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ORTHOGONAL POLYNOMIALS ON ELLIPSES AND THEIR RECURRENCE RELATIONS

Abstract. In this note we study the connection between orthogonal polynomials on an ellipse and orthogonal Laurent polynomials on the unit circle relative to some multiplicative measures and then establish the recurrence relations for orthogonal polynomials on an ellipse. The matrix representation of the operator of multiplication by coordinate function is obtained.

1. Introduction

Orthogonal polynomials on ellipses have been studied to a much less extent than their counterparts on the real line or even those on the unit circle. In recent decades there has been some work concerning orthogonal polynomials on ellipses. In a study of the invariant subspaces of a three-diagonal Toeplitz operator T, Duren [3, 4] used a sequence $\{p_n(\lambda)\}$ of orthogonal polynomials with respect to a measure $\omega(\lambda)|\,d\lambda|$, where $\omega(\lambda) \geq 0$ and $|\,d\lambda|$ is the arc-length measure on some ellipse $\mathcal{E}_T$, to describe the lattice of invariant subspaces of the three-diagonal operator T. Such polynomials were obtained as part of the computation of the point spectrum of the operator T and they turn out to satisfy the three-term recurrence equation

$$p_{n+1}(\lambda) = \lambda\, p_n(\lambda) - b\, p_{n-1}(\lambda). \tag{1}$$

In [5], Duren proved that, relative to a measure $\omega(\lambda)\,|\,d\lambda|$ on an analytic curve $\mathcal{C}$ in which $\omega(\lambda)$ is a non-negative function, if the corresponding orthogonal polynomials satisfy a tree-term recurrence equation of the form

$$p_{n+1}(\lambda) = (\alpha_n\lambda + \beta_n)p_n(\lambda) + \gamma_n p_{n-1}(\lambda), \tag{2}$$

then the curve $\mathcal{C}$ is an ellipse.

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The orthogonal polynomials with respect to the harmonic measure along the boundary of a Caratheodory domain were studied by Dovgoshei [2] and were proved to satisfy the three-term recurrence equation (2) if and only if the curve is an ellipse.

More recently, Putinar and Stylianopoulos proved in [7] that, under some natural hypothesis, ellipses are the most general curves associated with a finite-term recurrence. We need to recall some terminology. A sequence of orthogonal polynomials with respect to a measure μ satisfies a finite-term recurrence if for every $k \geq 0$, there exists an $N(k) \geq 0$ such that

$$a_{k,n} := \langle \lambda p_n(\lambda), p_k(\lambda) \rangle_\mu = 0, \quad n \geq N(k),$$

while the sequence satisfies an $(N+1)$-term recurrence if

$$\lambda p_n(\lambda) = \sum_{k=0}^{N} a_{n+1-k,n} p_{n+1-k}(\lambda), \quad n \geq N-1.$$

Obviously, polynomials satisfying an $(N+1)$-term recurrence also satisfy a finite-term recurrence.

Thus Putinar and Stylianopoulos proved that for certain domains Ω, the corresponding Bergman orthogonal polynomials (that is, orthogonal polynomials with respect to the area measure), as well as Szegö polynomials (orthogonal polynomials with respect to the arc-length measure on $\partial\Omega$), satisfy a finite-term recurrence if and only if Dirichlet's problem for Ω with polynomial data on $\partial\Omega$ has a polynomial solution. As a consequence, they proved that if Bergman orthogonal polynomials on a Caratheodory domain Ω satisfy a three-term recurrence relation, then Ω must be an ellipsoid (that is a domain whose boundary is an ellipse), as well as: if $\partial\Omega$ is a subset of $\{(x, y) \in \mathbb{R}^2 : \psi(x, y) = 0\}$, where ψ is a polynomial with bounded zero set and if the Bergman orthogonal polynomials satisfy a finite-term recurrence, then the domain is an ellipsoid. A result along the same line was obtained by Khavinson and Stylianopoulos in [6], namely if Bergman orthogonal polynomials on a domain Ω with "nice" boundary satisfy an $(N+1)$-term recurrence with $N \geq 2$, then that the domain is an ellipsoid and $N = 2$.

In the present note, we begin a study of orthogonal polynomials with respect to a measure supported by an ellipse, which satisfies a certain multiplicativity property. The main goal of this note is to obtain the recurrence equations of such polynomials with respect to a finite positive definite Borel measure by studying their connection with the Laurent polynomials on the unit circle with respect to a corresponding measure via a certain transformation.

