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Vincent Grandjean

GRADIENT TRAJECTORIES FOR PLANE SINGULAR METRICS I: OSCILLATING TRAJECTORIES

Abstract. In this short note, we construct an example of a real plane analytic singular metric, degenerating only at the origin, such that any gradient trajectory (respectively to this singular metric) of some well chosen function spirals around the origin. The inversion mapping carries this example into an example of a gradient spiraling dynamics at infinity.

1. Introduction

In the early 60s, Thom asked about the behaviour of the (Euclidean) gradient flow of a given real analytic function nearby the critical locus of the function. He conjectured that any gradient trajectory with limit point a critical point $\mathbf{0}$ (the origin) should have a limit of secants at the origin. It took around thirty years to eventually prove that *Thom Gradient Conjecture* was true. This was achieved by Kurdyka, Mostowski and Parusiński [7], using intensively Łojasiewicz's result on the finiteness of the length of gradient trajectories in a neighbourhood of a limit point [8]. Nowadays questions around the dynamics of a gradient trajectory or of a pencil of trajectories nearby a limit point have switched to asking whether they are analytically oscillating or not. A gradient trajectory is analytically *non-oscillating* if for any semi-analytic subset, the intersection with the given gradient trajectory has finitely many connected components. Given any real analytic gradient differential equation, Moussu Theorem [10] ensures that there are always regular analytic curves through a singular point such that each open half-branch of the curve is a gradient trajectory. Such trajectories are obviously

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non-oscillating. In a few other particular cases [11, 2, 3, 6] the presence of other non-oscillating trajectories is not known.

Assume that $(\mathbb{R}^n, \mathbf{0})$ is equipped with any given real analytic Riemannian metric $\mathbf{g}$. Given a real analytic isolated surface singularity germ $(S, \mathbf{0})$ of $(\mathbb{R}^n, \mathbf{0})$, the regular part S_{reg} of S is equipped with $\mathbf{g}_S$ the restriction of the ambient Riemannian metric to the surface. For any real analytic function $f : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0)$, possibly singular at $\mathbf{0}$, we can consider its restriction f_S to the surface S, and consider the gradient vector field ∇f_S of the function f_S (relative to the metric $\mathbf{g}_S$). It is a real analytic vector field, defined only on S_{reg}. A trajectory of the restricted gradient ∇f_S will be called a *restricted gradient trajectory.* The presence of the singular point $\mathbf{0}$, at which the metric $\mathbf{g}_S$ cannot be extended, may à-priori considerably influence the dynamics of the restricted trajectories nearby $\mathbf{0}$. Nevertheless, this problem is completely understood in the joint work of the author and F. Sanz [6]. Our result states that restricted gradient trajectories do not oscillate at their limit point $\mathbf{0}$.

Since Łojasiewicz [9], we know that the germ $(S, \mathbf{0})$ is topologically a finite union of the closure of positive cones over a circle with vertex $\mathbf{0}$ and so the germ $((S \backslash \mathbf{0}), \mathbf{0})$ has finitely many connected components, each of which is a positive cone. In particular, close enough to the singular point $\mathbf{0}$ any restricted gradient trajectory of ∇f_S must stay in a single positive cone. Such a positive cone is just a punctured plane. From a purely topological dynamical point of view, the problem of the local behaviour of a restricted gradient trajectory nearby the singular point looks like a particular case of the local behaviour of the trajectories of a vector field which outside the point $\mathbf{0}$ is the gradient vector field of a real analytic Riemannian metric $\mathbf{h}$, outside the origin $\mathbf{0}$. If the metric $\mathbf{h}$ cannot be extended through $\mathbf{0}$ into a Riemannian metric, then the metric $\mathbf{h}$ is not positive definite at $\mathbf{0}$ and we will call $\mathbf{0}$ the *degeneracy locus* of $\mathbf{h}$. (The problem of having the 2-symmetric tensor $\mathbf{h}$ defined at the origin $\mathbf{0}$ is not of such importance for the corresponding gradient trajectories.) A trajectory of such a vector field is a *singular gradient trajectory.* Although, there are similarities with the result of the author's joint work [6], the local behaviour of trajectories of singular gradient differential equations at a point of the degeneracy locus of the singular metric is far wilder than that of restricted gradient trajectories, as we will see through a simple example below.

