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VISCOSITY APPROXIMATION METHODS FOR NONEXPANSIVE MULTI-VALUED NONSELF MAPPINGS AND EQUILIBRIUM PROBLEMS

Abstract. In this paper, strong convergence theorems by the viscosity approximation method for nonexpansive multi-valued nonself mappings and equilibrium problems are established under some suitable conditions in a Hilbert space. The obtained results extend and improve the corresponding results existed in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let D be a nonempty and convex subset of H and let $F : D \times D \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for F is to find $u \in D$ such that

$$F(u, y) \geq 0 \quad \forall y \in D. \tag{1.1}$$

The solutions set of (1.1) is denoted by $EP(F)$. Given a mapping $S : D \to H$, let $F(x, y) = \langle Sx, y - x \rangle$ for all $x, y \in D$. Then $z \in EP(F)$ if and only if $F(z, y) = \langle Sz, y - z \rangle$ for all $y \in D$, *i.e.*, z is a solution of the variational inequality. The equilibrium problem (1.1) includes as special cases numerous problems in physics, optimization, and economics. Some methods have been continuously constructed for solving the equilibrium problem (see, for example, [5–7, 9, 10, 13, 14, 19, 21, 24, 25, 28]). The set D is called *proximinal* if for each $x \in E$, there exists an element $y \in D$ such that $\|x - y\| = d(x, D)$, where $d(x, D) = \inf\{\|x - z\| : z \in D\}$. Let $CB(D), K(D)$ and $P(D)$ be the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D, respec-

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tively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max\big\{ \sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\big\},$$

for $A, B \in CB(D)$. A single-valued mapping $T : D \to D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multi-valued mapping $T : D \to CB(D)$ is said to be *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a *fixed point* of $T : D \to D$ (resp. $T : D \to CB(D)$) if $p = Tp$ (resp. $p \in Tp$). The fixed points set of T is denoted by $F(T)$.

For single-valued nonexpansive mappings, in 2000, Moudafi [16] proved the following strong convergence theorem:

Theorem M [16]. *Let C be a nonempty, closed and convex subset of a Hilbert space H and let S be a nonexpansive mapping of C into itself such that $F(S)$ is nonempty. Let f be a contraction of C into itself and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \frac{1}{1+\varepsilon_n} S x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n),$$

for all $n \in \mathbb{N}$, where $\{\varepsilon_n\} \subset (0, 1)$ satisfies

$$\lim_{n\to\infty} \varepsilon_n = 0, \qquad \sum_{n=1}^{\infty} \varepsilon_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} \left| \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right| = 0.$$

Then $\{x_n\}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Such a method is called the *viscosity approximation method*. Recently, motivated by Combettes–Hirstoaga [9], Moudafi [16] and Tada–Takahashi [28], Takahashi–Takahashi [30] introduced an iterative scheme by the viscosity approximation method for finding a common element of the solutions set of (1.1) and the fixed points set of a nonexpansive mapping in a Hilbert space, and proved the following strong convergence theorem which is connected with the result in [9, 31].

Theorem TT. [30] *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

(A1) *$F(x, x) = 0$, for all $x \in C$;*
(A2) *F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$;*
(A3) *for each $x, y, z \in C$,*

$$\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) *for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.*

Let $S : C \to H$ be a nonexpansive mapping such that $F(S) \cap EP(F) \neq \emptyset$, let $f : H \to H$ be a contraction and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S u_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)} f(z)$.

In recent years, the fixed point theory of nonlinear multi-valued mappings in various spaces has been intensively studied and considered by many authors (see, for example, [1, 4, 20, 23, 27] and the references cited therein).

One way for approximating the fixed point of nonlinear multi-valued mappings is to use the concept of the best approximation operator P_T which is defined by $P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\}$. It is remarked that Hussain–Khan [11], in 2003, employed the best approximation operator $P_T x$ to study the fixed points of *-nonexpansive multi-valued mapping and strong convergence of its iterates to a fixed point defined on a closed and convex subset of a Hilbert space. The fixed points of nonlinear multi-valued mappings by using the concept of the best approximation operator can be found in [12, 22, 32].

In 2010, Zegeye–Shahzad [32] studied the convergence of viscosity approximation process for nonexpansive nonself multi-valued mappings in Banach spaces.

