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DEMONSTRATIO MATHEMATICA
Vol. XLVII No 2 2014

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EXTENDED WEYL-TYPE THEOREMS FOR DIRECT SUMS

Abstract. In this paper, we study the stability of extended Weyl and Browder-type theorems for orthogonal direct sum $S \oplus T$, where S and T are bounded linear operators acting on Banach space. Two counterexamples shows that property (ab), in general, is not preserved under direct sum. Nonetheless, and under the assumptions that $\Pi_a^0(T) \subset \sigma_a(S)$ and $\Pi_a^0(S) \subset \sigma_a(T)$, we characterize preservation of property (ab) under direct sum $S \oplus T$. Furthermore, we show that if S and T satisfy generalized a-Browder's theorem, then $S \oplus T$ satisfies generalized a-Browder's theorem if and only if $\sigma_{SBF_+^-}(S \oplus T) = \sigma_{SBF_+^-}(S) \cup \sigma_{SBF_+^-}(T)$, which improves a recent result of [13] by removing certain extra assumptions.

1. Introduction

Throughout this paper, let X and Y be Banach spaces, let $L(X, Y)$ denote the set of bounded linear operators from X to Y, and abbreviate $L(X, X)$ to the Banach algebra $L(X)$. For $T \in L(X)$, we will denote by $\mathcal{N}(T)$ the null space of T, by $\alpha(T)$ the nullity of T, by $\mathcal{R}(T)$ the range of T, by $\beta(T)$ the defect of T and by T^* its dual. We will denote also by $\sigma(T)$ the spectrum of T, by $\sigma_a(T)$ the approximate point spectrum of T, by $\sigma_p(T)$ the point spectrum of T (the set of all eigenvalues), and by $\sigma_p^0(T)$ the set of all eigenvalues of T of finite multiplicity. If the range $\mathcal{R}(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an *upper semi-Fredholm* (resp. a *lower semi-Fredholm*) operator. If $T \in L(X)$ is either an upper or a lower semi Fredholm, then T is called a *semi-Fredholm* operator, and the *index* of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a *Fredholm* operator. An operator $T \in L(X)$ is called a *Weyl* operator if it is a Fredholm operator of index zero. The *Weyl spectrum* $\sigma_W(T)$ of T is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not a Weyl operator$\}$.

2010 *Mathematics Subject Classification*: Primary 47A53, 47A10, 47A11.
Key words and phrases: property (gab), property (gaw), direct sums, B-Weyl spectrum.
*Supported by Protars D11/16 and PGR-UMP.

DOI: 10.2478/dema-2014-0032

For a bounded linear operator T and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_{[0]} = T$). If for some integer n, the range space $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an *upper* (resp. a *lower*) *semi-B-Fredholm* operator. A *semi-B-Fredholm* operator T is an upper or a lower semi-B-Fredholm operator, and in this case the index of the semi-B-Fredholm operator T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [7]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm* operator, see [3]. An operator $T \in L(X)$ is said to be a *B-Weyl* operator if it is a B-Fredholm operator of index zero [4]. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not a B-Weyl operator$\}$.

Let $SBF_+(X)$ be the class of all upper semi-B-Fredholm operators, $SBF_+^-(X) = \{T \in SBF_+(X) : \mathrm{ind}(T) \leq 0\}$. The *essential semi-B-Fredholm spectrum* $\sigma_{SBF_+^-}(T)$ of T is defined by $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_+^-(X)\}$. Recall that the *ascent* $a(T)$, of an operator T, is defined by $a(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$ and the *descent* $\delta(T)$ of T, is defined by $\delta(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$, with $\inf \emptyset = \infty$. An operator $T \in L(X)$ is called an *upper semi-Browder* operator if it is an upper semi-Fredholm operator of finite ascent, and is called a Browder operator if it is a Fredholm operator of finite ascent and descent, or equivalently ([15, Theorem 7.9.3]) if T is a Fredholm operator and $T - \lambda I$ is invertible for all sufficiently small $\lambda \in \mathbb{C}$, $\lambda \neq 0$. The *upper semi-Browder spectrum* $\sigma_{ub}(T)$ of T is defined by $\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not an upper semi-Browder operator$\}$, and the *Browder spectrum* $\sigma_b(T)$ of T is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Browder$\}$.

Define also the set $LD(X)$ by $LD(X) = \{T \in L(X) : a(T) < \infty$ and $\mathcal{R}(T^{a(T)+1})$ is closed$\}$. Following [6], an operator $T \in L(X)$ is said to be *left Drazin invertible* if $T \in LD(X)$. The *left Drazin spectrum* $\sigma_{LD}(T)$ of T is defined by $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(X)\}$. We say that $\lambda \in \sigma_a(T)$ is a *left pole* of T if $T - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$. Let $\Pi_a(T)$ denote the set of all left poles of T and let $\Pi_a^0(T)$ denote the set of all left poles of T of finite rank. Following [6], we say that T satisfies *generalized Browder's theorem* if $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$ and it satisfies *Browder's theorem* if $\sigma_W(T) = \sigma(T) \setminus \Pi^0(T)$, where $\Pi(T)$ is the set of all poles of the resolvent of T and $\Pi^0(T) = \{\lambda \in \Pi(T) : \alpha(T - \lambda I) < \infty\}$.

