neq x$, where $x_n \rightharpoonup x$ denotes that $\{x_n\}$ converges weakly to x.

In order to prove the main results of this paper, we need the following statements:

Lemma 2.1. [3] *Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of X and $T : K \to K$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is semi-closed at zero, i.e., for each sequence $\{x_n\}$ in K, if $\{x_n\}$ converges weakly to $p \in K$ and $\{(I-T)x_n\}$ converges strongly to 0, then $(I-T)p = 0$.*

Lemma 2.2. [8] *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be sequences of nonnegative real numbers satisfying the following conditions:* $\forall n \geq 1$, $a_{n+1} \leq (1+\sigma_n)a_n + b_n$, *where* $\sum_{n=0}^{\infty} \sigma_n < \infty$ *and* $\sum_{n=0}^{\infty} b_n < \infty$. *Then* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 2.3. [6] *Let* K *be a nonempty closed convex subset of a uniformly convex Banach space* X *and* $\{r_n\}$ *be a real sequence in* $[\delta, 1-\delta]$, *for some* $\delta \in (0,1)$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in* K *such that*

$$\limsup_{n\to\infty} \|x_n\| \leq d, \qquad \limsup_{n\to\infty} \|y_n\| \leq d$$

and

$$\limsup_{n\to\infty} \|r_n x_n + (1-r_n)y_n\| = d$$

hold for some $d \geq 0$. *Then*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Let X be a Banach space and K be a nonempty closed convex subset of X. Let $T_1, T_2, \ldots, T_N$ and $S_1, S_2, \ldots, S_N$ be asymptotically nonexpansive self-mappings on K. There exists a sequence $\{k_n^i\} \subset [1,\infty)$ with $k_n^i \to 1$ as $n \to \infty$ such that $\|T_i^n x - T_i^n y\| \leq k_n^i \|x-y\|$, for all $x, y \in K$ and $n \geq 1$. In similar way, there exists another sequence $\{\mu_n^i\} \subset [1,\infty)$ with $\mu_n^i \to 1$ as $n \to \infty$ such that $\|S_i^n x - S_i^n y\| \leq \mu_n^i \|x-y\|$, for all $x, y \in K$ and $n \geq 1$.

Letting $h_n = \max\{k_n^1, k_n^2, \ldots, k_n^N, \mu_n^1, \mu_n^2, \ldots, \mu_n^N\}$, we have that $\{h_n\} \subset [1,\infty)$ with $h_n \to 1$ as $n \to \infty$ and $\|T_i^n x - T_i^n y\| \leq k_n^i \|x-y\| \leq h_n \|x-y\|$ and $\|S_i^n x - S_i^n y\| \leq \mu_n^i \|x-y\| \leq h_n \|x-y\|$, for all $x, y \in K$ and for each $i = 1, \ldots N$.

Moreover, let $T_1, T_2, \ldots, T_N$ and $S_1, S_2, \ldots, S_N$ be asymptotically nonexpansive self-mappings on K. $\{T_i : i \in \{1, \ldots, N\}\}$ and $\{S_i : i \in \{1, \ldots, N\}\}$ are *uniformly Lipschitzian* with a Lipschitzian constant $L \geq 1$ if there exists Lipschitzian constant $L \geq 1$ with $L = \sup\{h_n : n \geq 1\}$ such that

$$\|T_i^n x - T_i^n y\| \leq L\|x-y\| \quad \text{and} \quad \|S_i^n x - S_i^n y\| \leq L\|x-y\|,$$

for all $x, y \in K$ and $n \geq 1$.

Definition 2.4. Let $T_1, T_2, \ldots, T_N$ and $S_1, S_2, \ldots, S_N$ be asymptotically nonexpansive self-mappings on K with $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$ are said to satisfy condition (A) on K if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$, for all $r \in [0,\infty)$ such that $\max_{1\leq \ell \leq N}\{\frac{1}{2}(\|x - T_\ell x\| + \|x - S_\ell x\|)\} \geq f(d(x,\mathcal{F}))$, for all $x \in K$.