Theorem ZS. [32] *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E, and $T : D \to K(D)$ a multimap such that P_T is nonexpansive. For given $x_0 \in D$, $y_0 \in P_T x_0$, let $\{x_n\}$ be generated by the algorithm (see, e.g., [27])*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad n \geq 1, \\ y_n \in P_T(x_n) \text{ such that } \|y_{n-1} - y_n\| = d(y_{n-1}, P_T(x_n)), \ n \geq 1, \end{cases}$$

where $f : D \to D$ is a contraction with constant β and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*
(iii) $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$.

If $F(T) \neq \emptyset$ then $\{x_n\}$ converges strongly to a fixed point of T.

In 2011, Song–Cho [26] gave the example for a multi-valued mapping T which is not necessary nonexpansive but P_T is nonexpansive. It would be interesting to study the convergence of multi-valued mapping by using the best approximation operator.

Let E be a Banach space and D a subset of E. A multi-valued mapping $T : D \to CB(E)$ is said to satisfy the condition (A), if $\|x - p\| = d(x, Tp)$ for all $x \in E$ and $p \in F(T)$.

It is easy to see that T satisfies the condition (A) if and only if $Tp = \{p\}$ for all $p \in F(T)$. The best approximation operator P_T satisfies the condition (A).

Motivated by Takahashi–Takahashi [30], Zegeye–Shahzad [32], we introduce the viscosity approximation method for solving the equilibrium problems and the fixed points problem of multi-valued nonself mappings in a Hilbert space.

2. Preliminaries and lemmas

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. When $\{x_n\}$ is a sequence in H, $x_n \rightharpoonup x$ implies that $\{x_n\}$ converges weakly to x and $x_n \to x$ means the strong convergence. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0,1]$. Let D be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in D, denoted by $P_D x$, such that

$$\|x - P_D x\| \leq \|x - y\|, \quad \forall y \in D.$$

P_D is called the *metric projection* of H onto D. We know that P_D is a nonexpansive mapping of H onto D.

The following lemmas will be used for the proof of our main results in the sequel.

Lemma 2.1. [15, 29] *Let D be a closed and convex subset of a real Hilbert space H and let P_D be the metric projection from H onto D. Given $x \in H$ and $z \in D$. Then $z = P_D x$ if and only if*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in D.$$

Lemma 2.2. [5] *Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let $r > 0$ and $x \in H$. Then, there exists $z \in D$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \textit{for all } y \in D.$$

Lemma 2.3. [9] *For $r > 0$, $x \in H$, define the mapping $T_r : H \to D$ as follows:*

$$T_r(x) = \left\{ z \in D : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \text{for all } y \in D \right\}.$$

Then the followings hold:

(1) *T_r is single-value;*
(2) *T_r is firmly nonexpansive,* i.e., *for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(F)$;*
(4) *$EP(F)$ is closed and convex.*

Lemma 2.4. [18] *Each Hilbert space H satisfies Opial's condition,* i.e., *for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for each $y \in H$ with $x \neq y$.

Lemma 2.5. [2] *Let D be a nonempty and weakly compact subset of a Banach space E with the Opial condition and $T : D \to K(E)$ a nonexpansive mapping. Then $I - T$ is demiclosed.*

Lemma 2.6. [3] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n,$$

for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7. [8] *Let D be a closed and convex subset of a real Hilbert space H. Let $T : D \to CB(D)$ be a nonexpansive multi-valued map with $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. Then $F(T)$ is a closed and convex subset of D.*

Using the above results, we study convergence of the following iteration (2.1). Let D be a nonempty, closed and convex subset of a Hilbert space H. Let $T : D \to K(H)$ be a multi-valued nonself mapping, $f : H \to H$ a contraction and $F : D \times D \to \mathbb{R}$ a bifunction. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ and $\{r_n\}$ a sequence in $(0,\infty)$. For a given $x_0 \in H$, we compute

$$u_0 \in D \text{ such that } F(u_0, y) + \frac{1}{r_0}\langle y - u_0, u_0 - x_0\rangle \geq 0, \ \forall y \in D,$$

then we let $z_0 \in Tu_0$ and define $x_1 \in D$ by

$$x_1 = \alpha_0 f(x_0) + (1 - \alpha_0)z_0.$$

We next compute

$$u_1 \in D \text{ such that } F(u_1, y) + \frac{1}{r_1}\langle y - u_1, u_1 - x_1\rangle \geq 0, \ \forall y \in D.$$

From Nadler Theorem (see [17]), there exists $z_1 \in Tu_1$ such that $\|z_1 - z_0\| \leq H(Tu_1, Tu_0)$. Inductively, we construct the sequence $\{x_n\}$ as follows:

$$\begin{cases} u_n \in D \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in D, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) z_n, \ n \geq 0, \end{cases} \tag{2.1}$$

where $z_n \in Tu_n$ such that $\|z_{n+1} - z_n\| \leq H(Tu_{n+1}, Tu_n)$.