Note that the related work of Dinh, Kurdyka and Orro, about the dynamics of horizontal sub-Riemannian gradient at their singular points of polynomial functions [1], guarantees, for generic polynomials, some asymptotic behaviours similar to the Riemannian case.

The question of the oscillation of singular gradient trajectories was simultaneously presented to us by Prof. K. Bekka and Prof. L. Paunescu.

The paper is organized as follows:

Section 2 introduces in a wider context, the problem of the (singular) gradient trajectories of a function relative to a singular metric near the degeneracy locus of the metric, namely, the locus of points, where the corresponding 2-symmetric tensor is not positive definite, is not empty.

Sections 3 and 4 are devoted to building an example of singular gradient of a function relative to a singular metric degenerating only at the origin, such that the corresponding (singular) gradient trajectories spiral, thus oscillate, in a neighbourhood of a point of the degeneracy locus. We proceed along the following lines:

We will build a real analytic Riemannian metric $\mathbf{h}$ onto the punctured unit ball $\mathbf{B}_1^* := \mathbf{B}_1 \setminus \mathbf{0}$ which extends into a real analytic 2-symmetric tensor through the origin. To achieve that, we first build a real analytic 2-symmetric tensor $\mathbf{g}$ on the spherical blowing-up $[\mathbf{B}_1, \mathbf{0}]$ of the disk $\mathbf{B}_1$ such that it is a Riemannian metric onto the pull-back of the punctured disk and is only positive semi-definite along the boundary circle, exceptional locus of the blowing-up. Thus, we find a real analytic function on $[\mathbf{B}_1, \mathbf{0}]$ whose gradient trajectory accumulates along the whole boundary circle. Up to a rescaling of the singular metric $\mathbf{g}$ by a non-negative function vanishing only on the boundary circle, we blow-down this singular metric to find the singular metric $\mathbf{h}$, and then we blow-down the function. Both are real analytic. Since (singular) gradient differential equation are only sensitive to the conformal structure of the (singular) metric, we are guaranteed that the singular gradient trajectory of the blown-down function spiral around the origin (Proposition 4.1).

Section 5, although short, exploits the previous counter-example using the inversion mapping, to exhibit a plane smooth semi-algebraic metric in a neighborhood of infinity for which there exists a smooth semi-algebraic function with a spiraling gradient dynamics at infinity (Proposition 5.1).

In the last section, we speculate about which properties of the metric at the singular point could cause the oscillating phenomenon, when the geometry of the function is too special in regards of that of the singular metric.

2. On singular gradient differential equations

Let M be a real analytic connected manifold. A real analytic 2-symmetric tensor $\mathbf{h}$ defined on M is called a *singular metric* on M, if there exists a real analytic subset Y of M of codimension larger than or equal to 1, such that $\mathbf{h}$ is positive definite on $M \setminus Y$, and degenerates along Y, that is at each point $y \in Y$, the quadratic form $\mathbf{h}(y)$ is only semi-positive definite. The subset Y is called *the degeneracy locus of the singular metric* $\mathbf{h}$.

Given a real analytic function $f : M \to \mathbb{R}$, we consider the vector field $\nabla_{\mathbf{h}} f$ defined on $M \backslash Y$ as dual of the differential $\mathrm{d}f$, for a given singular metric $\mathbf{h}|_{M\backslash Y}$. By definition, we obtain

$$\mathrm{d}_x f \cdot u = \langle \nabla_{\mathbf{h}} f(x), u \rangle_{\mathbf{h}}, \ \forall x \in M\backslash Y, \ \forall u \in T_x M, \tag{2.1}$$

where $\langle \, , \rangle_{\mathbf{h}}$ denotes the scalar product coming from $\mathbf{h}$. Once are given some coordinates x nearby a point $\mathbf{x}_0$ of M, the quadratic form $\mathbf{h}(x)$ is given by a matrix $H(x)$. The vector field $\nabla_{\mathbf{h}} f$ is given in the local coordinates by $H^{-1}(x)\partial f(x)$, where ∂f is the vector fields of the partial derivative of f in the local coordinates. Let H^* be the adjoint matrix of the matrix H. We recall that $H(x) \cdot H(x)^* = H(x)^* \cdot H(x) = \det H(x) Id = \det \mathbf{h}(x) Id$. In local coordinates we can define the vector field

$$\xi_{\mathbf{h}} f(x) := H^*(x) \cdot \partial f(x) = \det \mathbf{h}(x) \nabla_{\mathbf{h}} f(x). \tag{2.2}$$