It is well known that orthogonal polynomials on the real line with respect to a non-negative measure μ satisfy a three-term recurrence equation and the matrix representation of the operator of multiplication by coordinate function gives rise to a symmetric three-diagonal matrix (traditionally called a Jacobi matrix), which is a bounded self-adjoint operator. In the case of the unit circle, the polynomials are dense in $L^2(\mu)$ if the measure μ satisfies a certain property (Szegö condition), and thus the multiplication operator by coordinate function restricted to $P^2(\mu)$ (the closure of the polynomials in $L^2(\mu)$-norm) is a subnormal operator whose matrix representation is a Hessenberg (a matrix whose only nonzero sub-diagonal is the one right below main diagonal and all entries of that sub-diagonal are equal to 1). During last decade it has been proven that the sequence of monic Laurent polynomials obtained by applying the Gramm–Schmidt procedure to the sequence $1, \frac{1}{z}, z, \frac{1}{z^2}, z^2, \ldots$ satisfies a five-term recurrence equation and the operator of multiplication by coordinate function has a "staircase" matrix representation, called CMV matrix, (e.g., cf. [1], [8], [10]). Our goal here is to obtain the equivalent of the CMV matrix for orthogonal polynomials on ellipses.

2. Recurrence equations and matrix representation

For $r > 0$, let $\phi_r(z) := z + \frac{r}{z}$, $\mathbb{T} = \{z : \ |z| = 1\}$ and let $\mathcal{E}_r := \phi_r(\mathbb{T})$ be an ellipse of foci $\pm 2\sqrt{r}$ and x-intercepts $\pm(1 + r)$. For a finite non-negative Borel measure ξ with infinite support on $\mathcal{E}_r$, let $P_n(\lambda)$, $n \in \mathbb{N}$ be the sequence of unique monic polynomials such that the degree of $P_n(\lambda)$ is n and

$$\langle P_n(\lambda), P_m(\lambda)\rangle_\xi = \mu_n^{-2}\ \delta_{nm}, \qquad \mu_n > 0,$$

where $\langle f(\lambda), g(\lambda)\rangle_\xi = \int_{\mathcal{E}_r} f(\lambda) \cdot \overline{g(\lambda)}\, d\xi(\lambda)$. We can assume that $\xi(\mathcal{E}_r) = 1$ since monic orthogonal polynomials will be the same with respect to the normalized measure. Substituting $\lambda \in \mathcal{E}_r$ with $z + \frac{r}{z}$, $z \in \mathbb{T}$, there exists a finite non-negative Borel measure μ_ξ on $\mathbb{T}$ (it will be denoted in what follows by μ) such that

$$\langle P_n(\lambda), P_m(\lambda)\rangle_\xi = \langle \Phi_{2n}(z), \Phi_{2m}(z)\rangle_\mu,$$

where

$$\Phi_{2n}(z) := P_n\left(z + \frac{r}{z}\right),$$

and of course $\langle h(z), k(z)\rangle_\mu := \int_{\mathbb{T}} h(z) \cdot \overline{k(z)}\, d\mu(z)$. Thus, $\Phi_{2n}(z)$ is a Laurent polynomial in which the exponents of z vary between $-n$ and n. We define for $n \geq 1$

$$\Phi_{2n-1}(z) := \overline{\Phi_{2n}}\left(\frac{1}{z}\right),$$

where $\overline{\Phi_{2n}}(\cdot)$ is the Laurent polynomial $\Phi_{2n}(\cdot)$ in which the complex conjugate is applied only to its coefficients; thus for $z \in \mathbb{T}$, $\Phi_{2n-1}(z) = \overline{\Phi_{2n}(z)}$

and consequently,

$$\langle \Phi_{2n}(z), \Phi_{2m}(z) \rangle_\mu = \langle \Phi_{2n-1}(z), \Phi_{2m-1}(z) \rangle_\mu = \mu_n^{-2} \delta_{nm}.$$

We will be interested in measures μ that will give rise to a sequence $\{\Phi_k(z)\}_{k=0}^\infty$ of orthogonal Laurent polynomials.