3. Main results

In this section, we prove a strong convergence theorem of the iteration (2.1) to find a common element of the solutions set of an equilibrium problem and the fixed points set of a multi-valued nonself mapping.

Theorem 3.1. *Let D be a nonempty, closed and convex subset of a Hilbert space H. Let F be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and T a nonexpansive multi-valued mapping of D into $K(H)$ such that $F(T) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself. Let $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ be sequences satisfied the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*;*
(ii) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$*.*

If T satisfies the condition (A)*, then the sequences $\{x_n\}$ and $\{u_n\}$ generated by* (2.1) *converge strongly to $z \in F(T) \cap EP(F)$, where $z = P_{F(T)\cap EP(F)} f(z)$.*

Proof. Using Lemma 2.3(4) and Lemma 2.7, we can define $Q = P_{F(T)\cap EP(F)}$. Since f is a contraction, there exists a constant $\alpha \in [0, 1)$ such that $\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in H$. Hence Qf is a contraction of H into itself. So there exists a unique element $z \in H$ such that $z = Qf(z)$. We next divide the proof into five steps.

Step 1. Show that $\{x_n\}$ is bounded.

Let $p \in F(T) \cap EP(F)$. Then from $u_n = T_{r_n} x_n$, we have

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\|,$$

for all $n \in \mathbb{N}$. It follows by the nonexpansiveness of T that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + (1 - \alpha_n)\|z_n - p\| \\ &\leq \alpha_n \big(\|f(x_n) - f(p)\| + \|f(p) - p\|\big) + (1 - \alpha_n) d(z_n, Tp) \\ &\leq \alpha_n \big(\alpha\|x_n - p\| + \|f(p) - p\|\big) + (1 - \alpha_n) H(Tu_n, Tp) \end{aligned}$$

$$\begin{aligned}
&\le \alpha_n\big(\alpha\|x_n - p\| + \|f(p) - p\|\big) + (1-\alpha_n)\|u_n - p\| \\
&\le \big(1 - \alpha_n(1-\alpha)\big)\|x_n - p\| + \alpha_n(1-\alpha)\frac{1}{(1-\alpha)}\|f(p) - p\| \\
&\le \max\left\{\|x_n - p\|, \frac{1}{(1-\alpha)}\|f(p) - p\|\right\}.
\end{aligned}$$

By induction, we have

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{1}{(1-\alpha)}\|f(p) - p\|\right\},\ \forall n \ge 0.$$

Hence $\{x_n\}$ is bounded. So are $\{u_n\}$, $\{z_n\}$ and $\{f(x_n)\}$.

Step 2. Show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

From the definition of $\{x_n\}$, there exist $z_{n+1} \in Tu_{n+1}$ and $z_n \in Tu_n$ such that $\|z_{n+1} - z_n\| \le H(Tu_{n+1}, Tu_n)$. Put $K = \sup_{n\ge 0}\{\|f(x_n)\| + \|z_n\|\}$. Then, we have

$$\begin{aligned}
(3.1)\qquad &\|x_{n+2} - x_{n+1}\| \\
&= \|\alpha_{n+1}f(x_{n+1}) - \alpha_{n+1}f(x_n) + \alpha_{n+1}f(x_n) - \alpha_n f(x_n) \\
&\quad + (1-\alpha_{n+1})z_{n+1} - (1-\alpha_{n+1})z_n + (1-\alpha_{n+1})z_n - (1-\alpha_n)z_n\| \\
&\le \alpha_{n+1}\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|f(x_n)\| + (1-\alpha_{n+1})\|z_{n+1} - z_n\| \\
&\quad + |\alpha_{n+1} - \alpha_n|\|z_n\| \\
&\le \alpha_{n+1}\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|f(x_n)\| \\
&\quad + (1-\alpha_{n+1})H(Tu_{n+1}, Tu_n) + |\alpha_{n+1} - \alpha_n|\|z_n\| \\
&\le \alpha_{n+1}\alpha\|x_{n+1} - x_n\| + 2|\alpha_{n+1} - \alpha_n|K \\
&\quad + (1-\alpha_{n+1})\|u_{n+1} - u_n\|.
\end{aligned}$$

