DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 1 2014

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SLICE THEOREM FOR DIFFERENTIAL SPACES AND REDUCTION BY STAGES

Abstract. We show that the space P/G of orbits of a proper action of a Lie group G on a locally compact differential space P is a locally compact differential space with quotient topology. Applying this result to reduction of symmetries of Hamiltonian systems, we prove the reduction by stages theorem.

1. Introduction

A symplectic manifold is a pair (M, ω), where M is a manifold and ω is a closed non-degenerate 2-form on M. We denote by $C^\infty(M)$ the ring of smooth functions on M. Since ω is non-degenerate, for each $h \in C^\infty(M)$, there exists a unique vector field X_h on M, called the Hamiltonian vector field of X_h, such that

$$X_h \lrcorner \omega = -dh, \tag{1}$$

where $\lrcorner$ denotes the left interior product (contraction) of vector fields and forms. If M is the phase space of a Hamiltonian system with Hamiltonian $h \in C^\infty(M)$, then integral curves of the Hamiltonian vector field X_h of h are trajectories of the system. In this case, equation (1) is equivalent to the Hamiltonian equations of motion of the system.

Let G be a locally compact, connected Lie group with a Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. The group G is a symmetry group of the Hamiltonian system (M, ω, h) if there is a smooth action

$$\Phi : G \times M \to M : (g, x) \mapsto \Phi_g(x) = gx$$

of G on (M, ω) that preserves the symplectic form ω and preserves the Hamiltonian h. In other words, we assume that for every $g \in G$,

$$\Phi_g^* \omega = \omega \quad \text{and} \quad \Phi_g^* h = h.$$

2010 *Mathematics Subject Classification*: 58A40, 70H33.
Key words and phrases: differential space, proper action, reduction by stages, slice.

DOI: 10.2478/dema-2014-0016

Definition 1. The action of G on M is proper if for every convergent sequence (x_n) in M, and a sequence (g_n) in G such that the sequence $(g_n x_n)$ is convergent, there exists a convergent subsequence, (g_{n_k}) of (g_n) and

$$\lim_{n\to\infty}(g_n x_n) = (\lim_{k\to\infty} g_{n_k})(\lim_{n\to\infty} x_n).$$

For each $x \in M$, the isotropy group of x is $G_x = \{g \in G \mid gx = x\}$. If the action of G on M is proper, then all isotropy groups are compact. For every compact subgroup K of G, the set of points of symmetry type K is $M_K = \{x \in M \mid G_x = K\}$. Similarly, the set of points of orbit type K is $M_{(K)} = \{x \in M \mid G_x \text{ is conjugate to } K\}$. The properness of the action of G implies that connected components of M_K and of $M_{(K)}$ are submanifolds of M. We denote by $\mathfrak{M}$ the family of all connected components of sets $M_{(K)}$ of orbit type K, for all compact subgroups K of G.

In order to describe the geometric structure of the space of G-orbits in M, we have to define what we mean by a stratified space. Let T be a topological space and let $\mathfrak{N}$ be a locally finite family of locally closed manifolds N contained in T that cover T. In other words, we assume that each $N \in \mathfrak{N}$ is a locally closed connected subset of T carrying the structure of a smooth manifold such that the manifold topology of N is induced by the inclusion map $N \hookrightarrow T$. Moreover, we assume that $T = \cup_{N\in\mathfrak{N}} N$, and for each $x \in T$, there exists an open neighbourhood U of x that intersects only a finite number of manifolds N in $\mathfrak{N}$. We say that T is stratified by the family $\mathfrak{N}$ if the following condition is satisfied[1].

Condition 2. *For $N, N' \in \mathfrak{N}$, if $N' \cap \overline{N} \neq \varnothing$, then either $N' = N$ or $N' \subset \overline{N}\backslash N$.*

Manifolds $N \in \mathfrak{N}$ are called strata of the stratification of T defined by $\mathfrak{N}$.

Let $R = M/G$ be the space of orbits of a proper action of G on M and let $\rho : M \to R$ be the orbit map. Consider the family $\mathfrak{N}$ consisting of the projections to R of manifolds in $\mathfrak{M}$. In other words, elements of $\mathfrak{N}$ are of the form $N = \rho(C)$, where C is a connected component of $M_{(K)}$ for some compact subgroup K of G.

Theorem 3. *For a proper action of a connected Lie group G on a manifold M, the family $\mathfrak{N}$ is locally finite, consists of locally closed manifolds, and it defines a stratification of the orbit space $R = M/G$.*

Proof. Bierstone [2]. ∎

[1] In the literature, there are a variety of definitions of the notion of "stratification". For example, Mather defines *stratification* of a topological space S as a map from S to the sheaf of germs of manifolds satisfying certain conditions, [12]. Our definition is equivalent to Mather's in the case when S is a differential space. It is more convenient because it does not require the introduction of sheaves.

The stratification of R defined by $\mathfrak{N}$ is called the orbit type stratification of the orbit space.

Theorem 4. *If the proper action of G on M preserves the symplectic form ω, then each stratum of the orbit space $R = M/G$ is a Poisson manifold singularly foliated by symplectic manifolds.*

Proof. See Cushman and Bates [5], and Libermann and Marle [9]. ∎

The existence of symmetries of a Hamiltonian system usually simplifies solving equations of motion. Since the Hamiltonian h is G-invariant, it pushes forward to a function $\overline{h}$ on the orbit space R. In other words, $h = \rho^*\overline{h}$. Suppose that $c : I \to M$ is an integral curve of X_h. Then, the projection $\overline{c} = \rho \circ c$ is contained in a symplectic leaf of a stratum of R and it is an integral curve of the Hamiltonian vector field of $\overline{h}$ defined in terms of the symplectic form on that leaf. In other words, if L is a symplectic leaf of R containing $\overline{c}$, and ω_L is the symplectic form on L, then $\overline{c} : I \to L$ is an integral curve of a vector field $X^L_{\overline{h}}$ on L such that

$$X^L_{\overline{h}} \lrcorner \omega_L = -d\overline{h}_{|L}, \tag{2}$$

where $\overline{h}_{|L}$ is the restriction of $\overline{h}$ to L. Usually, $\dim L$ is smaller than $\dim M$, and the differential equation satisfied by $\overline{c}$ has a smaller number of dependent variables than the equation satisfied by c.

In mechanics, the passage from M to the space $R = M/G$ of G-orbits in M is called reduction of symmetries. In applications, the symmetry group G has often a normal subgroup H. It may be convenient to reduce first the symmetries of the system given by H, and to pass to the space $P = M/H$ of H-orbits in M. Let $\pi : M \to P = M/H$ denote the orbit map. Since the action of G on M is proper, the action of H on M is proper, and by the theorems above, P is a stratified space, each stratum of P is a Poisson manifold singularly foliated by symplectic manifolds.

The quotient group G/H acts on P. Let $Q = P/(G/H)$ be the space of (G/H)-orbits in P and $\eta : P \to Q$ be the orbit map. We have the following identifications $R = M/G = (M/H)/(G/H) = P/(G/H) = Q$, which means that there is a bijection $\beta : R \to Q$ such that

$$\beta \circ \rho = \eta \circ \pi. \tag{3}$$

Equation (3) is an equality of maps in the category of sets. However, our sets have structures and individual maps may preserve these structures. For example, if we consider the orbit type stratifications of M corresponding to the action of G and H, respectively, then ρ and π are morphisms of stratified spaces. On the other hand, β and η are continuous maps. It would be nice if all maps involved here, were morphisms in a category such that equation (3) guarantees that reduction by stages gives the same structure as the reduction of all symmetries at once.