The vector field $\xi_{\mathbf{h}} f$ is independent of the coordinates chosen. It is a real analytic vector field on the whole of M, co-linear to $\nabla_{\mathbf{h}} f$ and vanishing on the subset Y.

With an obvious abuse of language we call the next differential equation the *singular gradient differential equation of the function f relative to the singular metric* $\mathbf{h}$:

$$\dot{x}(t) = \xi_{\mathbf{h}} f(x(t)), \ x(0) = x_0 \notin Y. \tag{2.3}$$

We would like to inquire about the behaviour of the singular gradient trajectories in a neighbourhood of a point y of Y.

In the present paper, we are going to provide a simple example of such a situation where the degeneracy locus Y is the origin of M, the real plane, and such that all the trajectories accumulate at this point in spiraling.

3. Plane counter-example: how to make it

We provide an example of a singular metric $\mathbf{h}$ on the real plane whose degeneracy locus consists just of the origin, and we find a function for which all singular gradient trajectories spiral around the origin.

In order to do so, we will work on a spherical blowing-up of the plane. We will produce there a singular metric $\mathbf{g}$ degenerating only on the boundary circle, that is the pre-image of the degeneracy locus of $\mathbf{h}$ under the spherical blowing-up. The singular metric $\mathbf{g}$, we will choose, will be, up to the multiplication by a function vanishing only along the boundary circle, the pull-back under the spherical blowing-up of the singular metric $\mathbf{h}$ we want to produce. The rescaling factor of the lifting of our singular metric $\mathbf{h}$ is of no importance since what the foliation induced by a (singular) gradient differential equation uses from the (singular) metric is its conformal structure.

Such a gradient foliation is insensitive to the sole change of the measure of the length.

Let us consider the spherical blowing up of the real plane:

$$\beta : \mathbf{S}^1 \times \mathbb{R}_{\geq 0} \to \mathbb{R}^2, \text{ defined as } (\mathbf{u}, r) \to r\mathbf{u}.$$

The pre-image $\beta^{-1}(\mathbf{0})$ of the origin $\mathbf{0}$ is thus the boundary circle $\mathbf{S}^1 \times 0$. Instead of working exactly on $\mathbf{S}^1 \times \mathbb{R}_{\geq 0}$, we are going to work on its universal covering $\mathbb{R} \times \mathbb{R}_{\geq 0}$ to exhibit the metric and see very well the spiraling behaviour around the boundary circle of the singular gradient trajectories, relatively to the singular metric we will consider. Thus the boundary circle $\mathbf{S}^1 \times 0$ is replaced by the boundary line $\mathbb{R} \times 0$.

We consider the following 2-symmetric tensor on $\mathbb{R} \times [0, 1[$:

$$\mathbf{g} = \mathrm{d}r^2 + 2r^3 \mathrm{d}r\mathrm{d}\varphi + r^4 \mathrm{d}\varphi^2, \tag{3.1}$$

in the coordinates (φ, r) in $\mathbb{R} \times [0, 1[$.

Given any $(u, v) \in \mathbb{R}^2 \backslash \mathbf{0}$, we check easily that for any $r \in]0, 1[$, the real number $u^2 + 2r^3 uv + r^4 v^2$ is positive.

The determinant of this metric is $r^4(1 - r^2)$ and thus vanishes on $r = 0$. Thus $\mathbf{g}$ is a Riemannian metric on $\{0 < r < 1\}$ and degenerates along the boundary line.