On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \tag{3.2}$$

for all $y \in D$ and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0, \tag{3.3}$$

for all $y \in D$. Setting $y = u_{n+1}$ in (3.2) and $y = u_n$ in (3.3), we have

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \ge 0$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0.$$

It follows from (A2) that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number a such that $r_n > a > 0$ for all $n \geq 0$. Then, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}\end{aligned}$$

and hence

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n| M,\end{aligned} \tag{3.4}$$

where $M = \sup\{\|u_n - x_n\| : n \geq 0\}$. Combining (3.1) and (3.4), we obtain

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| &\leq \alpha_{n+1}\alpha\|x_{n+1} - x_n\| + 2|\alpha_{n+1} - \alpha_n|K \\ &\quad + (1 - \alpha_{n+1})\left(\|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|M\right) \\ &= (1 - \alpha_{n+1} + \alpha_{n+1}\alpha)\|x_{n+1} - x_n\| + 2|\alpha_{n+1} - \alpha_n|K \\ &\quad + (1 - \alpha_{n+1})\frac{1}{a}|r_{n+1} - r_n|M \\ &\leq \big(1 - \alpha_{n+1}(1 - \alpha)\big)\|x_{n+1} - x_n\| + 2|\alpha_{n+1} - \alpha_n|K \\ &\quad + \frac{M}{a}|r_{n+1} - r_n|.\end{aligned}$$

By conditions (i) and (ii), we have $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ using Lemma 2.6.

Step 3. Show that $\lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|u_n - z_n\| = 0$.

From (3.4) and (ii), we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.5}$$

Since $x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)z_n$,

$$\|x_{n+1} - z_n\| = \alpha_n\|f(x_n) - z_n\|.$$

From $\alpha_n \to 0$ as $n \to \infty$, we have $\|x_{n+1} - z_n\| \to \infty$ as $n \to \infty$. This implies that

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \to 0 \tag{3.6}$$

as $n \to \infty$. For $p \in F(T) \cap EP(F)$, we see that

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\
&\leq \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle \\
&= \langle u_n - p, x_n - p\rangle \\
&= \frac{1}{2}\Big(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\Big),
\end{aligned}$$

which yields

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2.$$

Therefore, from the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) - (1-\alpha_n)z_n - p\|^2 \\
&\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&= \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)d(z_n, Tp)^2 \\
&\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)H(Tu_n, Tp)^2 \\
&\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\Big(\|x_n - p\|^2 - \|x_n - u_n\|^2\Big) \\
&\leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - (1-\alpha_n)\|x_n - u_n\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
(1-\alpha_n)\|x_n - u_n\|^2 &\leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\leq \alpha_n\|f(x_n) - p\|^2 + \|x_{n+1} - x_n\|\big(\|x_n - p\| + \|x_{n+1} - p\|\big).
\end{aligned}$$

It follows from (i) and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ that

$$\|x_n - u_n\| \to 0 \tag{3.7}$$

as $n \to \infty$. It follows from (3.6) that

$$\|z_n - u_n\| \leq \|z_n - x_n\| + \|x_n - u_n\| \to 0 \tag{3.8}$$

as $n \to \infty$.

Step 4. Show that $\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leq 0$, where $z = P_{F(T)\cap EP(F)}f(z)$.

Firstly, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle = \limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle$$

and $x_{n_i} \rightharpoonup q \in D$. From $\|x_n - u_n\| \to 0$, we obtain $u_{n_i} \rightharpoonup q$. Let us show $q \in EP(F)$. From $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in D.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}).$$

Since $\frac{u_{n_i}-x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup q$, from (A4) we have

$$0 \geq F(y, q),$$

for all $y \in D$. For t with $0 < t \leq 1$ and $y \in D$, let $y_t = ty + (1-t)q$. Since $y \in D$ and $q \in D$, $y_t \in D$. Hence $F(y_t, q) \leq 0$. So, from (A1) and (A4) we get

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, q) \leq tF(y_t, y)$$

and hence $0 \leq F(y_t, y)$. So $0 \leq F(q, y)$ for all $y \in D$ by (A3) and hence $q \in EP(F)$. Since $\lim_{n\to\infty} \|z_n - u_n\| = 0$, $u_{n_i} \rightharpoonup q$ and $I - T$ is demiclosed at 0, we obtain that $q \in F(T)$. Therefore $q \in F(T) \cap EP(F)$. Since $z = P_{F(T)\cap EP(F)}f(z)$, by Lemma 2.1,

$$\begin{aligned} \limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle &= \lim_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle \\ &= \langle f(z) - z, q - z\rangle \leq 0. \end{aligned} \tag{3.9}$$

Step 5. Show that $x_n \to z$ as $n \to \infty$.