Definition 1.1. [10] Let $S \in L(X)$ and $T \in L(Y)$. We will say that S and T have the *same stable sign index* if for each $\lambda \in \rho_{SBF}(T)$ and $\mu \in \rho_{SBF}(S)$, $\mathrm{ind}(T - \lambda I)$ and $\mathrm{ind}(S - \mu I)$ have the same sign, where $\rho_{SBF}(T) =$

$\mathbb{C}\backslash\sigma_{SBF}(T)$ and

$$\sigma_{SBF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a semi-B-Fredholm operator}\}.$$

Several authors had been concerned with the problem of giving conditions on the direct summands to ensure that Weyl-Browder type theorems and properties (generalized or not) hold for the direct sum, see for example [10–14, 17]. In the present paper, we study the preservation under direct sum of the properties (ab), (gab), (aw) and (gaw), and the results we obtain can be summarized as follows. In the second section, we give two counterexamples which show that property (gab) and property (ab) are not transferred in general from the direct summands $S \in L(X)$ and $T \in L(Y)$ to the direct sum $S \oplus T \in L(X \oplus Y)$. Nonetheless, and under the extra assumptions that $\Pi_a(S) \subset \sigma_a(T)$ and $\Pi_a(T) \subset \sigma_a(S)$, we characterize the stability of property (gab) under direct sum via union of B-Weyl spectra of its components. We obtain also an analogous result for property (ab). Moreover, we characterize the preservation of generalized a-Browder's theorem under direct sum via union of essential semi-B-Fredholm spectra of its components, extending [13, Theorem 2.7] by removing certain extra hypothesis. In the third section, we characterize the stability of properties (aw) and (gaw) under direct sum via union of Weyl or B-Weyl spectra of its summands, and under the assumption of equality of their point spectrums.

2. Properties (ab) and (gab) for direct sums

We will say that $T \in L(X)$ has the *single valued extension property* at λ_0, (SVEP for short) if for every open neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f : \mathcal{U} \longrightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$, is the function $f = 0$. An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$ (see [16]). For $T \in L(X)$, let $\Delta(T) = \sigma(T)\backslash\sigma_W(T)$, $\Delta^g(T) = \sigma(T)\backslash\sigma_{BW}(T)$.

According to [8, Definition 2.1], an operator $T \in L(X)$ is said to possess *property* (ab) if $\Delta(T) = \Pi_a^0(T)$ and is said to possess *property* (gab) if $\Delta^g(T) = \Pi_a(T)$. Property (gab) is an extension to the context of B-Fredholm theory of the property (ab). We refer the reader to [8, 9] for more details about these properties.

In a first step, we begin with the following useful lemma.

Lemma 2.1. *Let $S \in L(X)$ and let $T \in L(Y)$. Then $\sigma_{LD}(S\oplus T) = \sigma_{LD}(S)\cup \sigma_{LD}(T)$.*

Proof. Let $\lambda \notin \sigma_{LD}(S) \cup \sigma_{LD}(T)$ be arbitrary. Without loss of generality, we can assume that $\lambda = 0$. Then $a(T) < \infty$, $a(S) < \infty$, the range spaces $\mathcal{R}(T^{a(T)+1})$ and $\mathcal{R}(S^{a(S)+1})$ are closed. As $a(S \oplus T) = \max\{a(T), a(S)\}$

then $a(S \oplus T) < \infty$. If $a(S \oplus T) = a(T)$ then $\mathcal{R}((S \oplus T)^{a(S \oplus T)+1}) = \mathcal{R}(S^{a(T)+1}) \oplus \mathcal{R}(T^{a(T)+1})$. Since $a(S) < \infty$ and $\mathcal{R}(S^{a(S)+1})$ is closed then by [18, Lemma 12], we conclude that $\mathcal{R}(S^{a(T)+1})$ is also closed. So $\mathcal{R}(S^{a(T)+1}) \oplus \mathcal{R}(T^{a(T)+1})$ is closed. Similarly, if $a(S \oplus T) = a(S)$, then $\mathcal{R}((S \oplus T)^{a(S \oplus T)+1})$ is closed. Thus $0 \notin \sigma_{LD}(S \oplus T)$ and $\sigma_{LD}(S \oplus T) \subset \sigma_{LD}(S) \cup \sigma_{LD}(T)$.