The category of differential spaces and its subcategory of subcartesian differential spaces provide the required setting. We assume that the reader is familiar with the techniques of the theory of differential spaces. A comprehensive bibliography of the literature on differential spaces during the period 1965–1992 is given in [3].

Theorem 5. *The space $R = M/G$ of G-orbits of a proper action of a connected Lie group on a manifold M, endowed with the differential structure*

$$C^{\infty}(R) = \{f : R \to \mathbb{R} \mid \rho^{*} f \in C^{\infty}(M)\},$$

is a subcartesian space.

Proof. Cushman and Śniatycki [7], and Śniatycki [18]. See also Cushman, Duistermaat and Śniatycki [6]. ∎

Theorem 6. *Strata of the orbit type stratification of the orbits space $R = M/G$ of a proper action of G on M are orbits of the family of all vector fields on R.*

Proof. Lusala and Śniatycki [10], and Śniatycki [19]. See also Cushman, Duistermaat and Śniatycki [6]. ∎

Theorems 5 and 6 show that the stratification structure of the orbit space R of a proper action of a connected Lie group G on a manifold M is encoded in the differential structure $C^{\infty}(R)$. Hence, the stratification structure is invariant under diffeomorphisms of differential spaces.

The orbit maps $\rho : M \to R = M/G$ and $\pi : M \to P = M/G$ are smooth in the category of differential spaces. Our aim in this paper is to show that the orbit map $\eta : P \to Q = P/G$ is smooth and that $\beta : R \to Q$ is a diffeomorphism. This will justify reduction by stages.

2. Properness of the action of G/H

Since H is a normal Lie subgroup of G, the quotient $K = G/H$ is a Lie group. The action of K on $P = M/H$, induced by the action of G on M, is given by $(G/H) \times (M/H) \to M/H : (Hg, Hx) \mapsto Hgx$. With this notation, we can rewrite the expression for the action. Our aim in this section is to prove that the properness of the action of G on M implies that the action of G/H on M, given above, is proper. Our main tool for this task is Palais' Slice Theorem [14], which we shall state presently. Here, we need the Slice Theorem for the action of H on M, and we formulate the definition of the slice accordingly.

For each $x \in M$, the isotropy group H_x of x is given by

$$H_x = \{g \in H \mid gx = x\}. \tag{4}$$

Since H is a closed subgroup of G, the assumed properness of the action of G on M implies that the action of H on M is proper. Therefore, for every $x \in M$, the isotropy group H_x of x is compact.

Definition 7. A slice through $x \in M$, for an action of a Lie group H on M, is a submanifold S_x of M containing x such that

1. S_x is transverse and complementary to the orbit Hx of H through x. In other words,
$$T_xM = T_xS_x \oplus T_x(Hx).$$
2. For every $x' \in S_x$, the manifold S_x is transverse to the orbit Hx', that is
$$T_{x'}M = T_{x'}S_x + T_{x'}(Hx').$$
3. S_x is H_x-invariant.
4. For any the following holds: If $gx' \in S_x$ then $g \in H_x$.

Theorem 8. *Let $H \times M \to M : (g, x) \mapsto gx$ be a proper action of a Lie group H on a manifold M. For every $x \in M$, there exists a slice S_x for the action of H on M.*

Proof. Palais [14]. ∎

Remark 9. Let S_x be a slice at x for the action of H. Shrinking S_x, if necessary, we may assume that HS_x is an H-invariant open neighborhood of x in M. Moreover, for any $x' \in S_x$, the orbit Hx' of H through x' intersects S_x along the orbit H_xx' of H_x through x'. For details, see [14].

Consider a convergent sequence (p_n) of orbits in $P = M/H$. Let x be a point in M contained in the limit orbit $p = \lim_{n\to\infty} p_n$. We can write $p = Hx = \pi(x)$. Let $H_x = \{g \in H \mid gx = x\}$ be the isotropy group of x. By Theorem 8, there exists a slice S_x through x for the action of H on M such that HS_x is an H-invariant neighbourhood of x in M. Since the sequence of H-orbits p_n converges to p, it follows that there exists $N > 0$ such that for every $n > N$, the orbit p_n intersects HS_x. Without loss of generality, for each $n \in \mathbb{N}$, we can choose a point x_n contained in the orbit p_n and such that $x_n \in S_x$ whenever $n > N$. Note that for $n > N$, the point $x_n \in S_x$ is determined up to the action of $g \in H_x$. Property 1 of Definition 7 implies that $S_x \cap Hx = \{x\}$. Since $p = \lim_{n\to\infty} p_n$, for every neighbourhood U of x in S_x, there exists $N_U > 0$ such that $p_n \in HU$ for all $n > N_U$. Therefore, $x_n \in U$ for all $n > N_U$. This implies that $x = \lim_{n\to\infty} x_n$.

Suppose now that (k_n) is a sequence in G/H such that the sequence (k_np_n) converges. As before, let y be a point in the orbit $q = \lim_{n\to\infty} k_np_n$, and let S_y be a slice at y for the action of H on M. We may construct a sequence (y_n) in M such that y_n is in the intersection of the orbit $k_n p_n$ and S_y for all n greater than some constant N'. The same argument as before proves that $y = \lim_{n\to\infty} y_n$.

Elements of G/H are H-orbits in G. Hence, there exists a sequence $g_n \in G$ such that $k_n = g_n H$, and

$$Hy_n = k_n p_n = g_n H p_n = g_n H H x_n = H g_n x_n,$$

where the last equality follows from the fact that H is a normal subgroup of G. In other words, there exist elements $\tilde{g}_n \in H$ such that $y_n = \tilde{g}_n g_n x_n$. We have shown above that the sequences x_n and y_n converge in M. Since the action of G on M is proper, it implies that there exists a convergent subsequence $\tilde{g}_{n_k} g_{n_k}$ of $\tilde{g}_n g_n$ such that

$$y = \lim_{n\to\infty} y_n = (\lim_{k\to\infty} \tilde{g}_{n_k} g_{n_k})(\lim_{n\to\infty} x_n) = gx, \tag{5}$$

where $g = (\lim_{k\to\infty} \tilde{g}_{n_k} g_{n_k})$. Hence, the sequence $(\tilde{g}_{n_k} g_{n_k} H)$ converges in G/H to gH. Since H is a normal subgroup of G, and $\tilde{g}_{n_k} \in H$, it follows that $\tilde{g}_{n_k} g_{n_k} = g_{n_k} \tilde{g}'_{n_k}$ for some $\tilde{g}'_{n_k} \in H$. Hence,

$$\tilde{g}_{n_k} g_{n_k} H = g_{n_k} \tilde{g}'_{n_k} H = g_{n_k} H = k_{n_k},$$

and the sequence k_{n_k} converges to gH in G/H. Let $\kappa : G \to K = G/H$ denote the quotient map. Then $gH = \lim_{k\to\infty} k_{n_k}$, and equation (5) projected to P gives

$$\begin{aligned}\lim_{n\to\infty} k_{n_k} p_n = q = \pi(y) = \pi(gx) = \kappa(g)\pi(x) &= (\lim_{k\to\infty} k_{n_k})p \\ &= (\lim_{k\to\infty} k_{n_k})(\lim_{n\to\infty} p_n).\end{aligned}$$

Thus, we have proved the following result

Proposition 10. *Let H be a normal Lie subgroup of a Lie group G. If G has a smooth proper action on a manifold M, then the induced action of G/H on M/H is proper.*

3. The Slice Theorem of Palais

In the preceding section, we used Palais' Slice Theorem in the formulation adapted for a proper action of a Lie group on a manifold. The original result of Palais is also valid for locally compact topological spaces [14]. In this section, we review the original formulation of Palais' results. Definitions and Theorems are quoted from reference [14]. Remarks are added by the authors.