Proposition 1. *The sequence $\{\Phi_k(z)\}_{k=0}^\infty$ is orthogonal with respect to the measure μ if and only if the measure μ is multiplicative on the sequence $\{\Phi_{2n}(z)\}_{n=0}^\infty$, that is,*

$$\int_{\mathbb{T}} \Phi_{2n}(z) \cdot \Phi_{2m}(z)\, d\mu(z) = 0,\ n, \quad m \geq 1.$$

Proof. Assume first that the measure μ is multiplicative on the sequence $\{\Phi_{2n}(z)\}_{n=0}^\infty$. This is equivalent to $\langle \Phi_{2n}(z), \Phi_{2m-1}(z) \rangle_\mu = 0$, for $n \geq 0$, $m \geq 1$. Since the sequence $\{\Phi_{2n}(z)\}_{n=0}^\infty$ is orthogonal (by construction) and the measure μ is non-negative, it implies the entire sequence $\{\Phi_k(z)\}_{k=0}^\infty$ is orthogonal.

Conversely, if the sequence $\{\Phi_k(z)\}_{k=0}^\infty$ is orthogonal, then

$$\begin{aligned} 0 = \langle \Phi_{2n}(z), \Phi_{2m-1}(z) \rangle_\mu &= \int_{\mathbb{T}} \Phi_{2n}(z) \cdot \overline{\Phi_{2m-1}(z)}\, d\mu(z) \\ &= \int_{\mathbb{T}} \Phi_{2n}(z) \cdot \Phi_{2m}(z)\, d\mu(z), \quad n, m \geq 1. \ \blacksquare \end{aligned}$$

A question that arises naturally is whether such measures exist. Obviously, Dirac measures δ_z, $z \in \mathbb{T}$ are multiplicative measures. On the other hand, since the conversion between the measure ξ on the ellipse and the corresponding measure μ on the unit circle preserves the multiplicativity property, then the measure, say μ_0, that arises from the harmonic measure ξ_0, is another example of such measure.

An interesting and useful question is to describe all measures that satisfy such multiplicative property, but it is not the purpose of the note.

The remainder of this note will be a narrative construction in which we obtain the recurrence equations and a CMV-type of matrix representation of the operator of multiplication by coordinate function and will be concluded with a formal statement.

The assumption that we will make in the remainder of the note is that the measure μ is multiplicative in the sense stated in Proposition 1.

We define a sequence of *standard* Laurent polynomials that will be used to construct some subspaces, as follows:

$$l_k(z) = z^k + \left(\frac{r}{z}\right)^k, \quad \text{and} \quad l_k^*(z) = (rz)^k + \left(\frac{1}{z}\right)^k, \quad k \geq 0.$$

Let

$$H^0 := \vee\{l_0\} = \vee\{\Phi_0\}, \quad H^1 := \vee\{l_0, l_1^*\} = \vee\{\Phi_0, \Phi_1\},$$
$$H^2 := \vee\{l_0, l_1^*, l_1\} = \vee\{\Phi_0, \Phi_1, \Phi_2\},$$

and in general

$$H^{2k-1} := \vee\{l_0, l_1^*, l_1, \dots, l_k^*\} = \vee\{\Phi_0, \Phi_1, \Phi_2, \dots, \Phi_{2k-1}\}, \qquad k \geq 1,$$

and

$$H^{2k} := \vee\{l_0, l_1^*, l_1, \dots, l_k^*, l_k\} = \vee\{\Phi_0, \Phi_1, \Phi_2, \dots, \Phi_{2k-1}, \Phi_{2k}\}, \qquad k \geq 0,$$

where the symbol $\vee$ denotes the linear span generated by finitely many vectors. Since $\Phi_{2n}(z)$ arises from a monic polynomial of degree n in variable λ, one can write

$$\Phi_{2n}(z) = l_n(z) + \sum_{k=1}^{n} \alpha_{n-k,n}\, l_{n-k}(z). \tag{3}$$

We will prove that

$$l_1(z)\,\Phi_{2n}(z) = \Phi_{2n+2}(z) - \sum_{k=2n-2}^{2n} \beta_{k,n}\,\Phi_k(z). \tag{4}$$

The following relations, easily verifiable, will be used in what follows:

$$l_1(z)\,l_k(z) = l_{k+1}(z) + r\,l_{k-1}(z), \qquad k \geq 1, \tag{5}$$

and

$$\begin{aligned} l_1^*(z)\,l_k(z) = {} & c_1(r,k)\,l_{k+1}(z) + c_2(r,k)\,l_{k+1}^*(z) \\ & + c_3(r,k)\,l_{k-1}(z) + c_4(r,k)\,l_{k-1}^*(z), \qquad k \geq 1, \end{aligned} \tag{6}$$

with $c_i(r,k) \neq 0,\ i = 1, \dots, 4$.