Given any smooth function $(\varphi, r) \to f(\varphi, r)$ defined over $\mathbb{R} \times \mathbb{R}_{\geq 0}$, the gradient vector field $\nabla_{\mathbf{g}} f$ of f for the degenerate metric $\mathbf{g}$ is

$$r^4(1 - r^2)\nabla_{\mathbf{g}} f = [r^4 \partial_r f - r^3 \partial_\varphi f]\partial_r + [-r^3 \partial_r f + \partial_\varphi f]\partial_\varphi.$$

Thus, the gradient differential equation associated with f, up to multiplication by r^4 reads

$$\begin{cases} \dot{r} = r^4 \partial_r f - r^3 \partial_\varphi f, \\ \dot{\varphi} = -r^3 \partial_r f + \partial_\varphi f. \end{cases} \tag{3.2}$$

Let us see how the solution of this differential equation does behave nearby $\{r = 0\}$ in the very simple case of $f(\varphi, r) = -r$.
Namely it reduces to

$$\begin{cases} \dot{r} = -r, \\ \dot{\varphi} = 1. \end{cases} \tag{3.3}$$

We deduce that any trajectory from a point (φ_0, r_0) with $r_0 > 0$ never ends-up on a point $(\varphi_1, 0)$ since $r = 0$ is a trajectory of the above differential equation.

We can check that for $r > 0$ this differential equation reads

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = -\frac{1}{r}. \tag{3.4}$$

Thus, any trajectory from a point (φ_0, r_0) is a graph in r of a function $\varphi(r) = C_0 - \ln(r)$ and thus, $\varphi(r)$ tends to $+\infty$ as r tends to 0.

Remark 3.1. In the present case, we observe that any function g of the form $g := f + r^4 h$, for any real analytic function h defined in a neighbourhood of the boundary circle, will provide singular gradient trajectories which accumulate at a point of $\mathbb{R} \times 0$ only at infinity.

4. Plane singular gradient trajectories spiraling around the origin

The beginning of Section 3 explained how to provide the singular metric on the plane. We will use the spherical blowing-down mapping β, in polar coordinates, in order to find the singular metric that will give the spiraling-around-the-origin behaviour of the whole phase portrait of the singular gradient trajectories of the function Euclidean distance to the origin for the singular metric $\mathbf{h}$.

Let (x, y) be coordinates in $\mathbb{R}^2$, and let us write $x = r\cos\overline{\varphi}$ and $y = r\sin\overline{\varphi}$, for $(r, \overline{\varphi}) \in \mathbb{R}_{\geq 0} \times [0, 2\pi]$.

We thus find

$$r\mathrm{d}r = x\mathrm{d}x + y\mathrm{d}y,$$
$$r^2\mathrm{d}\overline{\varphi} = x\mathrm{d}y - y\mathrm{d}x.$$

Defining $\mathbf{h}(x, y)$ as $r^2\mathbf{g}(r, \overline{\varphi})$, we find

$$\begin{aligned}\mathbf{h} &= (x\mathrm{d}x + y\mathrm{d}y)^2 + 2r^2(x\mathrm{d}x + y\mathrm{d}y)(x\mathrm{d}y - y\mathrm{d}x) + r^2(x\mathrm{d}y - y\mathrm{d}x)^2 \\ &= [x^2 + r^2(-2xy + y^2)]\mathrm{d}x^2 + 2[xy + r^2(x^2 - y^2 - 2xy)]\mathrm{d}x\mathrm{d}y + \\ &\quad [y^2 + r^2(2xy + x^2)]\mathrm{d}y^2.\end{aligned}$$

Thus $\mathbf{h}$ defines a real analytic 2-symmetric tensor on $\mathbb{R}^2$ whose degeneracy locus is the origin at which it is the null quadratic form. Note that $\mathbf{h}$ is positive definite on $\mathbb{R}^2 \cap \{0 < r < 1\}$. Consequently, we restrict our attention to the open unit ball $\mathbf{B}_1$ of $\mathbb{R}^2$, which we equip with the singular Riemannian metric $\mathbf{h}$ just defined above.

Let us consider the universal covering of the spherical blowing-up of $\mathbf{B}_1$, namely, $\widetilde{\beta} : [0, 1[\times\mathbb{R} \to \mathbf{B}_1$, defined as $(r, \varphi) \to (r\cos\varphi, r\sin\varphi)$.