We consider a continuous action $\Phi : G \times P \to P : (g, p) \mapsto \Phi_g(p) = gp$ of a Lie group G on a locally compact topological space P.

Definition 11. P is a proper G-space if each point $p \in P$ has a neighbourhood U such that for every $q \in P$, there exists a neighbourhood V of q for which the closure of the set $\{g \in K \mid gU \cap V \neq \emptyset\}$ is compact.

In the following we assume that P is a proper G-space.

Definition 12. Let H be a closed subgroup of G. A subset Σ of P is an H-kernel if there exists an equivariant map $\varphi : G\Sigma \to G/H$ such that $\varphi^{-1}(H) = \Sigma$.

Remark 13. *The map φ is uniquely determined by Σ.*

Proof. Let $y \in G\Sigma$. Then $y = gs$ for some $g \in G$ and $s \in \Sigma$. Moreover, $\varphi(y) = \varphi(gs) = g\varphi(s) = gH$ since $\varphi(s) = H$ for every $s \in \Sigma$. If $h \in H$, then for each $s \in \Sigma$, $\varphi(hs) = h\varphi(s) = hH = H$. Since, $\Sigma = \varphi^{-1}(H)$, it follows that Σ is H-invariant.

Suppose now that $g_1 s_1 = g_2 s_2$ for $g_1, g_2 \in G$ and $s_1, s_2 \in S$. Then, $g_1 H = \varphi(g_1 s_1) = \varphi(g_2 s_2) = g_2 \varphi(s_2) = g_2 H$. Hence, $\varphi : G\Sigma \to G/H : gs \mapsto gH$ is well defined and uniquely determined by Σ. ∎

Theorem 14. *Let H be a closed subgroup of G. If Σ is an H-kernel in P, then*

1. *Σ is closed in $G\Sigma$.*
2. *Σ is invariant under H.*
3. *$g\Sigma \cap \Sigma \neq \emptyset$ implies that $g \in H$.*

If H is compact, then in addition

4. *Σ has a neighbourhood U in P such that the set $\{g \in G \mid gU \cap U \neq \emptyset\}$ has compact closure.*

Conversely, if the above conditions hold then H is compact and Σ is an H-kernel in P.

Proof. See Theorem 2.14 in [14]. ∎

Definition 15. Let a subset Σ of P be an H-kernel. If $G\Sigma$ is open in P, the set Σ is called an H-slice in P. If in addition $G\Sigma = P$ then Σ is a global H-slice in P.

Since P is a proper G-space, for each $p \in P$, the isotropy group $G_p = \{g \in G \mid gp = p\}$ of p is compact.

Definition 16. A subset Σ of P is a slice at p if Σ is a G_p-slice containing p.

In the following, we shall denote a slice at $p \in P$ by Σ_p or Σ.

Theorem 17. *If P is a proper G-space, then for every point $p \in P$ there exists a slice at p.*

Proof. See Proposition 2.3.1 in [14]. ∎

Next, we show that the notions of a "proper action", given in Definition 1, and of a "proper G-space", given in Definition 11, are equivalent.

Remark 18. *A locally compact topological space P is a proper G-space if and only if the action of G on P is proper.*

Proof. Given $p_0 \in P$, let U be a neighbourhood of p_0 in P with compact closure. Take any $q_0 \in P$ and let V be a neighbourhood of q_0 with compact closure. We want to show that the set $W = \{g \in G \mid gU \cap V \neq \varnothing\}$ has compact closure. In other words, if g_n is a sequence of points in $\overline{W}$ then there exists a convergent subsequence. Each $g_n \in \overline{W}$ is the limit point of a sequence $g_{n,m} \in W$. That is, for each n, m, there exists $p_{n,m} \in U$ such that $g_{n,m}p_{n,m} \in V$. Since V has compact closure, there exists a subsequence $g_{n,m_j}p_{n,m_j}$ convergent to some $q_n \in \overline{V}$. Similarly, since U has compact closure, there exists a subsequence of p_{n,m_k} convergent to $p_n \in \overline{U}$ in the limit as $k \to \infty$. Without loss of generality, we may assume that $g_{n,m}p_{n,m} \to q_n$ and $p_{n,m} \to p_n$ as $m \to \infty$. By construction $g_{n,m} \to g_n$ as $m \to \infty$. The assumption that the action is proper implies $q_n = g_n p_n$ for every $n \in \mathbb{N}$.

Let us now consider sequences $g_n \in \overline{W}$, $p_n \in \overline{U}$ and $q_n \in \overline{V}$ such that $q_n = g_n p_n$ for all $n \in \mathbb{N}$. Since $\overline{U}$ and $\overline{V}$ are compact, without loss of generality, we may assume that these sequences are convergent to $p \in \overline{U}$ and $q \in \overline{V}$, respectively. Properness of the action of G on P implies that there is a subsequence convergent to $g \in G$ such that $q = gp$. However, $\overline{W}$ is closed which implies that $g \in \overline{W}$. Hence, $\overline{W}$ is compact.

Conversely, suppose that P is a proper G-space. Let p_n be a sequence of points in P convergent to p and g_n be a sequence in G such that the sequence $g_n p_n$ converges to $q \in P$. Let U and V be a neighbourhoods of p and q, respectively, such that $\overline{U}$, $\overline{V}$ and $\overline{W}$ are compact, where $W = \{g \in G \mid gU \cap V \neq \varnothing\}$. Since $p_n \to p$ and $g_n p_n \to q$, there exists $N > 0$ such that $g_n \in W \subseteq \overline{W}$ for all $n > N$. Compactness of $\overline{W}$ ensures that the sequence g_n has a convergent subsequence g_{n_m} with limit $g \in \overline{W}$. Since the action of G on P is continuous, it follows that $q = \lim_{n\to\infty} g_n p_n = \lim_{m\to\infty} g_{n_m} p_{n_m} = (\lim_{m\to\infty} g_{n_m})(\lim_{m\to\infty} p_{n_m}) = gp$. This implies that the action Φ of G on P is proper. ∎

4. Proper actions on subcartesian spaces

In this section, we assume that P is a locally compact, subcartesian differential space, and that the action of G on P is smooth. Let H be a compact subgroup of G. We begin with a lemma, which will be needed in the following.

Lemma 19. *Consider an action $\Phi : H \times P \to P : (g, p) \mapsto \Phi_g(p) = gp$ of a compact Lie group H on a subcartesian differential space P. Let $d\mu$ be a*

Haar measure on H, normalized so that the total volume of H is 1. For each $f \in C^\infty(P)$, the H-average

$$\tilde{f} = \int_H \Phi_g^* f \; d\mu(g)$$

is a smooth function on P.