3. Convergence of implicit iteration for two finite families asymptotically nonexpansive mappings

Let X be a Banach space and K be a nonempty closed convex subset of X. Let $T_1, T_2, \ldots, T_N$ and $S_1, S_2, \ldots, S_N$ be asymptotically nonexpansive self-mappings on K. For any $x_{n-1} \in K$, and $\{\alpha_n\}, \{\beta_n\} \in [0,1]$, define $W_n : K \to K$

$$W_n(x) = \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} [\beta_n x_{n-1} + (1-\beta_n) S_{i(n)}^{k(n)} x + v_n] + u_n.$$

For any $x, y \in K$

$$\begin{aligned}
&\|W_n(x) - W_n(y)\| \\
&\quad = \|\alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} [\beta_n x_{n-1} + (1-\beta_n) S_{i(n)}^{k(n)} x + v_n] + u_n \\
&\qquad - [\alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} [\beta_n x_{n-1} + (1-\beta_n) S_{i(n)}^{k(n)} y + v_n] + u_n]\| \\
&\quad \leq (1-\alpha_n) L \|\beta_n x_{n-1} + (1-\beta_n) S_{i(n)}^{k(n)} x + v_n - [\beta_n x_{n-1} + (1-\beta_n) S_{i(n)}^{k(n)} y + v_n]\| \\
&\quad \leq (1-\alpha_n) L[(1-\beta_n) \|S_{i(n)}^{k(n)} x - S_{i(n)}^{k(n)} y\|] \leq (1-\alpha_n) L[(1-\beta_n) L \|x-y\|] \\
&\quad \leq (1-\alpha_n) L[(1-\beta_n) L] \|x-y\| \leq (1-\alpha_n)(1-\beta_n) L^2 \|x-y\|.
\end{aligned}$$

If $L^2 < \frac{1}{(1-\alpha_n)(1-\beta_n)}$ for all $n \geq 1$, then W_n is a contraction. By Banach contraction mapping principal, there exists a unique fixed point $x_n \in K$ such that

$$x_n = W_n(x_n) = \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} [\beta_n x_{n-1} + (1-\beta_n) S_{i(n)}^{k(n)} x_n + v_n] + u_n.$$

That is, the implicit iterative process (1.1) is well defined.

Lemma 3.1. *Let X be a real Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in \{1, \dots, N\}\}$ and $\{S_i : i \in \{1, \dots, N\}\}$ be two finite families of asymptotically nonexpansive self-mappings of K with $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$. Let $\{u_n\}$, $\{v_n\}$ be two bounded sequences in K and $\{\alpha_n\}, \{\beta_n\}$ be two real sequences in $[0,1]$. Let $\{h_n\}$ be a sequence such that $\{h_n\} \subset [1, \infty)$ with $h_n \to 1$ as $n \to \infty$ and $L = sup\{h_n : n \geq 1\} \geq 1$. Let $x_0 \in K$ be any given point, $\{x_n\}$ be the sequence defined by (1.1), satisfying the following conditions:*

(1) $\sum_{n=1}^{\infty} (h_n - 1) < \infty$,
(2) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that* $0 < \tau_1 < (1-\alpha_n)$, $(1-\beta_n) < \tau_2$, *for all* $n \geq 1$,
(3) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

Then $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$.

Proof. From (1.1), we have for any $p \in \mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$,

$$\begin{aligned}
(3.1) \qquad \|y_n - p\| &= \|(\beta_n x_{n-1} + (1-\beta_n) S_{i(n)}^{k(n)} x_n + v_n) - p\| \\
&\leq \beta_n \|x_{n-1} - p\| + (1-\beta_n) \|S_{i(n)}^{k(n)} x_n - p\| + \|v_n\|
\end{aligned}$$

$$\leq \beta_n\|x_{n-1}-p\| + (1-\beta_n)h_n\|x_n-p\| + \|v_n\|.$$

On the other hand, we have that

$$\begin{aligned}\|x_n-p\| &= \|(\alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)}y_n + u_n) - p\| \\ &\leq \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\|T_{i(n)}^{k(n)}y_n - p\| + \|u_n\| \\ &\leq \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)h_n\|y_n-p\| + \|u_n\|.\end{aligned} \tag{3.2}$$

