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Arindam Bhattacharyya, Buddhadev Pal

VOLUME OF A TETRAHEDRON REVISITED

Abstract. The volume of a tetrahedron is represented in terms of the twelve face angles, inradii of the faces of tetrahedron, circumradii of the faces and the radius of the sphere circumscribing the tetrahedron.

1. Introduction

Let $v_0, v_1, v_2, \ldots, v_n$ be independent points in n-dimensional Euclidean space $\mathbf{E}^n$. Let $[v_0, v_1, \ldots, v_n]$, V_n and r denote an n-simplex generated by $v_0, v_1, v_2, \ldots, v_n$, the volume of $[v_0, v_1, \ldots, v_n]$ and the inradius of 2-simplex, respectively. Now, for any 2-simplex

$$\begin{aligned} a &= r(\cot B/2 + \cot C/2), \\ b &= r(\cot A/2 + \cot C/2), \\ c &= r(\cot A/2 + \cot B/2). \end{aligned}$$

Therefore, the area V_2 is given by

$$\begin{aligned} V_2 &= \frac{1}{2} bc \sin A \\ &= r^2 (\cos A/2 \cos B/2 \cos C/2)/(\sin A/2 \sin B/2 \sin C/2) \\ &= 4Rr \cos A/2 \cos B/2 \cos C/2. \end{aligned} \tag{1}$$

On the other hand, in terms of circumradius and face angles of 2-simplex, V_2 is given by

$$V_2 = 2R^2 \sin A \sin B \sin C. \tag{2}$$

In this paper, we first derive a three dimensional version of formula (1) for tetrahedron in terms of its twelve face angles, inradius of each faces and the radius of the sphere circumscribing the tetrahedron. Next, we derive a three dimensional version of formula (2) in terms of its twelve face angles,

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circumradius of each faces of tetrahedron and the radius of the sphere circumscribing the tetrahedron. We know that Cayley–Menger determinant gives the volume of a simplex in j dimensions. If S is a j-simplex in $\mathbf{E}^n$ with vertices $v_0, v_1, v_2, \ldots, v_j$ and $B = (\beta_{ik})$ denotes the $(j+1) \times (j+1)$ matrix given by

$$\beta_{ik} = |v_i - v_k|^2,$$

then the content V_j is given by

$$V_j^2(S) = \frac{(-1)^{j+1}}{2^j (j\,!)^2} \det(\hat{B}),$$

where $\hat{B}$ is the $(j+2) \times (j+2)$ matrix obtained from B. Here $det(\hat{B})$ is the Cayley–Menger determinant.

2. Volume of a tetrahedron in terms of r_i, R and θ_{ij}

For $j = 3$, the content of the 3-simplex (i.e., volume of the general tetrahedron) is given by

$$V_3^2 = \frac{(-1)^{3+1}}{2^3 (3!)^2} \det(\hat{B}) = \frac{1}{288} \det(\hat{B}),$$

where

$$\det(\hat{B}) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a_{01}^2 & a_{02}^2 & a_{03}^2 \\ 1 & a_{10}^2 & 0 & a_{12}^2 & a_{13}^2 \\ 1 & a_{20}^2 & a_{21}^2 & 0 & a_{23}^2 \\ 1 & a_{30}^2 & a_{31}^2 & a_{32}^2 & 0 \end{vmatrix}.$$

It is known that [1, section 9.7] the circumradius R is given by

$$R^2 = -\frac{1}{2} \frac{\det(\bar{B})}{\det(\hat{B})},$$

where

$$\det(\bar{B}) = \begin{vmatrix} 0 & a_{01}^2 & a_{02}^2 & a_{03}^2 \\ a_{10}^2 & 0 & a_{12}^2 & a_{13}^2 \\ a_{20}^2 & a_{21}^2 & 0 & a_{23}^2 \\ a_{30}^2 & a_{31}^2 & a_{32}^2 & 0 \end{vmatrix}.$$