Proof. The pull-back $\Phi^* f$ of $f \in C^\infty(P)$ by the action Φ is a smooth function on $H \times P$ such that $\Phi^* f(g,p) = f(gp) = \Phi_g^* f(p)$. For each $p \in P$, the function $g \mapsto \Phi_g^* f(p)$ on H is smooth. Hence, the integral

$$\tilde{f}(p) = \int_H \Phi_g^* f(p) d\mu(g) = \int_H \Phi^* f(g,p) d\mu(g)$$

exists and $\tilde{f}$ is a function on P. We need to show that $\tilde{f}$ is smooth.

Since P is subcartesian, for each $p \in P$, there exists a neighbourhood V_p of p and a diffeomorphism φ_p of V_p onto a subset of $\mathbb{R}^{n_p}$. Hence,

$$id \times \varphi_p : H \times V_p \to H \times \varphi_p(V_p) : (g,q) \mapsto (g, \varphi(q)) \in H \times \mathbb{R}^{n_p}$$

is a diffeomorphism. This implies that there exists a function $F_p \in C^\infty(H \times \mathbb{R}^{n_p})$ such that $((id \times \varphi_p)^{-1})^*(\Phi^* f_{|H \times V_p}) = F_{p|H \times \varphi_p(V_p)}$. Therefore, for every $(g,q) \in H \times V_p$, $\Phi^* f(g,q) = F_p(g, \varphi_p(q))$. Integrating this equation over H, we get for each $q \in V_p$,

$$\tilde{f}_{|V_p}(q) = \int_H \Phi^* f(g,q) d\mu(g) = \int_H F_p(g, \varphi_p(q)) d\mu(g).$$

Since $F_p \in C^\infty(H \times \mathbb{R}^{n_p})$ and H is compact, it follows that $\tilde{f}_{|V_p}$ is smooth. This means that there exists a function $h_p \in C^\infty(P)$ such that $\tilde{f}_{|V_p} = h_{p|V_p}$. This holds for every $p \in P$, which ensures that $\tilde{f} \in C^\infty(P)$. ∎

Let Σ be an H-slice at p for the action of G on P. By definition of the slice, Σ is invariant under the action of H on P. Hence, we have an action of H on Σ

$$H \times \Sigma \to \Sigma : (g,s) \mapsto gs = \Phi_g s. \tag{6}$$

Let Σ/H be the space of H orbits in Σ and let $\rho_\Sigma : \Sigma \to \Sigma/H$ denote the orbit map. The differential structure $C^\infty(\Sigma)$ of Σ is generated by restrictions to Σ of smooth functions on P. We consider the orbit space Σ/H as a differential space with the differential structure $C^\infty(\Sigma/H) = \{f : \Sigma \to \mathbb{R} \mid \rho_\Sigma^* f \in C^\infty(\Sigma)\}$.

By definition of the slice, the space $G\Sigma = \{gs \in P \mid g \in G \text{ and } s \in \Sigma\}$ is an open G-invariant neighbourhood of $p \in P$. Its differential structure is generated by the restrictions to $G\Sigma$ of smooth functions on P. We denote the space of G-orbits in $G\Sigma$ by $G\Sigma/G$ and the orbit map by $\rho_{G\Sigma} : G\Sigma \to$

$G\Sigma/G$. The differential structure $C^\infty(G\Sigma/G)$ of $G\Sigma/G$ consists of functions $f : G\Sigma/G \to \mathbb{R}$ such that $\rho_{G\Sigma}^* f \in C^\infty(G\Sigma)$.

Let $\iota_\Sigma : \Sigma \hookrightarrow G\Sigma$ be the inclusion map. For each $s \in \Sigma$, the H-orbit Hs extends to the unique G-orbits through s. Thus, we have a one-to-one map $\beta : \Sigma/H \to G\Sigma/G : Hs \mapsto Gs$. Moreover, every G-orbit in $G\Sigma$ intersects Σ along a unique H-orbit, which implies that β is invertible. We have the following commutative diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{\iota_\Sigma} & G\Sigma \\ {\scriptstyle \rho_\Sigma}\downarrow & & \downarrow{\scriptstyle \rho_{G\Sigma}} \\ \Sigma/H & \xrightarrow[\beta]{} & G\Sigma/G \end{array} .$$

Theorem 20. *The bijection $\beta : \Sigma/H \to G\Sigma/G : Hs \mapsto Gs$ is a diffeomorphism.*

Proof. For every G-invariant function $f \in C^\infty(G\Sigma)$, the restriction of f to Σ is H-invariant. It implies that $\beta : \Sigma/H \to G\Sigma/G$ is smooth.

In order to demonstrate that $\beta^{-1} : G\Sigma/G \to \Sigma/H$ is smooth, we have to show that every H-invariant function h on Σ extends to a G-invariant function on $G\Sigma$. Since each point $q \in G\Sigma$ can be presented as $q = gs$ for some $g \in G$ and $s \in \Sigma$, we can define a function f on $G\Sigma$ by

$$f(gs) = h(s). \tag{7}$$

If (g_1, s_1) and $(g_2, s_2) \in G \times \Sigma$ are such that $g_1 s_1 = g_2 s_2$, then $s_2 = g_2^{-1} g_1 s_1$, which implies that $g_2^{-1} g_1 \in H$. The H-invariance of h implies that $h(s_2) = h(g_2^{-1} g_1 s_1) = h(s_1)$. Hence, f is well defined by equation (7).

Next, we need to show that f is smooth. For each ξ in the Lie algebra $\mathfrak{g}$ of G, let X^ξ be the vector field on P corresponding to the action of $\exp t\xi$ on P. Since $G\Sigma$ is G-invariant, the restriction $X^\xi_{|G\Sigma}$ of X^ξ to $G\Sigma$ is a vector field on $G\Sigma$. By assumption, P is subcartesian, which implies that $G\Sigma \subseteq P$ is subcartesian. Hence, for each $q \in G\Sigma$, there exists an open neighbourhood U_q of q in $G\Sigma$ and a diffeomorphism φ_q of U_q onto a subset of $\mathbb{R}^{n_q}$. For each $\xi \in \mathfrak{g}$, $\varphi_{q*} X^\xi_{|U_q}$ is a vector field on $\varphi(U_q)$. Consider the following system of differential equations on $\varphi(U_q)$ for functions $F_q \in C^\infty(\mathbb{R}^{n_q})$:

$$\varphi_{q*} X^\xi_{|U_q}(F_q) = 0 \ \forall \ \xi \in \mathfrak{g}, \text{ and } F_{q|\varphi_q(U_q \cap \Sigma)} = (\varphi_q^{-1})^*(f_{|U_q \cap \Sigma}).$$

Since every G-orbit in $G\Sigma$ intersects Σ, there exists a unique solution of this system, and it satisfies the condition $\varphi_q^*(F_q)_{|U_q \cap U_{q'}} = \varphi_{q'}^*(F_{q'})_{|U_q \cap U_{q'}}$ for every $q, q' \in G\Sigma$. Hence, there exists a unique smooth function on $G\Sigma$ which coincides with $\varphi_q^*(F_q)_{|U_q}$ for every $q \in G\Sigma$. It is easy to see that this function is the function f defined above. ∎

Next we show that the quotient and differential space topologies of our orbit spaces coincide.