On the other hand, if $0 \notin \sigma_{LD}(S \oplus T)$ then $S \oplus T$ is left Drazin invertible, so that $a(S \oplus T)$ is finite and $\mathcal{R}((S \oplus T)^{a(S \oplus T)+1})$ is closed. Clearly, $a(S)$ and $a(T)$ are finite, and it is easily seen that $\mathcal{R}(T^{a(T)+1})$ and $\mathcal{R}(S^{a(S)+1})$ are closed. Thus $0 \notin \sigma_{LD}(S) \cup \sigma_{LD}(T)$. Hence $\sigma_{LD}(S \oplus T) = \sigma_{LD}(S) \cup \sigma_{LD}(T)$. ■

If $S \in L(X)$ and $T \in L(Y)$ both possess property (gab), then it is not guaranteed that their (orthogonal) direct sum $S \oplus T \in L(X \oplus Y)$ possesses property (gab). For instance, let R be the unilateral right shift operator defined on the Hilbert space $\ell^2(\mathbb{N})$, then $\sigma(R) = D(0,1)$ the closed unit disc in $\mathbb{C}$, $\sigma_a(R) = C(0,1)$ the unit circle of $\mathbb{C}$, $\sigma_{BW}(R) = D(0,1)$ and $\Pi_a(R) = \emptyset$. It follows that $\Delta^g(R) = \Pi_a(R)$, i.e. R possesses property (gab). If we let $T = 0$ then $\sigma_a(T) = \sigma(T) = \Pi_a(T) = \{0\}$ and $\sigma_{BW}(T) = \emptyset$. So $\Delta^g(T) = \Pi_a(T)$, i.e. T possesses property (gab). Now consider the direct sum $R \oplus T$ on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ then $\sigma(R \oplus T) = D(0,1)$, $\sigma_{BW}(R \oplus T) = D(0,1)$ and $\Pi_a(R \oplus T) = \{0\}$. Thus $R \oplus T$ does not possess property (gab). We notice that $\Pi_a(T) \not\subset \sigma_a(R)$ and $\Pi_a(R) \subset \sigma_a(T)$. Summing up: $R \oplus T$ is an operator for which property (gab) does not hold, although property (gab) holds for both of its direct summands.

Nonetheless, we give in the following result sufficient conditions on T and S under which the property (gab) will be transferred from the direct summands to the direct sum.

Theorem 2.2. *Suppose that $S \in L(X)$ and $T \in L(Y)$ be such that $\Pi_a(T) \subset \sigma_a(S)$ and $\Pi_a(S) \subset \sigma_a(T)$. If S and T both possess property (gab), then the following assertions are equivalent.*

(i) *$S \oplus T$ possesses property (gab);*
(ii) *$\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$.*

Proof. (ii)⇒(i) Suppose that $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$. Since we know that $\sigma_a(S \oplus T) = \sigma_a(S) \cup \sigma_a(T)$ for every pair of operators, and by Lemma 2.1, we have $\sigma_{LD}(S \oplus T) = \sigma_{LD}(S) \cup \sigma_{LD}(T)$ then

$$\begin{aligned}\Pi_a(S \oplus T) &= \sigma_a(S \oplus T) \backslash \sigma_{LD}(S \oplus T)\\ &= [\sigma_a(S) \cup \sigma_a(T)] \backslash [\sigma_{LD}(S) \cup \sigma_{LD}(T)]\\ &= [\Pi_a(S) \cap \rho_a(T)] \cup [\Pi_a(T) \cap \rho_a(S)] \cup [\Pi_a(S) \cap \Pi_a(T)],\end{aligned}$$

where $\rho_a(.) = \mathbb{C} \backslash \sigma_a(.)$. As by assumption, $\Pi_a(S) \subset \sigma_a(T)$ and $\Pi_a(T) \subset \sigma_a(S)$ then $\Pi_a(S) \cap \rho_a(T) = \emptyset$ and $\Pi_a(T) \cap \rho_a(S) = \emptyset$. Hence $\Pi_a(S \oplus T) =$

$\Pi_a(S) \cap \Pi_a(T)$. On the other hand, since T and S both possess property (gab) then

$$[\sigma(S) \cup \sigma(T)]\backslash[\sigma_{BW}(S) \cup \sigma_{BW}(T)] \\ = [\Pi_a(S) \cap \rho(T)] \cup [\Pi_a(T) \cap \rho(S)] \cup [\Pi_a(S) \cap \Pi_a(T)],$$

where $\rho(.) = \mathbb{C}\backslash\sigma(.)$. Again by hypothesis we have $\Pi_a(S) \cap \rho(T) = \emptyset$ and $\Pi_a(T) \cap \rho(S) = \emptyset$. Hence

$$[\sigma(S) \cup \sigma(T)]\backslash[\sigma_{BW}(S) \cup \sigma_{BW}(T)] = \Pi_a(S \oplus T).$$

As we know that $\sigma(S\oplus T) = \sigma(S)\cup\sigma(T)$ for any pair of operators, and by hypothesis we have $\sigma_{BW}(S\oplus T) = \sigma_{BW}(S)\cup\sigma_{BW}(T)$ then $\sigma(S\oplus T)\backslash\sigma_{BW}(S\oplus T) = \Pi_a(S \oplus T)$, and $S \oplus T$ possesses property (gab).