Therefore, another form of the volume of tetrahedron,

$$V_3 = \frac{\sqrt{-\det(\bar{B})}}{24R}.$$

Notation: Suppose x_0, x_1, x_2 and x_3 are four independent points in $\mathbf{E}^3$. T is the tetrahedron $[x_0, x_1, x_2, x_3]$. F_0, F_1, F_2, F_3 are the four faces of tetrahedron given by $x_i \notin F_i$ for $i = 0, 1, 2, 3$. Let r_i be the inradii of the corresponding faces F_i for $i = 0, 1, 2, 3$, and θ_{ik} are the face angles corresponding to the face F_i and point x_k, $k \in \{0, 1, 2, 3\} \backslash \{i\}$. Also a_{ij} are the edge length of tetrahedron, $a_{ij} = [x_i, x_j]$ for every i, $j \in \{0, 1, 2, 3\}$, $i \neq j$ and $a_{ij} = a_{ji}$, $\forall i, j$.

Theorem 1. *Let r_i be the inradii of faces F_i, $i \in \{0, 1, 2, 3\}$ and θ_{ij}, for $i, j \in \{0, 1, 2, 3\}$, $i \neq j$, be the face angles of a tetrahedron. Then the volume V of the tetrahedron is given by*

$$V = \frac{r_3^2 r_1^2}{24RD^2}\sqrt{-(M)},$$

where

$$\begin{aligned} M &= A^4 + B^4 + C^4 - 2A^2B^2 - 2B^2C^2 - 2C^2A^2, \\ A &= \frac{1}{8}\sin\theta_{32}\sin\theta_{21}\sin\theta_{10}, \\ B &= \frac{1}{8}\sin\theta_{31}\sin\theta_{20}\sin\theta_{12}, \\ C &= \frac{1}{8}\sin\theta_{30}\sin\theta_{21}\sin\theta_{12}, \\ D &= \sin\frac{\theta_{32}}{2}\sin\frac{\theta_{30}}{2}\sin\frac{\theta_{31}}{2}\sin\frac{\theta_{21}}{2}\sin\frac{\theta_{10}}{2}\sin\frac{\theta_{13}}{2}\sin\frac{\theta_{12}}{2}\cos\frac{\theta_{21}}{2} \end{aligned}$$

and R is the radius of the sphere circumscribing the tetrahedron.

Proof. It is obvious that

$$a_{ij} = \frac{r_k \cos\frac{\theta_{kp}}{2}}{\sin\frac{\theta_{ki}}{2}\sin\frac{\theta_{kj}}{2}}, \tag{3}$$

for every $i, j \in \{0, 1, 2, 3\}$, $i \neq j$, $k \in \{0, 1, 2, 3\} \backslash \{i, j\}$ and $p \in S_1 \backslash \{k\}$, where $S_1 = \{0, 1, 2, 3\} \backslash \{i, j\}$.

Now, it is clear that $a_{ij} = a_{ji}$, $\forall i, j \in \{0, 1, 2, 3\}$, $i \neq j$.

Then, we get six relations

$$\frac{r_d}{r_s} = \frac{\cos\frac{\theta_{sd}}{2}\sin\frac{\theta_{dq_1}}{2}\sin\frac{\theta_{dq_2}}{2}}{\cos\frac{\theta_{ds}}{2}\sin\frac{\theta_{sq_1}}{2}\sin\frac{\theta_{sq_2}}{2}}, \tag{4}$$

for every $d, s \in \{0, 1, 2, 3\}$, $d \neq s$ and $q_1, q_2 \in \{0, 1, 2, 3\} \backslash \{d, s\}$, $q_1 \neq q_2$.