To prove (4), denote $\Phi_{2n+2}(z) - l_1(z)\,\Phi_{2n}(z)$ by $\Psi_n(z)$ and, according to (3) and (5), we have

$$\Psi_n(z) \in \vee\{l_0, l_1, \dots, l_n\} \subset H^{2n}.$$

We prove next

$$\langle \Psi_n(z), \Phi_k(z)\rangle_\mu = 0, \qquad \text{for} \quad k \leq 2n-3.$$

First we prove that for $k = 2s \leq 2n-4$, $\langle \Psi_n(z), \Phi_{2s}\rangle_\mu = 0$. Indeed,

$$\langle \Phi_{2n+2}(z), \Phi_{2s}(z)\rangle_\mu = 0,$$

and

$$\langle l_1(z)\,\Phi_{2n}(z), \Phi_{2s}(z)\rangle_\mu = \langle \Phi_{2n}(z), l_1^*(z)\Phi_{2s}(z)\rangle_\mu.$$

According to (6), $l_1^*(z)\,\Phi_{2s}(z) \in H^{2s+2}$ and consequently,

$$\langle \Phi_{2n}(z), l_1^*(z)\Phi_{2s}(z)\rangle_\mu = 0, \quad \text{for} \quad 2s+2 < 2n,$$

and therefore

$$\langle \Psi_n(z), \Phi_{2s}(z)\rangle_\mu = 0, \quad \text{for} \;\; 2s \le 2n-4.$$

Next, we prove $\langle \Psi_n(z), \Phi_{2s-1}\rangle_\mu = 0, \quad \text{for} \;\; 2s-1 \le 2n-3$. Indeed,

$$\langle \Phi_{2n+2}(z), \Phi_{2s-1}(z)\rangle_\mu = 0,$$

and

$$\langle l_1(z)\, \Phi_{2n}(z), \Phi_{2s-1}(z)\rangle_\mu = \langle \Phi_{2n}(z), l_1^*(z)\Phi_{2s-1}(z)\rangle_\mu.$$

According to (6), $l_1^*(z)\,\Phi_{2s-1}(z) \in \vee\{l_1^*, l_2^*, \ldots, l_{s+1}^*\} \subset H^{2s+1}$, and therefore

$$\langle \Phi_{2n}(z), l_1^*(z)\Phi_{2s-1}(z)\rangle_\mu = 0$$

as long as $2n > 2s+1$, which implies that

$$\langle \Psi_n(z), \Phi_{2s-1}\rangle_\mu = 0, \quad \text{for} \;\; 2s-1 \le 2n-3,$$

and consequently $\Psi_n(z) \in H^{2n} \ominus H^{2n-3}$, i.e. relation (4) is proved.

Since eventually we will be interested in the matrix representation, with respect to an orthonormal basis, of the operator of multiplication by variable λ on $L^2(\xi)$, that is, $M_\lambda : L^2(\xi) \to L^2(\xi)$ defined by $(M_\lambda f)(\lambda) = \lambda\, f(\lambda)$, we prove also that

$$l_1^*(z)\,\Phi_{2n}(z) = \Phi_{2n+2}(z) - \sum_{k=2n-3}^{2n+2} \gamma_{k,n}\,\Phi_k(z). \tag{7}$$

The proof of relation (7) follows the same circle of ideas, and for sake of completeness, we include it here. Denote $\Phi_{2n+2}(z) - l_1^*(z)\,\Phi_{2n}(z)$ by $\Lambda_n(z)$, and observe that $\Lambda_n(z) \in H^{2n+2}$. We note that since $\Lambda_n(z)$ belongs to larger subspace than $\Psi_n(z)$ belongs to, the recurrence relation will contain more terms. We prove that $\Lambda_n(z) \in H^{2n+2} \ominus H^{2n-4}$, that is

$$\langle \Lambda_n(z), \Phi_k(z)\rangle_\mu = 0, \quad \text{for} \;\; k \le 2n-4.$$

If $k = 2s \le 2n-4$, then obviously $\langle \Phi_{2n+2}(z), \Phi_{2s}(z)\rangle_\mu = 0$. Furthermore,

$$\langle l_1^*(z)\,\Phi_{2n}(z), \Phi_{2s}(z)\rangle_\mu = \langle \Phi_{2n}(z), l_1(z)\,\Phi_{2s}(z)\rangle_\mu,$$

and according to (5), we have $l_1(z)\,\Phi_{2s}(z) \in H^{2s+2}$, which implies

$$\langle \Phi_{2n}(z), l_1(z)\,\Phi_{2s}(z)\rangle_\mu = 0.$$

On the other hand,

$$\langle \Phi_{2n+2}(z), \Phi_{2s-1}(z)\rangle_\mu = 0,$$

and

$$\langle l_1^*(z)\,\Phi_{2n}(z), \Phi_{2s-1}(z)\rangle_\mu = \langle \Phi_{2n}(z), l_1(z)\,\Phi_{2s-1}(z)\rangle_\mu.$$