Thus, for any interval $I = [a, a + 2\pi] \subset \mathbb{R}$, for any real number a, the restriction of $\widetilde{\beta}$ to $[0, 1[\times I$ induces a diffeomorphism $]0, 1[\times I$ onto the punctured ball $\mathbf{B}_1^*$.

We obviously check that $\widetilde{\beta}^*(\mathbf{h}) = r^2\mathbf{g}$. We want to understand the asymptotic behaviour of the gradient differential equation $\dot{p} = \nabla_{\mathbf{h}}\delta(p)$, defined on $\mathbf{B}_1^*$, nearby the boundary of this domain, namely the origin $\mathbf{0}$. In the coordinates (x,y), this differential equation reads as

$$\begin{cases} \dot{x} = -2x[y^2 + r^2(2xy + x^2)] + 2y[xy + r^2(x^2 - y^2 - 2xy)], \\ \dot{y} = 2x[xy + r^2(x^2 - y^2 - 2xy)] - 2y[x^2 + r^2(-2xy + y^2)]. \end{cases}$$

When pulled back by $\widetilde{\beta}$, this differential equation transforms into the differential equation of the gradient of $(r,\varphi) \to r^2$, that is the differential equation given by the vector field $2r\nabla_{\mathbf{g}}r$. Thus its trajectories are the same as that of $\nabla_{\mathbf{g}}r$ in $\{r > 0\}$.

Moreover, any non stationary trajectory of $\nabla_{\mathbf{h}}\delta$ is lifted by $\widetilde{\beta}$ in a unique trajectory of $\nabla_{\mathbf{g}}r$ lying in $\{r > 0\}$, the converse is also true.

But as we have already checked in the third section, any trajectory of $\nabla_{\mathbf{g}}r$ with initial data lying in $r > 0$ is a curve of the form $r \to (C_0 - \ln r, r)$. Thus, the image of such gradient trajectory will be mapped by $\widetilde{\beta}$ on a gradient trajectory of $\nabla_{\mathbf{h}}\delta$ lying in $\mathbf{B}_1^*$ and parameterized as $r \to (r\cos(C_0 - \ln r), r\sin(C_0 - \ln r))$ which spirals around the origin as r goes to 0. Thus, we have proved the following

Proposition 4.1. *Any singular gradient trajectory (respectively to the singular Riemannian metric* $\mathbf{h}$*) of the function*

$$\delta : \mathbf{B}_1 \to \mathbb{R} \text{ defined as } (x,y) \to \delta(x,y) := -(x^2 + y^2) = -r^2,$$

with initial data $(x_0, y_0) \neq \mathbf{0}$*, spirals around the origin as the "time" goes to* $+\infty$.

Remark 4.2. To echo Remark 3.1, any function g, analytic or not, of the form $-r^2 + r^5 h$, for h a C^1 function in a neighbourhood of $\mathbf{0}$, will provide singular gradient trajectories, relative to the singular metric $\mathbf{g}$ above, which spiral around the origin $\mathbf{0}$.

5. Example of a spiraling gradient dynamics at infinity

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 semi-algebraic function. The main result of [5] is that any Euclidean gradient trajectory of the function f leaving any compact subset of $\mathbb{R}^n$, has a limit of secants at infinity.

If the space $\mathbb{R}^n$ is equipped with a Riemannian analytic metric, the behaviour of a half-gradient curve is either to accumulate on a point in $\mathbb{R}^n$ or to leave any compact subset of $\mathbb{R}^n$. The image of $\mathbb{R}^n$ under the smooth semi-algebraic diffeomorphism $p \to \frac{p}{\sqrt{1+|p|^2}}$ is the open Euclidean unit ball $\mathbf{B}_1$. The sphere bounding this unit ball will be referred to as *the sphere at infinity.* The behaviour of the given Riemannian metric nearby the sphere at infinity is of great importance for the respective gradient curve leaving any compact subset.

If we restrict our attention to the plane, intuition and reason command that there should exist many metrics in the neighbourhood of the circle at infinity for which we should find globally subanalytic analytic functions whose gradient trajectories leave any compact and spiral, in other words accumulates at each point of the boundary circle at infinity. We are going to give such an example below.