Applying (3.1) to (3.2), we obtain

$$\begin{aligned}\|x_n-p\| \leq\ & \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)h_n\beta_n\|x_{n-1}-p\| \\ & + (1-\alpha_n)(1-\beta_n)h_n^2\|x_n-p\| \\ & + (1-\alpha_n)h_n\|v_n\| + \|u_n\|, \\ \leq\ & (\alpha_n + (1-\alpha_n)h_n\beta_n)\|x_{n-1}-p\| \\ & + (1-\alpha_n)(1-\beta_n)h_n^2\|x_n-p\| + L\|v_n\| + \|u_n\|.\end{aligned} \tag{3.3}$$

Here we take $\kappa_n = (h_n-1)$ and $\delta_n = 2\kappa_n + \kappa_n^2 = (h_n^2-1)$ and set (3.3). It follows that

$$\begin{aligned}\|x_n-p\| &\leq \frac{(1+\kappa_n)}{(1-\tau_2-\delta_n)}\|x_{n-1}-p\| + \frac{L\|v_n\|+\|u_n\|}{1-\tau_2-\delta_n} \\ &\leq \left(1 + \frac{\tau_2+\kappa_n+\delta_n}{1-\tau_2-\delta_n}\right)\|x_{n-1}-p\| + \frac{L\|v_n\|+\|u_n\|}{1-\tau_2-\delta_n}.\end{aligned} \tag{3.4}$$

By condition (1), $\sum_{k=1}^{\infty}\kappa_n < \infty$, and so $\sum_{k=1}^{\infty}\delta_n < \infty$, $\delta_n \to 0$. Therefore, there exists a positive integer n_0 such that $\delta_n \leq \frac{1-\tau_2}{2}$, for all $n \geq n_0$.

This implies that

$$\begin{aligned}\|x_n-p\| &\leq \left(1 + \frac{2(\tau_2+\kappa_n+\delta_n)}{1-\tau_2}\right)\|x_{n-1}-p\| + \frac{2L\|v_n\|+2\|u_n\|}{1-\tau_2} \\ &\leq (1+\sigma_n)\|x_{n-1}-p\| + \varsigma_n, \quad \forall n \geq n_0,\end{aligned} \tag{3.5}$$

where $\sigma_n = \frac{2(\tau_2+\kappa_n+\delta_n)}{1-\tau_2}$ and $\varsigma_n = \frac{2L\|v_n\|+2\|u_n\|}{1-\tau_2}$. Taking infimum over all $p \in \mathcal{F}$, we have

$$d(x_n,\mathcal{F}) \leq (1+\sigma_n)d(x_{n-1},\mathcal{F}) + \varsigma_n.$$

It follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_n-p\|$ exists, $\lim_{n\to\infty}d(x_n,\mathcal{F})$ exists for each $p \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i)\cap F(S_i)$. ∎

Theorem 3.2. *Let X be a real Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in \{1,\ldots,N\}\}$ and $\{S_i : i \in \{1,\ldots,N\}\}$ be two finite families of asymptotically nonexpansive self-mappings of K with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)\cap F(S_i) \neq \emptyset$. Let $\{u_n\}$, $\{v_n\}$ be two bounded sequences in K and $\{\alpha_n\},\{\beta_n\}$ be two real sequences in $[0,1]$. Let $\{h_n\}$ be a sequence such that $\{h_n\} \subset [1,\infty)$ with $h_n \to 1$ as $n \to \infty$ and $L = sup\{h_n : n \geq 1\} \geq 1$. Let*

$x_0 \in K$ be any given point, $\{x_n\}$ be the sequence defined by (1.1) *satisfying the following conditions:*

(1) $\sum\limits_{n=1}^{\infty}(h_n - 1) < \infty$,

(2) *there exist constants $\tau_1, \tau_2 \in (0,1)$ such that $0 < \tau_1 < (1-\alpha_n)$, $(1-\beta_n) < \tau_2$, for all $n \geq 1$,*

(3) $\sum_{n=1}^{\infty}\|u_n\| < \infty$, $\sum_{n=1}^{\infty}\|v_n\| < \infty$.