(i)$\Rightarrow$(ii) Since $S\oplus T$ possesses property (gab) and both of S and T possess property (gab), we have $\sigma_{BW}(S) \cup \sigma_{BW}(T) = \sigma_{BW}(S \oplus T)$. To see this, from [8, Corollary 2.6], $S \oplus T$ satisfies generalized Browder's theorem, so $\sigma_D(S \oplus T) = \sigma_{BW}(S \oplus T)$. Since $\sigma_D(S \oplus T) = \sigma_D(S) \cup \sigma_D(T)$ is always true (see [10, Theorem 2.4]) then $\sigma_{BW}(S) \cup \sigma_{BW}(T) \subset \sigma_D(S) \cup \sigma_D(T) = \sigma_D(S\oplus T) = \sigma_{BW}(S\oplus T)$, that is $\sigma_{BW}(S) \cup \sigma_{BW}(T) \subset \sigma_{BW}(S\oplus T)$. Since by [10, Lemma 2.2] $\sigma_{BW}(S) \cup \sigma_{BW}(T) \supset \sigma_{BW}(S\oplus T)$ then $\sigma_{BW}(S\oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$. ∎

Remark 2.3. In Theorem 2.2, the symmetric conditions $\Pi_a(T) \subset \sigma_a(S)$ and $\Pi_a(S) \subset \sigma_a(T)$ alone do not imply the property (gab) for $S\oplus T$, although property (gab) holds for both of its direct summands. Indeed, let R be the unilateral right shift operator defined on the Hilbert space $\ell^2(\mathbb{N})$ and let L its adjoint (the left shift operator on $\ell^2(\mathbb{N})$). As it had been already mentioned, we have R possesses property (gab). Since $\sigma_a(L) = \sigma(L) = D(0,1)$, $\sigma_{BW}(L) = D(0,1)$ and $\Pi_a(L) = \emptyset$ then L possesses property (gab). Moreover, we have $\Pi_a(R) \subset \sigma_a(L)$ and $\Pi_a(L) \subset \sigma_a(R)$, but $R \oplus L$ does not possess property (gab). Indeed, as $\alpha(R \oplus L) = \beta(R \oplus L) = 1$ then $0 \notin \sigma_W(R\oplus L)$, and since $a(R\oplus L) = \infty$, it follows that $R\oplus L$ does not have the SVEP at 0. From [1, Theorem 2.2], $R \oplus L$ does not satisfy Browder's theorem, and since we know from [2, Theorem 2.1] that Browder's theorem is equivalent to generalized Browder's theorem, it then follows that $R \oplus L$ does not satisfy generalized Browder's theorem. Hence by [8, Corollary 2.6], it does not possess property (gab). Here the inclusion $\sigma_{BW}(R\oplus L) \subset \sigma_{BW}(R) \cup \sigma_{BW}(L)$ is proper, because $\sigma_{BW}(R) \cup \sigma_{BW}(L) = D(0,1)$ and $0 \notin \sigma_{BW}(R\oplus L)$.

The (bounded linear) operator $A \in L(X,Y)$ is said to be *quasi-invertible* if it is injective and has dense range. Two bounded linear operators $T \in L(X)$ and $S \in L(Y)$ on complex Banach spaces X and Y are *quasisimilar* provided there exist quasi-invertible operators $A \in L(X, Y)$ and $B \in L(Y, X)$ such that $AT = SA$ and $BS = TB$. A bounded linear operator T acting on a Hilbert space $\mathcal{H}$ is said to be hyponormal if $T^*T - TT^* \geq 0$ (or equivalently $\|T^*x\| \leq \|Tx\|$ for all $x \in \mathcal{H}$).

Corollary 2.4. *Let $S \in L(\mathcal{H})$ and $T \in L(\mathcal{H})$ be quasisimilar hyponormal operators. If S and T both possess property (gab), then $S \oplus T$ possesses property (gab).*

Proof. As S and T are quasisimilar hyponormal operators, then by [10, Lemma 2.8] we have $\Pi(T) = \Pi(S)$. The property (gab) for S and for T entails that $\Pi(T) = \Pi_a(T)$ and $\Pi(S) = \Pi_a(S)$. So $\Pi_a(S) \subset \sigma_a(T)$ and $\Pi_a(T) \subset \sigma_a(S)$. On the other hand, since it is well known that every hyponormal operator has the SVEP, then from [5, Theorem 2.5], we deduce that S and T are of stable sign index. This implies from [10, Proposition 2.3] that $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$. But this is equivalent by Theorem 2.2 to saying that $S \oplus T$ possesses property (gab). ∎

Similarly to Theorem 2.2, we prove the following preservation result of property (ab) under direct sums.

Theorem 2.5. *Suppose that $S \in L(X)$ and $T \in L(Y)$ be such that $\Pi_a^0(S) \subset \sigma_a(T)$ and $\Pi_a^0(T) \subset \sigma_a(S)$. If S and T both possess property (ab), then the following statements are equivalent.*

(i) *$S \oplus T$ possesses property (ab);*
(ii) *$\sigma_W(S \oplus T) = \sigma_W(S) \cup \sigma_W(T)$.*