Proposition 21. *The differential space topology of $C^\infty(\Sigma/H)$ coincides with the quotient topology.*

Proof. Taking into account reference [15], in order to prove that the topology of the orbit space Σ/H induced by $C^\infty(\Sigma/H)$ coincides with the quotient topology, it suffices to show that for each set V in Σ/H, which is open in the quotient topology, and each $y \in V$, there exists $h \in C^\infty(\Sigma/H)$ such that $h(y) \neq 0$ and $h_{|(\Sigma/H)\backslash V} = 0$.

For $y \in \Sigma/H$, choose $q \in \Sigma$ such that $\rho_\Sigma(q) = y$. Since $G\Sigma$ is open in P, and P is locally compact and Hausdorff, it follows that there exists an open neighbourhood W of q in $G\Sigma$ with closure $\overline{W}$ contained in $\rho^{-1}(V)$, where $\rho : P \to P/G$ is the orbit map. Moreover, there exists a non-negative function $f \in C^\infty(G\Sigma)$ such that $f(q) > 0$, and $f_{|G\Sigma\backslash\overline{W}} = 0$, where $G\Sigma\backslash\overline{W}$ denotes the complement of $\overline{W}$ in $G\Sigma$; see [17], p. 78.

By Lemma 19, the H-average

$$\tilde{f} = \int_H \Phi_g^* f \; d\mu(g)$$

of f over H is in $C^\infty(G\Sigma)$. The assumption that f is non-negative and $f(q) > 0$ implies that $\tilde{f}(q) > 0$. Since $f_{|G\Sigma\backslash\overline{W}} = 0$, it follows that $\tilde{f}_{|G\Sigma\backslash H\overline{W}} = 0$. The compactness of $\overline{W}$ and H imply that the union $H\overline{W}$ of all H-orbits through $\overline{W}$ is compact, and $H\overline{W} = \overline{HW}$, where HW is the union of all H-orbits through W. Moreover, the assumption that $\overline{W} \subseteq \rho^{-1}(V)$ and the H-invarance of $\rho^{-1}(V)$, ensure that $\overline{HW} = H\overline{W} \subseteq \rho^{-1}(V)$. Thus, $\tilde{f}$ is an H-invariant smooth function on $G\Sigma$ such that $\tilde{f}(p) > 0$ and $\tilde{f}$ vanishes on $G\Sigma\backslash\rho^{-1}(V)$.

Let $\tilde{f}_{|\Sigma}$ be the restriction of $\tilde{f}$ to Σ. Since the differential structure $C^\infty(\Sigma)$ is induced by the restrictions to Σ of smooth functions on P, it follows that $\tilde{f}_{|\Sigma}$ is smooth. Moreover, $\tilde{f}_{|\Sigma}(q) = \tilde{f}(q) > 0$, because $q \in \Sigma$. On the other hand, $\tilde{f}$ vanishes on $G\Sigma\backslash\rho^{-1}(V)$. Hence, $\tilde{f}_{|\Sigma}$ vanishes on

$$(G\Sigma\backslash\rho^{-1}(V)) \cap \Sigma = \Sigma\backslash(\rho^{-1}(V) \cap \Sigma) = \Sigma\backslash\rho_\Sigma^{-1}(V).$$

Further, $\tilde{f}_{|\Sigma}$ is H-invariant because $\tilde{f}$ and Σ are H-invariant. By the definition of the differential structure $C^\infty(\Sigma/H)$ of the orbit space, there exists a function $h \in C^\infty(\Sigma/H)$ such that $\tilde{f}_{|\Sigma} = \rho_\Sigma^* h$. Clearly,

$$h(y) = h(\rho(q)) = \rho_\Sigma^* h(q) = \tilde{f}(q) > 0,$$

and

$$0 = (\tilde{f}_{|\Sigma})_{|\Sigma\backslash\rho_\Sigma^{-1}(V)} = (\rho_\Sigma^* h)_{|\Sigma\backslash\rho_\Sigma^{-1}(V)} = h_{|\rho_\Sigma(\Sigma)\backslash V} = h_{|(\Sigma/H)\backslash V},$$

which ensures that the quotient topology and the differential space topology of Σ/H coincide. ∎

Proposition 22. *For a proper action $\Phi : G \times P \to P$ of a Lie group G on a locally compact, subcartesian differential space P, the differential space topology of $C^\infty(P/G)$ coincides with the quotient topology.*

Proof. Let V be a neighbourhood of $y \in P/G$ that is open in the quotient topology. Choose $p \in P$ such that $\rho(p) = y$. The set $\rho^{-1}(V)$ is an open G-invariant neighbourhood of p in P.

Let Σ be a slice through p for the action of G on P. Then $G\Sigma$ is an open G-invariant neighbourhood of p in P. We denote the isotropy group of p by H, and the orbit map by $\rho_\Sigma : \Sigma \to \Sigma/H$.

Since P is locally compact and Hausdorff, there exists an open neighbourhood W of p in P with compact closure $\overline{W}$ contained in $\rho^{-1}(V) \cap G\Sigma$. Without loss of generality, we may assume that W is H-invariant; see the proof of Proposition 21. Then, the set $\rho^{-1}(\rho(W)) \cap \rho^{-1}(V) \cap G\Sigma$ is an open G-invariant neighbourhood of p in G. Hence,

$$\rho^{-1}(\rho(W) \cap V) \cap \Sigma = \rho^{-1}(\rho(W)) \cap (\rho^{-1}(V) \cap G\Sigma) \cap \Sigma$$

is an H-invariant open neighbourhood of p in Σ. Thus, $\rho_\Sigma(\rho^{-1}(W) \cap \rho^{-1}(V) \cap \Sigma)$ is an open neighbourhood of $\rho_\Sigma(p)$ in the quotient topology of Σ/H. By Proposition 21, the differential space topology of $C^\infty(\Sigma/H)$ coincides with the quotient topology. Therefore, there exists a smooth function $h \in C^\infty(\Sigma/H)$ that vanishes in the complement of $\rho_\Sigma(\rho^{-1}(\rho(W) \cap V) \cap \Sigma)$ in Σ/H and such that $h(\rho_\Sigma(p)) = 1$. Since G-orbits in $G\Sigma$ intersect Σ along orbits of H in Σ, and W is H-invariant, it follows that $\rho^{-1}(\rho(W)) \cap \Sigma = W \cap \Sigma$. Therefore, our function h vanishes on the complement of $\rho_\Sigma(\rho^{-1}(V) \cap W \cap \Sigma)$.

By Theorem 20, the map $\beta : \Sigma/H \to G\Sigma/G : Hs \mapsto Gs$ is a diffeomorphism. Therefore, $(\beta^{-1})^* h \in C^\infty(G\Sigma/G)$, and $\rho^*(\beta^{-1})^* h$ is a G-invariant smooth function on $G\Sigma$. By construction,

$$\rho^*(\beta^{-1})^* h(p) = (\beta^{-1})^* h(\rho(p)) = h(\beta^{-1}(\rho(p))) = h(\rho_\Sigma(p)) = 1,$$

because $\rho_{|G\Sigma} \circ \iota_\Sigma = \beta \circ \rho_\Sigma$, where $\iota_\Sigma : \Sigma \hookrightarrow P$ is the inclusion map. On the other hand, suppose that $q \in G\Sigma$ is in the complement of $\rho^{-1}(\rho(W) \cap V) \cap G\Sigma$. Hence, $\rho(q)$ is in the complement of

$$\rho(W) \cap V \cap \rho(G\Sigma) = \rho(W) \cap V \cap (G\Sigma/G) = \rho(W \cap \rho^{-1}(V) \cap \Sigma)$$

in $G\Sigma/G$. Since $\beta^{-1} : G\Sigma/G \to \Sigma/H$ is a diffeomorphism, $\beta^{-1}(\rho(q))$ is in the complement of

$$\beta^{-1} \circ \rho(W \cap \rho^{-1}(V) \cap \Sigma) = \rho_\Sigma(W \cap \rho^{-1}(V) \cap \Sigma)$$

in Σ/H. But h vanishes in the complement of $\rho_\Sigma(\rho^{-1}(V) \cap W \cap \Sigma)$ in Σ/H. Therefore, $\rho^*(\beta^{-1})^*h = h \circ \beta^{-1} \circ \rho$ vanishes on the complement of $\rho^{-1}(\rho(W) \cap V) \cap G\Sigma$ in $G\Sigma$. Hence, the support of $\rho^*(\beta^{-1})^*h$ is a closed set contained in $\rho^{-1}(\rho(W) \cap V) \cap G\Sigma$, which is open in P.