Then the implicit iterative sequence $\{x_n\}$ converges strongly to a common fixed point $p \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(S_i)$ if and only if

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0. \tag{3.6}$$

Proof. The necessity of condition (3.6) is obvious. Next, we show the sufficiency. For any $p \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(S_i)$, we have by (3.5), $\|x_n - p\| \leq (1+\sigma_n)\|x_{n-1} - p\| + \varsigma_n$, where $\sum_{n=1}^{\infty}\sigma_n < \infty$ and $\sum_{n=1}^{\infty}\varsigma_n < \infty$. Hence, we find

$$d(x_n, \mathcal{F}) \leq (1+\sigma_n)d(x_{n-1}, \mathcal{F}) + \varsigma_n, \tag{3.7}$$

for all $n \geq 1$. It follows from (3.7) and Lemma 2.2 that $\liminf_{n\to\infty} d(x_n, \mathcal{F})$ exists. By assumption $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, we obtain

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. Notice that $1 + t \leq \exp(t)$, for all $t > 0$. From (3.6), for any $p \in \mathcal{F}$, we have

$$\begin{aligned}\|x_{n+m} - p\| &\leq \exp\Big(\sum_{i=n+1}^{n+m}\sigma_i\Big)\|x_n - p\| + \exp\Big(\sum_{i=n+2}^{n+m}\sigma_i\Big)\Big(\sum_{i=n+1}^{n+m}\varsigma_i\Big)\\ &\leq M\|x_n - p\| + M\Big(\sum_{i=n+1}^{\infty}\varsigma_i\Big),\end{aligned}$$

for all natural numbers m, n, where $M = \exp\{\sum_{n=1}^{\infty}\sigma_n\} < +\infty$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, for any given $\epsilon > 0$, there exists a positive integer N_0 such that for all $n \geq N_0$, $d(x_n, \mathcal{F}) < \frac{\epsilon}{2(1+M)}$ and $\sum_{i=n+1}^{\infty}\varsigma_i < \frac{\epsilon}{2M}$. There exists $p* \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(S_i)$ such that $\|x_n - p*\| < \frac{\epsilon}{2(1+M)}$.

Hence, for all $n \geq N_0$ and $m \geq 1$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p*\| + \|x_n - p*\|\\ &\leq (1+M)\|x_n - p*\| + M\Big(\sum_{i=n+1}^{\infty}\varsigma_i\Big)\\ &\leq \left(\frac{(1+M)\epsilon}{2(1+M)} + \frac{M\epsilon}{2M}\right) = \epsilon,\end{aligned}$$

which shows that $\{x_n\}$ is a Cauchy sequence in K. Thus, the completeness of X implies that $\{x_n\}$ is convergent. Assume that $\{x_n\}$ converges to a point p. Then $p \in K$, because K is closed subset of X. The set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(S_i)$ is closed. $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ gives that $d(p, \mathcal{F}) = 0$. Thus $p \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(S_i)$. This completes the proof. ∎

Lemma 3.3. *Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in \{1, \dots, N\}\}$ and $\{S_i : i \in \{1, \dots, N\}\}$ be two finite families of asymptotically nonexpansive self-mappings of K with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(S_i) \neq \emptyset$. Let $\{u_n\}$, $\{v_n\}$ be two bounded sequences in K and $\{\alpha_n\}, \{\beta_n\}$ be two real sequences in $[0, 1]$. Let $\{h_n\}$ be a sequence such that $\{h_n\} \subset [1, \infty)$ with $h_n \to 1$ as $n \to \infty$ and $L = \sup\{h_n : n \geq 1\} \geq 1$. Let $x_0 \in K$ be any given point, $\{x_n\}$ be the sequence defined by (1.1), satisfying the following conditions:*