Proof. (ii)⇒(i) Suppose that $\sigma_W(S \oplus T) = \sigma_W(S) \cup \sigma_W(T)$. Since we know that the upper semi-Browder spectrum of a direct sum is the union of the upper semi-Browder spectra of its components, that is, $\sigma_{ub}(S \oplus T) = \sigma_{ub}(S) \cup \sigma_{ub}(T)$, then

$$\begin{aligned}\Pi_a^0(S \oplus T) &= \sigma_a(S \oplus T) \backslash \sigma_{ub}(S \oplus T) \\ &= [\sigma_a(S) \cup \sigma_a(T)] \backslash [\sigma_{ub}(S) \cup \sigma_{ub}(T)] \\ &= [\Pi_a^0(S) \cap \rho_a(T)] \cup [\Pi_a^0(T) \cap \rho_a(S)] \cup [\Pi_a^0(S) \cap \Pi_a^0(T)].\end{aligned}$$

Since by hypothesis $\Pi_a^0(T) \subset \sigma_a(S)$ and $\Pi_a^0(S) \subset \sigma_a(T)$ then $\Pi_a^0(S) \cap \rho_a(T) = \Pi_a^0(T) \cap \rho_a(S) = \emptyset$. Therefore $\Pi_a^0(S \oplus T) = \Pi_a^0(S) \cap \Pi_a^0(T)$. On the other hand, as T and S both possess property (ab) then

$$\begin{aligned}&[\sigma(S) \cup \sigma(T)] \backslash [\sigma_W(S) \cup \sigma_W(T)] \\ &\qquad = [\Pi_a^0(S) \cap \rho(T)] \cup [\Pi_a^0(T) \cap \rho(S)] \cup [\Pi_a^0(S) \cap \Pi_a^0(T)].\end{aligned}$$

Since $\Pi_a^0(S) \cap \rho(T) = \Pi_a^0(T) \cap \rho(S) = \emptyset$, it then follows that

$$[\sigma(S) \cup \sigma(T)] \backslash [\sigma_W(S) \cup \sigma_W(T)] = \Pi_a^0(S \oplus T).$$

As by hypothesis $\sigma_W(S\oplus T) = \sigma_W(S) \cup \sigma_W(T)$ then $\sigma(S\oplus T)\backslash\sigma_W(S\oplus T) = \Pi_a^0(S \oplus T)$, and $S \oplus T$ possesses property (ab).

(i)$\Rightarrow$(ii) The property (ab) for $S \oplus T$ implies with no other restriction on either S or T that $\sigma_W(S\oplus T) = \sigma_W(S)\cup\sigma_W(T)$. Indeed, by [8, Theorem 2.4], $S\oplus T$ satisfies Browder's theorem, so that $\sigma_b(S\oplus T) = \sigma_W(S\oplus T)$. As we have always $\sigma_b(S \oplus T) = \sigma_b(S) \cup \sigma_b(T)$ then $\sigma_W(S) \cup \sigma_W(T) \subset \sigma_b(S) \cup \sigma_b(T) = \sigma_b(S \oplus T) = \sigma_W(S \oplus T)$, that is $\sigma_W(S) \cup \sigma_W(T) \subset \sigma_W(S \oplus T)$. Since the inclusion $\sigma_W(S) \cup \sigma_W(T) \supset \sigma_W(S \oplus T)$ is always true, then $\sigma_W(S \oplus T) = \sigma_W(S) \cup \sigma_W(T)$. ∎

Remark 2.6. (1) The following example shows that if $S \in L(X)$ and $T \in L(Y)$ are Banach space operators possessing property (ab), then it does not necessarily follow that the direct sum $S \oplus T$ possesses property (ab). On the Hilbert space $\ell^2(\mathbb{N})$ we define S by $S(x_1, x_2, x_3, ...) = (0, x_2, x_3, ...)$ and let R the unilateral right shift operator. Then S possesses property (ab), since $\sigma(S) = \sigma_a(S) = \{0, 1\}$, $\sigma_W(S) = \{1\}$, $\Pi_a^0(S) = \{0\}$. We also have that R possesses property (ab), since $\sigma(R) = D(0, 1)$, $\sigma_W(R) = D(0, 1)$ and $\Pi_a^0(R) = \emptyset$. But $S \oplus R$ does not possess this property, since $\sigma(S \oplus R) = D(0, 1)$, $\sigma_W(S \oplus R) = D(0, 1)$ and $\Pi_a^0(S \oplus R) = \{0\}$. Here $\Pi_a^0(R) \subset \sigma_a(S)$ and $\sigma_W(S \oplus R) = \sigma_W(S) \cup \sigma_W(R) = D(0, 1)$, but $\Pi_a^0(S) \not\subset \sigma_a(R)$.

(2) Generally, Theorem 2.5 does not hold if we do not assume that $\sigma_W(S\oplus T) = \sigma_W(S) \cup \sigma_W(T)$. If we consider the operator $R \oplus L$ defined as in Remark 2.3, then R possesses property (ab). We also have that L possesses property (ab), since $\sigma(L) = D(0, 1)$, $\sigma_W(L) = D(0, 1)$ and $\Pi_a^0(L) = \emptyset$. But $R \oplus L$ does not possess property (ab) although $\Pi_a^0(R) \subset \sigma_a(L)$ and $\Pi_a^0(L) \subset \sigma_a(R)$. Indeed, as $R \oplus L$ does not satisfy Browder's theorem then by [8, Theorem 2.4], $R \oplus L$ does not possess property (ab). Notice that the inclusion $\sigma_W(R\oplus L) \subset \sigma_W(R)\cup\sigma_W(L)$ is proper, because $\sigma_W(R)\cup\sigma_W(L) = D(0, 1)$ and $0 \notin \sigma_W(R \oplus L)$.