According to (6), we have $l_1(z)\,\Phi_{2s-1}(z) \in H^{2s+2}$,

$$\langle \Phi_{2n}(z), l_1(z)\,\Phi_{2s-1}(z)\rangle_\mu = 0,$$

as long as $2s+2 < 2n$, or equivalently, for $2s-1 \le 2n-5$.

Re-denoting the coefficients in equation (4), that equation can be rewritten as follows,

$$\text{(8a)}\qquad \begin{aligned} l_1(z)\,\Phi_{2n}(z) = {} & \Phi_{2n+2}(z) + \alpha_{2n,2n}\,\Phi_{2n}(z) + \alpha_{2n-1,2n}\,\Phi_{2n-1}(z) \\ & + \alpha_{2n-2,2n}\,\Phi_{2n-2}(z), \end{aligned}$$

and after applying the operator "*" and then re-denoting the coefficients in equation (7), the equation can be rewritten as follows,

$$\text{(8b)}\qquad \begin{aligned} l_1(z)\,\Phi_{2n-1}(z) = {} & \alpha_{2n+2,2n-1}\,\Phi_{2n+2}(z) + \alpha_{2n+1,2n-1}\,\Phi_{2n+1}(z) \\ & + \alpha_{2n,2n-1}\,\Phi_{2n}(z) + \alpha_{2n-1,2n-1}\,\Phi_{2n-1}(z) \\ & + \alpha_{2n-2,2n-1}\,\Phi_{2n-2}(z) + \alpha_{2n-3,2n-1}\,\Phi_{2n-3}(z). \end{aligned}$$

The normalized sequence $\{\phi_k(z)\}_{k=0}^{\infty}$, that is

$$\phi_{2n}(z) = \mu_n\,\Phi_{2n}(z) \quad\text{and}\quad \phi_{2n-1}(z) = \mu_n\,\Phi_{2n-1}(z),$$

forms an orthonormal basis for $L^2(\mu)$ that satisfies the following relations:

$$\text{(9a)}\qquad \begin{aligned} l_1(z)\,\phi_{2n}(z) = {} & \frac{\mu_n}{\mu_{n+1}}\,\phi_{2n+2}(z) \\ & + \alpha_{2n,2n}\,\phi_{2n}(z) + \alpha_{2n-1,2n}\,\phi_{2n-1}(z) \\ & + \frac{\mu_n}{\mu_{n-1}}\,\alpha_{2n-2,2n}\,\phi_{2n-2}(z), \end{aligned}$$

and

$$\text{(9b)}\qquad \begin{aligned} & l_1(z)\,\phi_{2n-1}(z) \\ & = \frac{\mu_n}{\mu_{n+1}}\,\alpha_{2n+2,2n-1}\,\phi_{2n+2}(z) + \frac{\mu_n}{\mu_{n+1}}\,\alpha_{2n+1,2n-1}\,\phi_{2n+1}(z) \\ & \quad + \alpha_{2n,2n-1}\,\phi_{2n}(z) + \alpha_{2n-1,2n-1}\,\phi_{2n-1}(z) \\ & \quad + \frac{\mu_n}{\mu_{n-1}}\,\alpha_{2n-2,2n-1}\,\phi_{2n-2}(z) + \frac{\mu_n}{\mu_{n-1}}\,\alpha_{2n-3,2n-1}\,\phi_{2n-3}(z). \end{aligned}$$

If we denote by $p_n(\lambda) := \mu_n\,P_n(\lambda)$ and $p_n^*(\lambda) := \overline{p_n(\lambda)}$, then the above relations become

$$\text{(10a)}\qquad \begin{aligned} \lambda\,p_n(\lambda) = {} & \frac{\mu_n}{\mu_{n+1}}\,p_{n+1}(\lambda) + \alpha_{2n,2n}\,p_n(\lambda) + \alpha_{2n-1,2n}\,p_n^*(\lambda) \\ & + \frac{\mu_n}{\mu_{n-1}}\,\alpha_{2n-2,2n}\,p_{n-1}(\lambda), \end{aligned}$$