(1) $\sum_{n=1}^{\infty}(h_n - 1) < \infty$,
(2) *there exist constants* $\tau_1, \tau_2 \in (0, 1)$ *such that* $0 < \tau_1 < (1 - \alpha_n)$, $(1 - \beta_n) < \tau_2$, *for all* $n \geq 1$,
(3) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

Then $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(S_i) \neq \emptyset$. *The sequences* $\{x_n\}$ *and* $\{y_n\}$ *are defined by* (1.1), *the following conclusions hold:*

$$\lim_{n\to\infty} \|T_\ell x_n - x_n\| = \lim_{n\to\infty} \|S_\ell x_n - x_n\| = 0, \forall \ell = 1, 2, \dots, N.$$

Proof. By Lemma 3.1 for any $p \in \mathcal{F}$, $\lim_{n\to\infty} \|x_n - p\|$ exists. Let

$$\lim_{n\to\infty} \|x_n - p\| = d. \tag{3.8}$$

Now suppose $d > 0$. It follows from (3.8) and (1.1) that

$$\|x_n - p\| = \|\alpha_n[(x_{n-1} - p) + u_n] + (1 - \alpha_n)[(T_{i(n)}^{k(n)} y_n - p) + u_n]\|. \tag{3.9}$$

By virtue of (3.8) and the condition (3), we have

$$\limsup_{n\to\infty} \|[(x_{n-1} - p) + u_n]\| \leq \limsup_{n\to\infty} (\|x_{n-1} - p\| + \|u_n\|) \leq d. \tag{3.10}$$

$$\begin{aligned} \limsup_{n\to\infty} \|(T_{i(n)}^{k(n)} y_n - p) + u_n\| & \\ &\leq \limsup_{n\to\infty} \|T_{i(n)}^{k(n)} y_n - p\| + \|u_n\| \\ &\leq \limsup_{n\to\infty} (h_n \|y_n - p\| + \|u_n\| \\ &\leq \limsup_{n\to\infty} h_n \beta_n \|x_{n-1} - p\| + (1 - \beta_n) h_n^2 \|x_n - p\| \\ &\quad + \limsup_{n\to\infty} h_n \|v_n\| + \|u_n\| \leq d, \end{aligned} \tag{3.11}$$

and

(3.12)
$$\begin{aligned}\|x_n - p\| &= \|\alpha_n[(x_{n-1} - p) + u_n] \\ &\qquad + (1-\alpha_n)[(T_{i(n)}^{k(n)} y_n - p) + u_n]\| \to d, \quad \text{as } (n \to \infty).\end{aligned}$$

Therefore, from (3.9)–(3.12) and Lemma 2.3, we know that

$$\limsup_{n\to\infty} \|T_{i(n)}^{k(n)} y_n - x_{n-1}\| = 0. \tag{3.13}$$

From (1.1) and (3.13), we obtain

$$\begin{aligned}\|x_n - x_{n-1}\| &= \|\alpha_n x_{n-1} - x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} y_n + u_n\| \\ &\le (1-\alpha_n)\|T_{i(n)}^{k(n)} y_n - x_{n-1}\| + \|u_n\| \to 0, \quad \text{as } (n\to\infty).\end{aligned}$$

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0, \tag{3.14}$$

which implies that

$$\lim_{n\to\infty} \|x_n - x_{n+j}\| = 0, \quad \forall j = 1, 2, \dots, N. \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}\|x_n - p\| &= \|\alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} y_n + u_n - p\| \\ &\le \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\|T_{i(n)}^{k(n)} y_n - p\| + \|u_n\| \\ &\le \alpha_n \|x_{n-1} - T_{i(n)}^{k(n)} y_n\| + \|T_{i(n)}^{k(n)} y_n - p\| + \|u_n\|,\end{aligned}$$

which implies that $\|y_n - p\| \ge d$. Observe that

$$\lim_{n\to\infty} \|y_n - p\| = \|\beta_n(x_{n-1} - p + v_n) + (1-\beta_n)(S_{i(n)}^{k(n)} x_n - p + v_n)\|. \tag{3.16}$$