Since for every quasisimilar Banach spaces operators S and T, we have $\alpha(T - \lambda I) = \alpha(S - \lambda I)$, and since it is easily seen that if S and T are of stable sign index then $\sigma_W(S \oplus T) = \sigma_W(S) \cup \sigma_W(T)$, from Theorem 2.5 we obtain immediately the following corollary:

Corollary 2.7. *Let $S \in L(\mathcal{H})$ and $T \in L(\mathcal{H})$ be quasisimilar hyponormal operators. If S and T possess property (ab), then $S \oplus T$ possesses property (ab).*

According to [6], we say that T satisfies *generalized a-Browder's theorem* if $\sigma_{SBF_+^-}(T) = \sigma_a(T)\backslash\Pi_a(T)$. Generally, generalized a-Browder's theorem as well as property (gab), is not transferred from the direct summands to the direct sum. Indeed, if we consider the operator $R \oplus L$ defined as in Remark 2.3, then R satisfies generalized a-Browder's theorem, since $\sigma_a(R) = C(0,1)$, $\sigma_{SBF_+^-}(R) = C(0,1)$ and $\Pi_a(R) = \emptyset$. We also have that L satisfies generalized a-Browder's theorem, since $\sigma_a(L) = D(0,1)$, $\sigma_{SBF_+^-}(L) = D(0,1)$ and $\Pi_a(L) = \emptyset$. But as $R \oplus L$ does not satisfy generalized Browder's theorem, then from [6, Theorem 3.8], it does not satisfy generalized a-Browder's theorem. Observe that R is an upper semi-Fredholm operator with $\text{ind}(R) = -1$ and L is an upper semi-Fredholm operator with $\text{ind}(L) = 1$, $\sigma_{SBF_+^-}(R) \cup \sigma_{SBF_+^-}(L) = D(0,1)$ and $0 \notin \sigma_{SBF_+^-}(R \oplus L)$.

However, we have the following preservation result of generalized a-Browder's theorem under direct sum, extending [13, Theorem 2.7] which establishes that if S and T are a-polaroid operators acting on Hilbert spaces, satisfying generalized a-Browder's theorem and if $\sigma_{SBF_+^-}(S) \cup \sigma_{SBF_+^-}(T) = \sigma_{SBF_+^-}(S \oplus T)$, then $S \oplus T$ satisfies generalized a-Browder's theorem.

Theorem 2.8. *If $S \in L(X)$ and $T \in L(Y)$ satisfy generalized a-Browder's theorem, then the following assertions are equivalent.*

(i) *$S \oplus T$ satisfies generalized a-Browder's theorem;*
(ii) *$\sigma_{SBF_+^-}(S \oplus T) = \sigma_{SBF_+^-}(S) \cup \sigma_{SBF_+^-}(T)$.*

Proof. (i)⇒(ii) If generalized a-Browder's theorem holds for $S \oplus T$ then $\sigma_{SBF_+^-}(S \oplus T) = \sigma_{LD}(S \oplus T)$. As $\sigma_{LD}(S \oplus T) = \sigma_{LD}(S) \cup \sigma_{LD}(T)$ (see Lemma 2.1) then $\sigma_{SBF_+^-}(S) \cup \sigma_{SBF_+^-}(T) \subset \sigma_{LD}(S) \cup \sigma_{LD}(T) = \sigma_{LD}(S \oplus T) = \sigma_{SBF_+^-}(S \oplus T)$, that is $\sigma_{SBF_+^-}(S) \cup \sigma_{SBF_+^-}(T) \subset \sigma_{SBF_+^-}(S \oplus T)$. Since by [10, Lemma 2.2], we have always that $\sigma_{SBF_+^-}(S) \cup \sigma_{SBF_+^-}(T) \supset \sigma_{SBF_+^-}(S \oplus T)$ then $\sigma_{SBF_+^-}(S \oplus T) = \sigma_{SBF_+^-}(S) \cup \sigma_{SBF_+^-}(T)$.

(ii)⇒(i) Assume that $\sigma_{SBF_+^-}(S \oplus T) = \sigma_{SBF_+^-}(S) \cup \sigma_{SBF_+^-}(T)$. As generalized a-Browder's theorem holds for T and for S then $\sigma_{LD}(S) = \sigma_{SBF_+^-}(S)$ and $\sigma_{LD}(T) = \sigma_{SBF_+^-}(T)$. Thus $\sigma_{SBF_+^-}(S \oplus T) = \sigma_{LD}(S) \cup \sigma_{LD}(T) = \sigma_{LD}(S \oplus T)$. So $S \oplus T$ satisfies generalized a-Browder's theorem. ∎

Since we know from [10, Proposition 2.3] that

$$\sigma_{SBF_+^-}(S \oplus T) = \sigma_{SBF_+^-}(S) \cup \sigma_{SBF_+^-}(T)$$

whenever S and T are bounded Banach spaces operators of stable sign index, by Theorem 2.8 we have immediately the following corollary:

Corollary 2.9. *If $S \in L(X)$ and $T \in L(Y)$ are of stable sign index and satisfy generalized a-Browder's theorem, then $S \oplus T$ satisfies generalized a-Browder's theorem.*

3. Properties (aw) and (gaw) for direct sums

For $T \in L(X)$, let $E_a(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda I)\}$, $E_a^0(T) = \{\lambda \in E_a(T) : \alpha(T - \lambda I) < \infty\}$, where $\text{iso}\sigma_a(T)$ denotes the set of all isolated points of $\sigma_a(T)$. Following [8, Definition 3.1], an operator $T \in L(X)$ is said to possess *property* (aw) if $\Delta(T) = E_a^0(T)$ and is said to possess *property* (gaw) if $\Delta^g(T) = E_a(T)$, which extends property (aw) to the general context of B-Fredholm theory. It is shown in [8], that the properties (gaw) and (aw) imply the properties (gab) and (ab) respectively, but the converses do not hold in general. For more details about these properties we refer the reader to [8, 9].

In this section, we show that if S and T are Banach space operators possessing property (gaw), then it does not necessarily imply that their (orthogonal) direct sum $S \oplus T$ possesses property (gaw) (see Example 3.2). Moreover, we explore in the following theorem certain sufficient conditions on S and T to ensure that this property will be transferred from the direct summands to the direct sum.

Theorem 3.1. *Suppose that $S \in L(X)$ and $T \in L(Y)$ have the same point spectrum. If S and T both possess property (gaw), then the following statements are equivalent.*

(i) *$S \oplus T$ possesses property (gaw);*
(ii) *$\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$.*

Proof. (ii)⇒(i) Assume that $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$. Since T and S both possess property (gaw) then

$$\begin{aligned}\sigma(S \oplus T)\backslash\sigma_{BW}(S \oplus T) &= [\sigma(S) \cup \sigma(T)]\backslash[\sigma_{BW}(S) \cup \sigma_{BW}(T)] \\ &= [E_a(T) \cap \rho(S)] \cup [E_a(S) \\ &\qquad \cap \rho(T)] \cup [E_a(S) \cap E_a(T)].\end{aligned}$$

As by hypothesis $\sigma_p(T) = \sigma_p(S)$ then $E_a(T) \cap \rho(S) = \emptyset$ and $E_a(S) \cap \rho(T) = \emptyset$. Thus $\sigma(S \oplus T)\backslash\sigma_{BW}(S \oplus T) = E_a(S) \cap E_a(T)$. On the other hand, we have

$$\begin{aligned}E_a(S \oplus T) &= \text{iso}\sigma_a(S \oplus T) \cap \sigma_p(S \oplus T) \\ &= \text{iso}[\sigma_a(S) \cup \sigma_a(T)] \cap [\sigma_p(S) \cup \sigma_p(T)] \\ &= [E_a(S) \cap \rho_a(T)] \cup [E_a(T) \cap \rho_a(S)] \cup [E_a(S) \cap \text{iso}\sigma_a(T)] \\ &\quad \cup [E_a(T) \cap \text{iso}\sigma_a(S)] \\ &= E_a(S) \cap E_a(T),\end{aligned}$$

since $E_a(S) \cap \rho_a(T) = \emptyset$ and $E_a(T) \cap \rho_a(S) = \emptyset$. Hence $E_a(S \oplus T) = \sigma(S \oplus T)\backslash\sigma_{BW}(S \oplus T)$ and $S \oplus T$ possesses property (gaw).

(i)⇒(ii) If $S \oplus T$ possesses property (gaw), then from [8, Theorem 3.5], $S \oplus T$ possesses property (gab). By the same arguments used as in the proof of "(i) $\Rightarrow$(ii)" of Theorem 2.2, we obtain that $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$. ∎

Example 3.2. The condition $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$ assumed in Theorem 3.1 is not a sufficient condition on the direct sum to possess property (gaw). To see this, let $S = 0$ and T defined on the Hilbert space $\ell^2(\mathbb{N})$ by $T(x_1, x_2, x_3, \ldots) = (0, x_1/2, x_2/3, x_3/4, \ldots)$. Then $\sigma(S) = \{0\}$, $\sigma_{BW}(S) = \emptyset$ and $E_a(S) = \{0\}$. So $\sigma(S) \backslash \sigma_{BW}(S) = E_a(S)$, i.e. S possesses property (gaw). We also have that $\sigma(T) = \{0\}$, $\sigma_{BW}(T) = \{0\}$ and $E_a(T) = \emptyset$. So $\sigma(T) \backslash \sigma_{BW}(T) = E_a(T)$, i.e. T possesses property (gaw). But this property does not hold for $S \oplus T$, because $\sigma(S \oplus T) = \{0\}$, $\sigma_{BW}(S \oplus T) = \{0\}$ and $E_a(S \oplus T) = \{0\}$. Notice that $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$, but $\sigma_p(S) = \{0\}$ and $\sigma_p(T) = \emptyset$.