and

$$\text{(10b)}\qquad \begin{aligned} \lambda\,p_n^*(\lambda) = {} & \frac{\mu_n}{\mu_{n+1}}\,\alpha_{2n+2,2n-1}\,p_{n+1}(\lambda) + \frac{\mu_n}{\mu_{n+1}}\,\alpha_{2n+1,2n-1}\,p_{n+1}^*(\lambda) \\ & + \alpha_{2n,2n-1}\,p_n(\lambda) + \alpha_{2n-1,2n-1}\,p_n^*(\lambda) \\ & + \frac{\mu_n}{\mu_{n-1}}\,\alpha_{2n-2,2n-1}\,p_{n-1}(\lambda) + \frac{\mu_n}{\mu_{n-1}}\,\alpha_{2n-3,2n-1}\,p_{n-1}^*(\lambda). \end{aligned}$$

Since the sequence $\phi := \{\phi_k(z)\}_{k=0}^{\infty}$ is uniformly dense in the Banach algebra of the continuous functions on the unit circle, and this is dense in $L^2(\mu)$, the sequence ϕ forms an orthonormal basis for $L^2(\mu)$. Similarly, the sequence $p := \{p_0(\lambda), p_1^*(\lambda), p_1(\lambda), \dots\}$ forms an orthonormal basis for $L^2(\xi)$.

Remark. With some standard arguments, the multipicativity property of a measure μ on the sequence of the orthogonal Laurent polynomials generated by μ can be extended to the entire space $L^2(\mu)$.

With the notation used above, we summarize this note with the following.

Theorem. *The matrix representation of the operator* $M_{l_1} : L^2(\mu) \to L^2(\mu)$ *defined by* $(M_{l_1}h)(z) = l_1(z)\,h(z)$ *with respect to the orthonormal basis* $\{\phi_k(z)\}_{k\geq 0}$ *that arises from a multiplicative measure* μ *on on* $L^2(\mu)$*, has the following form:*

$$\begin{pmatrix}
\alpha_{0,0} & \frac{\mu_1}{\mu_0}\alpha_{0,1} & \frac{\mu_1}{\mu_0}\alpha_{0,2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \alpha_{1,1} & \alpha_{1,2} & \frac{\mu_2}{\mu_1}\alpha_{1,3} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\frac{\mu_0}{\mu_1} & \alpha_{2,1} & \alpha_{2,2} & \frac{\mu_2}{\mu_1}\alpha_{2,3} & \frac{\mu_2}{\mu_1}\alpha_{2,4} & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{\mu_1}{\mu_2}\alpha_{3,1} & 0 & \alpha_{3,3} & \alpha_{3,4} & \frac{\mu_3}{\mu_2}\alpha_{3,5} & 0 & 0 & 0 & \cdots \\
0 & \frac{\mu_1}{\mu_2}\alpha_{4,1} & \frac{\mu_1}{\mu_2} & \alpha_{4,3} & \alpha_{4,4} & \frac{\mu_3}{\mu_2}\alpha_{4,5} & \frac{\mu_3}{\mu_2}\alpha_{4,6} & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{\mu_2}{\mu_3}\alpha_{5,3} & 0 & \alpha_{5,5} & \alpha_{5,6} & \frac{\mu_4}{\mu_3}\alpha_{5,7} & 0 & \cdots \\
0 & 0 & 0 & \frac{\mu_2}{\mu_3}\alpha_{6,3} & \frac{\mu_2}{\mu_3} & \alpha_{6,5} & \alpha_{6,6} & \frac{\mu_4}{\mu_3}\alpha_{6,7} & \frac{\mu_4}{\mu_3}\alpha_{6,8} & \cdots \\
0 & 0 & 0 & 0 & 0 & \frac{\mu_3}{\mu_4}\alpha_{7,5} & 0 & \alpha_{7,7} & \alpha_{7,8} & \ddots \\
0 & 0 & 0 & 0 & 0 & \frac{\mu_3}{\mu_4}\alpha_{8,5} & \frac{\mu_3}{\mu_4} & \alpha_{8,7} & \alpha_{8,8} & \ddots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mu_4}{\mu_5}\alpha_{9,7} & 0 & \ddots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mu_4}{\mu_5}\alpha_{10,7} & \frac{\mu_4}{\mu_5} & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}.$$

Of course, the same representation is valid for the operator $M_\lambda : L^2(\xi) \to L^2(\xi)$ with respect to the orthonormal basis p that arises from a multiplicative measure ξ on $\mathcal{E}_r$.

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