Let $\mathbf{h}$ be the singular metric of Section 4 defined on $\mathbf{B}_1$. The smooth semi-algebraic function $-f : \mathbb{R}^2 \to \mathbb{R}$, where $f(p) := \frac{|p|^2}{1+|p|^2}$, has all its singular gradient trajectories respectively to $\mathbf{h}$ which accumulate onto the origin in spiraling. In other words given any singular gradient trajectory γ, for any unit vector $\mathbf{u} \in \mathbf{S}^1$, there exists a sequence $(p_k)_k$ of points of γ such that $p_k \to \mathbf{0}$ and $\frac{p_k}{|p_k|} \to \mathbf{u}$ as $k \to \infty$.

Let us consider the plane inversion mapping:

$$I : \mathbb{R}^2 \backslash \mathbf{0} \to \mathbb{R}^2 \backslash \mathbf{0} \text{ defined as } p \to \frac{p}{|p|^2}.$$

We find that $I^*\mathbf{h}$ is a smooth semi-algebraic Riemannian metric on the pre-image $I^{-1}(\mathbf{B}_1) = \mathbb{R}^2 \backslash \mathbf{clos}(\mathbf{B}_1)$. We find that $I^* f = f$ and also observe that the origin, seen as the boundary circle of the punctured unit disk, is "mapped" onto the boundary circle at infinity. Thus we deduce the following

Proposition 5.1. *Any gradient curve γ of the function $I^* f$ for the metric $I^*\mathbf{h}$, leaves any compact subset of $\mathbb{R}^2$ and acccumulates on the whole boundary circle at infinity, in other words for any unit vector $\mathbf{u} \in \mathbf{S}^1$, there exists a sequence $(p_k)_k$ of points of γ such that $|p_k| \to \infty$ and $\frac{p_k}{|p_k|} \to \mathbf{u}$ as $k \to \infty$.*

6. Remarks and speculations

1) Given a plane real analytic singular metric, there will be nevertheless uncountably many functions whose singular gradient trajectories will not spiral, for instance those taking positive and negative values close to the origin. Now, given a real analytic function vanishing only at the origin, finding necessary and sufficient conditions so that the pair singular metric and function does not rise a singular gradient differential equation with a spiraling dynamics around $\mathbf{0}$ seems, at the moment, complicated. If we are concerned only with the properties of the metric, this problem is partially solved in [5].

2) The topological description of the author's joint work [6] of restricted gradient on isolated surface singularity suggested that a singular metric degenerating only at a single point might have produced non-oscillating singular gradient trajectories. As we show here it is generally not true. In particular, this naive point of view is forgetting that in this description the degeneracy of the metric is forced by the space, since the restricted metric

can extend to the whole ambient space as a standard Riemannian metric. Consequently, the singular metric of [6] comes from the restriction of a Riemannian metric to a singular "cone". Note that the asymptotic behaviour at the singular point of this restricted metric is just the restriction of the ambient metric to the asymptotic behaviour of the singular "cone" at its tip, namely the limits at the tip of the tangent spaces to the surface.

3) The singular metric, we have exhibited here, presents two asymptotic behaviours at $\mathbf{0}$ which may play some role in the spiraling example presented. First, any limit at $\mathbf{0}$ of the normalized quadratic forms associated with the metric are of rank 1. Consequently to the remark of point 2), the example presented here clearly forbids the embedding of $\mathbf{B}_1$ in $\mathbb{R}^n$, so that the metric $\mathbf{h}$ is the restriction of an ambient metric. Moreover, in the projective space of quadratic forms, the sets of these limits are an embedded projective line. In other words, the "conformal" structure carried by the singular metric $\mathbf{h}$, which is the quality that matters for (singular) gradient trajectories, cannot be defined uniquely at the origin since there is an embedded projective line of such possible limits. Then some geometric properties of the function are such that its singular gradient trajectories spiral around the origin. Which are they relatively to the singular metric, we just do not know yet.

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Temporary Address:
DEPARTAMENTO DE MATEMÁTICA, UFC
Av. Humberto Monte s/n, Campus do Pici Bloco 914
CEP 60.455-760, FORTALEZA-CE, BRASIL
E-mail: vgrandje@fields.utoronto.ca

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