Consider now a point q in the boundary $\partial(G\Sigma)$ of $G\Sigma$ in P. Since $G\Sigma$ is open in P, it follows that $\partial(G\Sigma) = \overline{G\Sigma}\backslash G\Sigma$. We want to show that there exists an open neighbourhood U of q in P such that $U \cap G\Sigma$ is contained in the complement of $\rho^{-1}(\rho(W) \cap V) \cap G\Sigma$. Suppose that there is a sequence U_n of neighbourhoods of q such that $\cap_{n=1}^{\infty} U_n = \{q\}$ and $U_n \cap G\Sigma \cap \rho^{-1}(\rho(W) \cap V) \neq \varnothing$. Then, for each n there is a point $q_n \in U_n \cap G\Sigma \cap \rho^{-1}(\rho(W) \cap V)$, and the sequence q_n converges to q. Moreover, there exists $g_n \in G$ such that $g_n q_n \in W$. Since $\overline{W}$ is compact, there exists a subsequence $g_{n_k} q_{n_k}$ convergent to $\bar{q} = \lim_{k\to\infty} g_{n_k} q_{n_k} \in \overline{W}$. By the properness of the action, without loss of generality, we may assume that the sequence g_{n_k} is convergent to $\bar{g} \in G$ and $\bar{q} = \bar{g}q$. This implies that $\overline{W} \cap \partial(G\Sigma) \neq \varnothing$. But, by assumption, $\overline{W} \subseteq G\Sigma$, and $G\Sigma$ is open in P so that $G\Sigma \cap \partial(G\Sigma) = \varnothing$. Hence, we get a contradiction. This implies that there exists a function $f \in C^\infty(P)$ such that $f_{|G\Sigma} = \rho^*(\beta^{-1})^*h$ and $f_{|P\backslash G\Sigma} = 0$. Clearly, f is G-invariant and it pushes forward to a function $\rho_* f \in C^\infty(P/G)$. By construction $\rho_* f(y) = 1$ and $(\rho_* f)_{|(P/G)\backslash V} = 0$.

The argument above is valid for each point $y \in P/G$ and each neighbourhood V of y in P/G that is open in the quotient topology. Hence, the differential space topology of P/G coincides with its quotient topology [15]. ∎

We show now that the orbit space P/G is Hausdorff. Observe first, that the orbit map $\rho : P \to P/G$ is open. It can be seen as follows. Let U be an open subset of P. For each $g \in G$, $gU = \{gp \in P \mid p \in P\}$ is open, which implies that $GU = \cup_{g\in G} gU$ is open. Hence, $\rho(U) = \rho(GU)$ is open in P/G. Next, consider the relation

$$R = \{(p, q) \in P \times P \mid q = gp \text{ for some } g \in G\}$$

on P defined by the partition of P by orbits of G. Consider a convergent sequence of points in R. It can be written as $(p_n, q_n) = (p_n, g_n p_n)$, where the sequences (p_n) and $(g_n p_n)$ converge in P. Since the action of G on P is proper, there exists a convergent subsequence (g_{n_k}) in G and $\lim_{n\to\infty}(g_n p_n) = (\lim_{k\to\infty} g_{n_k})(\lim_{n\to\infty} p_n)$. Therefore, $\lim_{n\to\infty}(p_n, g_n p_n) \in R$, which implies that R is closed in $P \times P$. This ensures that P/G is Hausdorff; see Theorem 11 in Chapter 3 of reference [8].

In Theorem 20, we showed that the bijection $\beta : \Sigma/H \to G\Sigma/G : Hs \mapsto Gs$ is a diffeomorphism. If W is an H-invariant open subset of Σ, $\rho_\Sigma(W)$

is an open subset of Σ/H that consists of H-orbits contained in W. On the other hand, since $G\Sigma$ is an open subset of P, the set GW, consisting of G-orbits intersecting W, is a G-invariant open subset of G, and $\rho(GW)$ is an open subset of P/G. Moreover, $\rho(W) = \rho(GW) = \beta(\rho_\Sigma(W))$. We shall use these equalities in the arguments below.

Proposition 23. *Let $G \times P \to P$ be a proper action of a Lie group G on a locally compact subcartesian differential space P. The space P/G of G-orbits in P is locally compact.*

Proof. Proposition 22 ensures that P/G is a differential space with quotient topology. For $p \in P$, let H be the isotropy group of p and let V be an open neighbourhood of $\rho(p) \in P/G$. We want to show that there exists an open neighbourhood of $\rho(p)$ in P/G that has compact closure contained in V.

Let Σ be the slice at p for the action of G on P. By definition, $G\Sigma$ is an open G-invariant neighbourhood of p in P. Without loss of generality, we may assume that $\rho^{-1}(V) \subseteq G\Sigma$. Hence, we may consider V as an open subset of $G\Sigma/G$. By Theorem 20, $\beta : \Sigma/H \to G\Sigma/G : Hs \mapsto Gs$ is a diffeomorphism. Hence, $\beta^{-1}(V)$ is open in Σ/H.

Since P is Hausdorff and locally compact, there exists a neighbourhood U of p with compact closure $\overline{U}$ contained in $\rho^{-1}(V)$. Let

$$W = \{gs \in \Sigma \mid g \in H \text{ and } s \in U \cap \Sigma\} = \bigcup_{g \in H} g(U \cap \Sigma) = H(U \cap \Sigma).$$

Since $U \cap \Sigma$ is open in Σ and the action of H on Σ is continuous, it follows that $g(U \cap \Sigma)$ is open in Σ for each $g \in H$. Hence, W is an open H-invariant neighbourhood of p in Σ. Therefore, $\rho_\Sigma(W)$ is an open neighbourhood of $\rho_\Sigma(p)$ contained in $\beta^{-1}(V)$. This implies that $\rho(W) = \beta(\rho_\Sigma(W))$ is an open neigbourhood of $\rho(p)$ in $G\Sigma/G$ contained in V.