Corollary 3.3. *Let $S \in L(\mathcal{H})$ and $T \in L(\mathcal{H})$ be quasisimilar hyponormal operators. If S and T both possess property (gaw), then $S \oplus T$ possesses property (gaw).*

Proof. It is easily seen that the quasisimilarity of S and T implies that $\sigma_p(S) = \sigma_p(T)$. As S and T are hyponormal, then they are of stable sign index. Hence $\sigma_{BW}(S \oplus T) = \sigma_{BW}(S) \cup \sigma_{BW}(T)$, and since S and T both possess property (gaw), from Theorem 3.1 this is equivalent to say that $S \oplus T$ possesses property (gaw). ∎

Similarly to Theorem 3.1, we have the following result in the case of property (aw).

Theorem 3.4. *Suppose that $S \in L(X)$ and $T \in L(Y)$ are such that $\sigma_p^0(S) = \sigma_p^0(T)$. If S and T both possess property (aw), then the following statements are equivalent.*

(i) *$S \oplus T$ possesses property (aw);*
(ii) *$\sigma_W(S \oplus T) = \sigma_W(S) \cup \sigma_W(T)$.*

Proof. (ii)$\Rightarrow$ (i) Suppose that $\sigma_W(S \oplus T) = \sigma_W(S) \cup \sigma_W(T)$. As T and S both possess property (aw) then

$$\begin{aligned} \sigma(S \oplus T) \backslash \sigma_W(S \oplus T) &= [\sigma(S) \cup \sigma(T)] \backslash [\sigma_W(S) \cup \sigma_W(T)] \\ &= [E_a^0(T) \cap \rho(S)] \cup [E_a^0(S) \cap \rho(T)] \\ &\quad \cup [E_a^0(S) \cap E_a^0(T)]. \end{aligned}$$

Since by hypothesis $\sigma_p^0(T) = \sigma_p^0(S)$, then $E_a^0(T) \cap \rho(S) = \emptyset$ and $E_a^0(S) \cap \rho(T) = \emptyset$. Therefore $\sigma(S \oplus T) \backslash \sigma_W(S \oplus T) = E_a^0(S) \cap E_a^0(T)$. Since we know that $\sigma_p^0(S \oplus T) = \{\lambda \in \sigma_p^0(S) \cup \sigma_p^0(T) : \dim \mathcal{N}(S - \lambda I) + \dim \mathcal{N}(T - \lambda I) < \infty\}$, then

$$\begin{aligned} E_a^0(S \oplus T) &= \mathrm{iso}\sigma_a(S \oplus T) \cap \sigma_p^0(S \oplus T) \\ &= \mathrm{iso}[\sigma_a(S) \cup \sigma_a(T)] \cap \sigma_p^0(S) \\ &= [E_a^0(S) \cap \rho_a(T)] \cup [E_a^0(T) \cap \rho_a(S)] \cup [E_a^0(S) \cap E_a^0(T)] \\ &= E_a^0(S) \cap E_a^0(T), \end{aligned}$$

since $E_a^0(S) \cap \rho_a(T) = \emptyset$ and $E_a^0(T) \cap \rho_a(S) = \emptyset$. Hence $\sigma(S \oplus T) \backslash \sigma_W(S \oplus T) = E_a^0(S \oplus T)$ and $S \oplus T$ possesses property (aw).

(i)⇒(ii) If $S \oplus T$ possesses property (aw), then by [8, Theorem 3.6], $S \oplus T$ possesses property (ab). Hence the equality of the spectra $\sigma_W(S \oplus T)$ and $\sigma_W(S) \cup \sigma_W(T)$ follows from the proof of "(i)⇒(ii)" of Theorem 2.5. ∎

Example 3.5. The equality $\sigma_p^0(S) = \sigma_p^0(T)$ assumed in Theorem 3.4 plays a central role in establishing conditions for the direct sum to possess property (aw). Indeed, if we consider the operator $S \oplus R$ defined as in part (1) of Remark 2.6, then S possesses property (aw), since $\sigma(S) = \{0, 1\}$, $\sigma_W(S) = \{1\}$ and $E_a^0(S) = \{0\}$. We also have that R possesses property (aw), since $\sigma(R) = \sigma_W(R) = D(0, 1)$ and $E_a^0(R) = \emptyset$. But $S \oplus R$ does not possess property (aw), since $\sigma(S \oplus R) = \sigma_W(S \oplus R) = D(0, 1)$ and $E_a^0(S \oplus R) = \{0\}$. Here $\sigma_p^0(R) = \emptyset$, $\sigma_p^0(S) = \{0\}$ and $\sigma_W(S \oplus R) = \sigma_W(S) \cup \sigma_W(R)$.

As for every quasisimilar Banach spaces operators S and T, we have $\sigma_p^0(S) = \sigma_p^0(T)$, then from Theorem 3.4, we obtain immediately the following corollary:

Corollary 3.6. *Let $S \in L(\mathcal{H})$ and $T \in L(\mathcal{H})$ be quasisimilar hyponormal operators. If S and T both possess property (aw), then $S \oplus T$ possesses property (aw).*

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Received May 9, 2012; revised version September 10, 2012.

Communicated by V. Gupta.