Hence

$$\limsup_{n\to\infty} \|x_{n-1} - p + v_n\| \le d, \tag{3.17}$$

and

$$\limsup_{n\to\infty} \|S_{i(n)}^{k(n)} x_n - p + v_n\| \le \limsup_{n\to\infty} (\|S_{i(n)}^{k(n)} x_n - p\| + \|v_n\|) \le d. \tag{3.18}$$

From (3.17), (3.18) and Lemma 2.3, we know that

$$\lim_{n\to\infty} \|S_{i(n)}^{k(n)} x_n - x_{n-1}\| = 0. \tag{3.19}$$

In fact, since for each $n > N$, $n = (n-N)(\operatorname{mod} N)$ and $n = (k(n) - 1)N + i(n), i(n) \in \{1, \dots, N\}$, letting

$$\vartheta_n = \|S_{i(n)}^{k(n)} x_n - x_{n-1}\|$$

and from (3.19), $\lim_{n\to\infty} \vartheta_n = 0$ and

$$
\begin{aligned}
(3.20)\quad \|S_n x_n - x_{n-1}\| &\le \|S_{i(n)}^{k(n)} x_n - x_{n-1}\| + \|S_{i(n)}^{k(n)} x_n - S_n x_n\| \\
&\le \vartheta_n + \|S_{i(n)}^{k(n)} x_n - S_{i(n)} x_n\| \\
&\le \vartheta_n + L\|S_{i(n)}^{k(n)-1} x_n - x_n\| \\
&\le \vartheta_n + L[\|S_{i(n)}^{k(n)-1} x_n - S_{i(n-N)}^{k(n)-1} x_{n-N}\| \\
&\quad + \|S_{i(n)}^{k(n)-1} x_n - x_{(n-N)-1}\| + \|x_n - x_{(n-N)-1}\|].
\end{aligned}
$$

Since for each $n > N$, $n = (n-N)(\operatorname{mod} N)$ and $n = (k(n)-1)N + i(n)$, hence $n - N = ((k(n)-1)-1)N + i(n) = (k(n-N)-1)N + i(n-N)$, that is, $k(n-N) = k(n) - 1$ and $i(n-N) = i(n)$.
Then we have

$$
\begin{aligned}
(3.21)\quad \|S_{i(n)}^{k(n)-1} x_n - S_{i(n-N)}^{k(n)-1} x_{n-N}\| &= \|S_{i(n)}^{k(n)-1} x_n - S_{i(n)}^{k(n)-1} x_{n-N}\| \\
&\le L\|x_n - x_{(n-N)}\|
\end{aligned}
$$

and

$$
(3.22)\quad \|S_{i(n-N)}^{k(n)-1} x_{n-N} - x_{(n-N)-1}\| = \|S_{i(n-N)}^{k(n-N)} x_{n-N} - x_{(n-N)-1}\| = \vartheta_{n-N}.
$$

The substitution (3.21) and (3.22) into (3.20) yields that

$$
(3.23)\qquad \lim_{n\to\infty} \|S_n x_n - x_{n-1}\| = 0,
$$

and from (3.14) we have

$$
(3.24)\qquad \lim_{n\to\infty} \|S_n x_n - x_n\| = 0.
$$

Thus for any $j = 1, 2, \ldots, N$, from (3.15) and (3.25),

$$
(3.25)\qquad \lim_{n\to\infty} \|S_{n+j} x_n - x_n\| = 0,
$$

which yields that the sequence

$$
\bigcup_{j=1}^{N} \{\|S_{n+j} x_n - x_n\|\}_{n=1}^{\infty} \to 0, \quad \text{as} \quad (n \to \infty).
$$

Since for each $j = 1, 2, \ldots, N, \{\|S_{n+j} x_n - x_n\|\}_{n=1}^{\infty}$ is a subsequence of $\bigcup_{j=1}^{N} \{\|S_{n+j} x_n - x_n\|\}_{n=1}^{\infty}$, we have

$$
(3.26)\qquad \lim_{n\to\infty} \|S_\ell x_n - x_n\| = 0, \quad \forall \ell = 1, 2, \ldots, N.
$$