The closure $\overline{W}$ of W is the set of limit points of sequences in W. Suppose a sequence $(g_n s_n)$ in W converges to $q \in \overline{W}$. Since the sequence (s_n) is contained in $U \cap \Sigma \subseteq \overline{U} \cap \overline{\Sigma}$ and $\overline{U}$ is compact, there exists a subsequence of (s_{n_m}) convergent to $\bar{q}$ in $\overline{U} \cap \overline{\Sigma}$. Compactness of H implies that there is a subsequence $(g_{n_{m_k}})$ of (g_{n_m}) convergent to $\bar{g} \in H$, and $q = \bar{g}\bar{q} \in H(\overline{U} \cap \overline{\Sigma})$. Conversely, every point of $H(\overline{U} \cap \overline{\Sigma})$ can be presented as a limit of a sequence $g_n s_n$ for $g_n \in H$ and $s_n \in U \cap \Sigma$. Hence, $\overline{W} = H(\overline{U} \cap \overline{\Sigma})$. Since $\overline{U} \subseteq \rho^{-1}(V) \subseteq G\Sigma$, it follows that $\overline{W} = H(\overline{U} \cap \Sigma) \subseteq \rho^{-1}(V) \cap \Sigma$.

The action $\Phi : G \times P \to P$ is continuous. Its restriction $\Phi_H : H \times P \to P$ to an action of H on P is also continuous. Moreover, the set $H(\overline{U} \cap \overline{\Sigma}) = \Phi_H(H \times (\overline{U} \cap \overline{\Sigma}))$. Further, $\overline{U} \cap \overline{\Sigma}$ is compact as a closed subset of a compact set $\overline{U}$. Since the product of compact sets is compact and the image of a compact set under a continuous map is compact, it follows that $\overline{W}$ is compact. Thus, W is an H-invariant neighbourhood of p in Σ such that its closure $\overline{W}$ is compact and contained in $\rho^{-1}(V) \cap \Sigma$.

We have shown above that orbit maps of a proper action are open. Hence, $\rho_\Sigma(W)$ is an open neighbourhood of $\rho_\Sigma(p)$ in Σ/H. Moreover, $\rho_\Sigma(\overline{W})$ is a compact subset of Σ/H contained in $\beta^{-1}(V)$. Hence, $\rho_\Sigma(\overline{W})$ is closed in Σ/H and it contains the closure of $\rho_\Sigma(W)$. Since every point $s \in \overline{W}$ is the limit of a convergent sequence s_n in W. Hence, $\rho_\Sigma(s) = \lim_{n\to\infty} \rho_\Sigma(s_n) \in \overline{\rho_\Sigma(W)}$. Therefore, $\rho_\Sigma(\overline{W}) = \overline{\rho_\Sigma(W)}$, which implies that the closure of $\rho_\Sigma(W)$ is compact.

Since $\beta : \Sigma/H \to G\Sigma/G : Hs \mapsto Gs$ is a diffeomorphism, $\beta(\rho_\Sigma(W))$ is an open neighbourhood of $\rho_{G\Sigma}(p)$ in $G\Sigma/G$ with compact closure contained in $V \subseteq G\Sigma/G$. But, $G\Sigma$ is open in P, so that $G\Sigma/G = \rho(G\Sigma) \subseteq P/G$. Thus, $\beta(\rho_\Sigma(W))$ is an open neighbourhood of $\rho(p)$ with compact closure contained in $V \subseteq P$. This implies that P/G is locally compact. ∎

We have shown that the space of orbits of a proper action of a Lie group G on a locally compact subcartesian space P is a locally compact differential space P/G with the quotient topology. This result is somewhat disappointing if one compares it to the wealth of information we have about spaces of orbits of proper actions of Lie groups on manifolds. In both cases (manifolds and differential spaces) the starting point in an application of the Slice Theorem, which we have discussed here. In the case of smooth manifolds, the next step is Bochner's Linearization Lemma, see [6]. It would be of interest to find a class of subcartesian spaces for which there is an analogue of Bochner's Linearization Lemma.

5. Symplectic reduction by stages

We return to symmetries of Hamiltonian systems discussed in the introduction. We have a symplectic manifold (M, ω), and a proper symplectic action $\Phi : G \times M \to M : (g, x) \mapsto gx$ of G on M. The symplectic form ω defines on $C^\infty(M)$ the structure of a Poisson algebra as follows. For each $f \in C^\infty(M)$, the Hamiltonian vector field of f is the unique vector field X_f on M such that $X_f \lrcorner \omega = df$, where $\lrcorner$ denotes the left interior product (contraction). The Poisson bracket of $f_1, f_2 \in C^\infty(M)$ is given by $\{f_1, f_2\} = X_{f_1} f_2$. It is antisymmetric, satisfies the Jacobi identity $\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0$ and the Leibniz rule $\{\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2 \{f_1, f_3\}$, for every $f_1, f_2, f_3 \in C^\infty(M)$.

By Theorem 5 and Proposition 21, the orbit space $P = M/G$ with the quotient topology is a subcartesian differential space with the differential structure $C^\infty(P) = \{h : P \to \mathbb{R} \mid \pi^* h \in C^\infty(M)\}$, where $\pi : M \to P$ is the orbit map. The ring $C^\infty(P)$ is isomorphic to the ring $C^\infty(M)^G$ of G-

invariant smooth functions on M. Since the action Φ of G on M preserves the symplectic form ω, the induced action of G on $C^\infty(M)$ preserves the Poisson bracket. Hence, the space $C^\infty(M)^G$ is a Poisson sub-algebra of $C^\infty(M)$. Using the ring isomorphism $\pi^* : C^\infty(P) \to C^\infty(M)^G$, we can pull-back the Poisson algebra structure of $C^\infty(M)^G$ to $C^\infty(P)$. For each $h_1, h_2 \in C^\infty(P)$, the Poisson bracket $\{h_1, h_2\}$ is given by $\pi^*\{h_1, h_2\} = \{\pi^* h_1, \pi^* h_2\}$. Since the differential structure $C^\infty(P)$ is a Poisson algebra, we refer to P as a Poisson differential space.

For $h \in C^\infty(P)$, the Poisson derivation Y_h corresponding to h is given by $Y_h h' = \{h, h'\}$ for every $h' \in C^\infty(P)$. Each Poisson derivation Y_h is a vector fields on P in the sense that translations along integral curves of Y_h give a local one-parameter local group of local diffeomorphisms of P. We denote by $\mathfrak{P}(P)$ the family of all Poisson derivations of $C^\infty(P)$. In other words, $\mathfrak{P}(P) = \{Y_h \mid h \in C^\infty(P)\} \subset \mathfrak{X}(P)$, where $\mathfrak{X}(P)$ is the family of all vector fields on P.

By Theorem 3 and Theorem 6, P is a locally compact Hausdorff stratified space, and for each $p \in P$, the stratum S of P through p is a manifold and it is an orbit of the family $\mathfrak{X}(P)$ of all vector fields on P. The space $C^\infty(S)$ of smooth functions on S is generated by restrictions to S of functions in $C^\infty(P)$. The restrictions to S of vector fields on P are tangent to S, because S is an orbit of the family of all vector fields on P. Hence, $C^\infty(S)$ inherits from $C^\infty(P)$ the structure of a Poisson algebra. Thus, S is a Poisson manifolds. Moreover, since restrictions to S of vector fields on P are tangent to S, the space of restrictions to S of Poisson vector fields on P coincides with the space $\mathfrak{P}(S)$ of Poisson vector fields on S.