From $x_{n+1} - z = \alpha_n(f(x_n) - z) + (1-\alpha_n)(z_n - z)$, we have

$$(1-\alpha_n)^2\|z_n - z\|^2 \geq \|x_{n+1} - z\|^2 - 2\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle.$$

Hence

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1-\alpha_n)^2\|z_n - z\|^2 + 2\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle \\ &= (1-\alpha_n)^2 d(z_n, Tz)^2 + 2\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)^2 H(Tu_n, Tz)^2 + 2\alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)^2\|u_n - z\|^2 + 2\alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle \\ &\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\alpha\|x_n - z\|\|x_{n+1} - z\| \\ &\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)^2\|x_n - z\|^2 + \alpha_n\alpha\left\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right\} \\ &\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle. \end{aligned} \tag{3.10}$$

This implies that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \frac{(1-\alpha_n)^2+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-z\|^2+\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z)-z, x_{n+1}-z\rangle\\ &= \frac{1-2\alpha_n+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-z\|^2+\frac{\alpha_n^2}{1-\alpha_n\alpha}\|x_n-z\|^2\\ &\quad+\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z)-z, x_{n+1}-z\rangle\\ &= \left(1-\frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}\right)\|x_n-z\|^2\\ &\quad+\frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}\left\{\frac{\alpha_n}{2(1-\alpha)}\|x_n-z\|^2+\frac{1}{1-\alpha}\langle f(z)-z, x_{n+1}-z\rangle\right\}.\end{aligned}$$

Put $\gamma_n = \frac{\alpha_n}{2(1-\alpha)}\|x_n-z\|^2+\frac{1}{1-\alpha}\langle f(z)-z, x_{n+1}-z\rangle$. It follows from (i) and (3.9) that $\limsup_{n\to\infty}\gamma_n \le 0$. So $\lim_{n\to\infty}\|x_n-z\|^2 = 0$ by Lemma 2.6. This concludes that $\{x_n\}$ converges strongly to $z \in F(T)\cap EP(F)$. We can easily check that $\{u_n\}$ also converges strongly to z. We thus complete the proof. ∎

If $Tp = \{p\}$ for all $p \in F(T)$, then T satisfies the condition (A). We obtain the following results:

Corollary 3.2. *Let D be a nonempty, closed and convex subset of a Hilbert space H. Let F be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and T a nonexpansive multi-valued mapping of D into $K(H)$ such that $F(T)\cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself, and let $\{\alpha_n\}$ and $\{r_n\}$ be as in Theorem* 3.1. *If $Tp = \{p\}$ for all $p \in F(T)$, then the sequences $\{x_n\}$ and $\{u_n\}$ generated by* (2.1) *converge strongly to $z \in F(T)\cap EP(F)$, where $z = P_{F(T)\cap EP(F)}f(z)$.*

Since P_T satisfies the condition (A), we also obtain the following results:

Corollary 3.3. *Let D be a nonempty, closed and convex subset of a Hilbert space H. Let F be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and T a multi-valued mapping of D into $P(H)$ such that $F(T)\cap EP(F) \neq \emptyset$ and $F(T)$ is closed and convex. Let f be a contraction of H into itself, and let $\{\alpha_n\}$ and $\{r_n\}$ be as in Theorem* 3.1. *Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated as follows:*

$$(3.11)\quad \begin{cases} u_n \in D \text{ such that } F(u_n,y)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \ \forall y \in D,\\ x_{n+1} = \alpha_n f(x_n)+(1-\alpha_n)z_n,\end{cases}$$

where $z_n \in P_T u_n$ such that $\|z_{n+1}-z_n\| \le H(P_T u_{n+1}, P_T u_n)$.

If P_T is nonexpansive and $I - T$ is demiclosed at 0*, then the sequences* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in F(T) \cap EP(F)$*, where*

$$z = P_{F(T) \cap EP(F)} f(z).$$

Remark 3.4. The main results obtained in this paper extend those announced in [30] for multi-valued mappings.

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