Using a_{ij} from equation (3) and relation (4) in the following expansion [2] of $\det(\bar{B})$:

$$
\begin{aligned}
V &= \frac{\sqrt{-\det(\bar{B})}}{24R} \\
&= \frac{1}{24R}((a_{01}a_{23}+a_{02}a_{13}+a_{03}a_{12})(-a_{01}a_{23}+a_{02}a_{13}+a_{03}a_{12}) \\
&\quad \times (a_{01}a_{23}-a_{02}a_{13}+a_{03}a_{12})(a_{01}a_{23}+a_{02}a_{13}-a_{03}a_{12}))^{1/2},
\end{aligned}
$$

we get

$$
\begin{aligned}
V &= \frac{1}{24R}(\frac{r_3r_1}{D})^2\{(A+B+C)(-A+B+C)(A-B+C)(A+B-C)\}^{1/2} \\
&= \frac{r_3^2r_1^2}{24RD^2}\{-(A^4+B^4+C^4-2A^2B^2-2B^2C^2-2C^2A^2)\}^{1/2} \\
&= \frac{r_3^2r_1^2}{24RD^2}\sqrt{-(M)},
\end{aligned}
$$

from which the theorem follows. ∎

Note. One can also find the volume of a tetrahedron in terms of other inradii of faces of tetrahedron using the above relation (3) and (4).

3. Volume of a tetrahedron in terms of R_i, R and θ_{ij}

Let R_i be the circumradii of the corresponding faces F_i for $i = 0, 1, 2, 3$. In this section, we prove:

Theorem 2. *Let R_i be the circumradii of faces F_i, $i \in \{0, 1, 2, 3\}$ and θ_{ij}, for $i, j \in \{0, 1, 2, 3\}$, $i \neq j$, be the face angles of a tetrahedron. Then the volume V of the tetrahedron is given by*

$$
V = \frac{R_3^2R_1^2}{3R}\sqrt{-(N)},
$$

where

$$
\begin{aligned}
N &= A^4+B^4+C^4-2A^2B^2-2B^2C^2-2C^2A^2, \\
A &= \sin\theta_{32}\sin\theta_{10}, \\
B &= \frac{\sin\theta_{31}\sin\theta_{20}\sin\theta_{12}}{\sin\theta_{21}}, \\
C &= \sin\theta_{30}\sin\theta_{12},
\end{aligned}
$$

and R is the radius of the sphere circumscribing the tetrahedron.

Proof. It is obvious that

$$
a_{ij} = 2R_k\sin\theta_{kp}, \tag{5}
$$

for every $i, j \in \{0, 1, 2, 3\}$, $i \neq j$, $k \in \{0, 1, 2, 3\}\backslash\{i, j\}$ and $p \in S_2\backslash\{k\}$, where $S_2 = \{0, 1, 2, 3\}\backslash\{i, j\}$.

Now, it is clear that $a_{ij} = a_{ji}$, $\forall i, j \in \{0, 1, 2, 3\}$, $i \neq j$. Then, we get six relations

$$\frac{R_d}{R_s} = \frac{\sin\theta_{sd}}{\sin\theta_{ds}}, \tag{6}$$

for every $d, s \in \{0, 1, 2, 3\}$, $d \neq s$.

Using a_{ij} from equation (5) and relation (6) in the expansion of $\det(\bar{B})$, we get

$$\begin{aligned} V &= \frac{\sqrt{-\det(\bar{B})}}{24R} \\ &= \frac{16R_3^2R_1^2}{24R}\{(A+B+C)(-A+B+C)(A-B+C)(A+B-C)\}^{1/2} \\ &= \frac{2R_3^2R_1^2}{3R}\sqrt{-(N)}, \end{aligned}$$

from which the theorem follows. ■

Note. One can also find the volume of a tetrahedron in terms of other circumradii of faces of tetrahedron using the relations (5) and (6).

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DEPARTMENT OF MATHEMATICS
JADAVPUR UNIVERSITY
KOLKATA 700032, WEST BENGAL, INDIA
E-mail: bhattachar1968@yahoo.co.in,
buddha.pal@rediffmail.com

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