Poisson manifolds are foliated by symplectic leaves. The orbit through $p \in S$ of the family $\mathfrak{P}(S)$ of Poisson vector fields on S is the symplectic leaf (L, ω_L) through P. The symplectic form ω_L on L is given by the Poisson structure of $C^\infty(P)$ as follows. For every $h_1, h_2 \in C^\infty(P)$ and each $p \in L$, we have $\omega_L(Y_{h_1}(p), Y_{h_2}(p)) = \{h_1, h_2\}(p)$; see [19]. It should be noted that, for each $p \in P$, the symplectic form ω_L of the symplectic leaf L through p is completely determined by the Poisson algebra structure of $C^\infty(P)$.

We have described here the process of singular symplectic reduction; that is, the symplectic reduction in the case when the action of the symmetry group G on (M, ω) is proper following reference [19], for more details see [20]. If G has a normal subgroup H, then we may first reduce by the action of H obtaining the space $Q = M/H$ of H-orbits in M. As before, Q is a stratified subcartesian differential space with differential structure $C^\infty(Q) = \{h_Q : Q \to \mathbb{R} \mid \rho^* h_Q \in C^\infty(M)\}$, where $\rho : M \to Q$ is the orbit map. As before, $C^\infty(Q)$ has the structure of a Poisson algebra and, for each $q \in Q$, the stratum $\tilde{S}$ through q is a Poisson manifold singularly foliated by symplectic leaves. Symplectic leaves $\tilde{L}$ of $\tilde{S}$ are orbits of the family of Poisson vector fields on $\tilde{S}$, and the symplectic form $\omega_{\tilde{L}}$ on $\tilde{L}$ is uniquely determined by the Poisson structure of $C^\infty(Q)$.

Next, we consider the action of the quotient group G/H on $Q = M/H$. In Section 2, we have shown that the action of G/H on Q is proper. By the results of Section 4, the space $R = Q/(G/H)$ of (G/H)-orbits on Q is a locally compact differential space and its differential structure

$$C^\infty(R) = \{h_R : R \to \mathbb{R} \mid \sigma^* h_R \in C^\infty(Q)\},$$

where $\sigma : Q \to R$ is the orbit map, is compatible with the quotient topology.

But the definition of $C^\infty(R)$ implies that

$$C^\infty(R) = \{h_R : R \to \mathbb{R} \mid (\sigma \circ \rho)^* h_R \in C^\infty(M)\}, \tag{8}$$

so that $(\sigma \circ \rho)^* : C^\infty(R) \to C^\infty(M)^G$ is a ring isomorphism. Hence, we can use it to pull-back to $C^\infty(R)$ the Poisson algebra structure of $C^\infty(M)^G$.

Lemma 24. *There is a unique differential space isomorphism $\varphi : P \to R$ such that $\varphi \circ \pi = \sigma \circ \rho$. Moreover, $\varphi^* : C^\infty(R) \to C^\infty(P)$ is a Poisson algebra isomorphism.*

Proof. Both $\pi : M \to P$ and $(\sigma \circ \rho) : M \to R$ are epimorphisms. Given $x_0 \in M$, let $p_0 = \pi(x_0)$, $q_0 = \rho(x_0)$ and $r_0 = \sigma(q_0)$. The fibre $\pi^{-1}(p_0)$ is the orbit Gx_0 of G through x_0. The action of G/H on R associates to each class $[g] \in G/H$ and $q \in Q$ the point $[g]q = \rho(gx)$ for any $x \in \rho^{-1}(q)$. Hence,

$$(\sigma \circ \rho)^{-1}(r_0) = \rho^{-1}(\sigma^{-1}(r_0)) = \rho^{-1}((G/H)q_0) = \rho^{-1}(\{\rho(gx_0) \mid g \in G\}) = Gx_0.$$

This implies that $\pi^{-1}(\pi(x_0)) = (\sigma \circ \rho)^{-1}(\sigma(\rho(x_0))$ for every $x_0 \in M$. Hence, there is a unique bijection $\varphi : P \to R$ such that, $\varphi(\pi(x)) = \sigma(\rho(x))$ for every $x \in M$.

To prove that φ is smooth observe that equation (8) implies that, $(\sigma \circ \rho)^* h_R \in C^\infty(M)^G$ for every $h_R \in C^\infty(R)$. Therefore, there is a unique $h_P \in C^\infty(P)$ such that $(\sigma \circ \rho)^* h_R = \pi^* h_P$. But $\varphi \circ \pi = \sigma \circ \rho$ implies that $h_P = \varphi^* h_R$. Moreover, every $h_P \in C^\infty(P)$ of the form $h_P = \pi^* f$ for $f \in C^\infty(M)^G$. On the other hand, $f = (\sigma \circ \rho)^* h_R$ for some $h_R \in C^\infty(S)$. Hence, φ^* maps $C^\infty(R)$ onto $C^\infty(P)$ which implies that $\varphi : P \to R$ is a diffeomorphism.

The Poisson algebra structures on $C^\infty(P)$ and $C^\infty(R)$ are induced by the Poisson algebra structure on $C^\infty(M)^G$ by ring homomorphisms $\pi^* : C^\infty(P) \to C^\infty(M)^G$ and $(\sigma \circ \rho)^* : C^\infty(R) \to C^\infty(M)^G$, respectively. This implies that $\varphi^* : C^\infty(R) \to C^\infty(P)$ is also a Poisson algebra isomorphism. ■

Note that the result that $\varphi : P \to R$ is an isomorphism in the category of differential spaces implies that, if one of these spaces has special properties that can be described within the category of differential spaces, so has the other. We have seen it already in the Poisson algebra structure of both $C^\infty(P)$ and $C^\infty(R)$.

Theorem 25. *The space R of G/H orbits in $Q = M/H$ is subcartesian. For every $x \in M$, the restriction of φ to the symplectic leaf $M_{\pi(x)}$ of the stratum $S_{\pi(x)}$ of P is a symplectomorphism of $(M_{\pi(x)}, \omega_{\pi(x)})$ onto the symplectic leaf through $r = \varphi(\pi(x)) = \sigma(\rho(x))$ of the orbit through r of the the family $\mathfrak{X}(R)$ of all vector fields on R.*

Proof. (i) Both P and R are locally compact Hausdorff differential spaces that are diffeomorphic to each other. Since the subcartesian property of P is defined within the category of differential spaces, it follows that R is subcartesian.

(ii) The orbit type stratification of $P = M/G$ is given by orbits of the family of all vector fields $\mathfrak{X}(P)$ on P, which is a notion in the category of subcartesian differential spaces. By the argument above, both P and R are subcartesian. Hence the isomorphism $\varphi : P \to R$ in the category of differential spaces implies that P and R are isomorphic in the subcategory of subcartesian differential spaces. Hence, R is a stratified subcartesian space and strata of R are orbits of the family $\mathfrak{X}(R)$ of all vector fields on R. Since $C^\infty(R)$ is a Poisson algebra, each stratum of R is a Poisson manifold and it is foliated by symplectic leaves. Moreover, $\varphi^* : C^\infty(R) \to C^\infty(P)$ is a Poisson algebra isomorphism. This implies that φ maps symplectic leaves of strata of P symplectomorphically onto the corresponding symplectic leaves of strata of R. ∎

Results of this section can be paraphrased by saying that for a proper action of a Lie group on a symplectic manifold, reduction and reduction by stages are equivalent. Note, that we have not required the stages hypothesis that is needed in the usual approach to Poisson reduction by stages; see Section 15.3 of reference [11].

Acknowledgment. The authors are greatly indebted to the Referee whose comments and corrections have led to a significant improvement of this paper.

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Received June 6, 2011; revised version January 24, 2013.