We have also

$$
\begin{aligned}
&\|T_{i(n)}^{k(n)}x_n - x_n\| \\
&\quad \leq \|T_{i(n)}^{k(n)}x_n - T_{i(n)}^{k(n)}x_{n-1}\| + \|T_{i(n)}^{k(n)}y_n - x_{n-1}\| + \|x_{n-1} - x_n\| \\
&\quad \leq (1+h_n)\|x_n - x_{n-1}\| + h_n\|x_{n-1} - y_n\| + \|T_{i(n)}^{k(n)}y_n - x_{n-1}\| \\
&\quad = (1+h_n)\|x_n - x_{n-1}\| + h_n\|x_{n-1} - [\beta_n x_{n-1} + (1-\beta_n)S_{i(n)}^{k(n)}x_n \\
&\qquad + v_n)]\| + \|T_{i(n)}^{k(n)}y_n - x_{n-1}\| \\
&\quad = (1+h_n)\|x_n - x_{n-1}\| + h_n[\|\beta_n(x_n - x_{n-1}) + (1-\beta_n)(S_{i(n)}^{k(n)}x_n - x_{n-1}) \\
&\qquad + v_n\|] + \|T_{i(n)}^{k(n)}y_n - x_{n-1}\| \\
&\quad \leq (1+h_n)\|x_n - x_{n-1}\| + h_n[\beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|S_{i(n)}^{k(n)}x_n - x_{n-1}\| \\
&\qquad + \|v_n\|] + \|T_{i(n)}^{k(n)}y_n - x_{n-1}\|.
\end{aligned}
$$

Taking $\lim_{n\to\infty}$ on both sides in the above inequality, from (3.14), (3.13), (3.19) and condition (3), we obtain

$$\lim_{n\to\infty} \|T_{i(n)}^{k(n)}x_n - x_n\| = 0. \tag{3.27}$$

In a similar way, we have

$$\lim_{n\to\infty} \|T_\ell x_n - x_n\| = 0, \quad \forall \ell = 1, 2, \ldots, N. \tag{3.28}$$

Then the proof is completed. ∎

Theorem 3.4. *Let X be a uniformly convex Banach space satisfying Opial's condition, K be a nonempty closed convex subset of X, $\{T_i : i \in \{1,\ldots,N\}\}$ and $\{S_i : i \in \{1,\ldots,N\}\}$ be two finite families of asymptotically nonexpansive self-mappings of K with $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$. Let $\{u_n\}$, $\{v_n\}$ be two bounded sequences in K and $\{\alpha_n\},\{\beta_n\}$ be two real sequences in $[0,1]$. Let $\{h_n\}$ be a sequence such that $\{h_n\} \subset [1,\infty)$ with $h_n \to 1$ as $n \to \infty$ and $L = \sup\{h_n : n \geq 1\} \geq 1$. Let $x_0 \in K$ be any given point, $\{x_n\}$ be the sequence defined by (1.1) satisfying the following conditions:*

(1) $\sum\limits_{n=1}^{\infty}(h_n - 1) < \infty$,

(2) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that* $0 < \tau_1 < (1-\alpha_n)$, $(1-\beta_n) < \tau_2$, *for all* $n \geq 1$,

(3) $\sum_{n=1}^{\infty}\|u_n\| < \infty$, $\sum_{n=1}^{\infty}\|v_n\| < \infty$.

Then $\lim_{n\to\infty}\|x_n - p\|$ exists for all $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$. Suppose that for any given $x_0 \in K$, the sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.1), *converges weakly to a common fixed point of* $\{T_i : i \in \{1, \ldots, N\}\}$ *and* $\{S_i : i \in \{1, \ldots, N\}\}$.

Proof. Let $p \in \mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap F(S_i)$. Then, as in Lemma 3.1, it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists and so for $n \geq 1$, $\{x_n\}$ is bounded on K. Then by the reflexivity of X and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup p$ weakly. We can obtain $\omega(x_n) \subset F(T_i)$ and $\omega(x_n) \subset F(S_i)$, for all $i \in \{1, \ldots, N\}$ by Lemma 2.1. Thus we have $\omega(x_n) \subset \mathcal{F}$. Finally, we prove that $\{x_n\}$ converges to p. Suppose $p, q \in w(\{x_n\})$, where $w(\{x_n\})$ denotes the weak limit set of $\{x_n\}$. Let $\{x_{n_j}\}$ and $\{x_{m_j}\}$ be two subsequences of $\{x_n\}$ which converge weakly to p and q, respectively. Opial's condition ensures that $\omega(x_n)$ is a singleton. It follows that $p = q$. Thus $\{x_n\}$ converges weakly to a $p \in \mathcal{F}$. This completes the proof. ∎

Theorem 3.5. *Let X be a uniformly convex Banach space, K be a non-empty closed convex subset of X, $\{T_i : i \in \{1, \ldots, N\}\}$ and $\{S_i : i \in \{1, \ldots, N\}\}$ be two finite families of asymptotically nonexpansive self-mappings of K with $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$. Let $\{u_n\}$, $\{v_n\}$ be two bounded sequences in K and $\{\alpha_n\}, \{\beta_n\}$ be two real sequences in $[0, 1]$. Let $\{h_n\}$ be a sequence such that $\{h_n\} \subset [1, \infty)$ with $h_n \to 1$ as $n \to \infty$ and $L = \sup\{h_n : n \geq 1\} \geq 1$. Let $x_0 \in K$ be any given point, $\{x_n\}$ be the sequence defined by* (1.1) *satisfying the following conditions:*

(1) $\sum_{n=1}^{\infty} (h_n - 1) < \infty$,
(2) *there exist constants* $\tau_1, \tau_2 \in (0, 1)$ *such that* $0 < \tau_1 < (1 - \alpha_n)$, $(1 - \beta_n) < \tau_2$, *for all* $n \geq 1$,
(3) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$. Suppose that for any given $x_0 \in K$, the sequences $\{x_n\}$ and $\{y_n\}$ are defined by (1.1), *$\lim_{n\to\infty} \|x_n - p\|$ exists for $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$. Suppose that one of the mappings $\{T_i : i \in \{1, \ldots, N\}\}$ and one of the mappings $\{S_i : i \in \{1, \ldots, N\}\}$ are semi-compact or satisfy condition* (A). *Then the implicit iterative sequence $\{x_n\}$ defined by* (1.1) *converges strongly to a common fixed point of $\{T_i : i \in \{1, \ldots, N\}\}$ and $\{S_i : i \in \{1, \ldots, N\}\}$.*

Proof. Without loss of generality, we can assume that $\{T_1\}$ and $\{S_1\}$ are semi-compact or satisfy condition (A). It follows from (3.26) and (3.28) in Lemma 3.3 that $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0 = \lim_{n\to\infty} \|x_n - S_1 x_n\| = 0$. By semi-compactness of $\{T_1\}$ and $\{S_1\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\} \to p \in K$ strongly as $j \to \infty$. From (3.26) and (3.28) in Lemma 3.3 we have "

$$\lim_{j\to\infty} \|x_{n_j} - T_\ell x_{n_j}\| = \|p - T_\ell p\| = 0,$$

for all $\ell \in \{1, \dots, N\}$, and

$$\lim_{j\to\infty} \|x_{n_j} - S_\ell x_{n_j}\| = \|p - S_\ell p\| = 0,$$

for all $\ell \in \{1, \dots, N\}$. This implies that $p \in \mathcal{F}$. Since $\liminf_{n\to\infty} f(d(x_n, \mathcal{F})) = 0$, Lemma 3.1 guarantees that $\{x_n\}$ converges strongly to a common fixed point in $\mathcal{F}$. If $\{T_1\}$ and $\{S_1\}$ satisfy condition (A), then we have $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. From Lemma 3.1, we have that $\{x_n\}$ converges to a common fixed point in $\mathcal{F}$. This completes the proof. ∎

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DEPARTMENT OF MATHEMATICS
ART AND SCIENCE FACULTY
HARRAN UNIVERSITY
63200, SANLIURFA, TURKEY
E-mail: temirseyit@harran.edu.tr

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