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CROSSED PRODUCT OF A C^* -ALGEBRA
BY A SEMIGROUP OF INTERACTIONS

Abstract. The paper presents a construction of the crossed product of a C^* -algebra by a commutative semigroup of bounded positive linear maps generated by partial isometries. In particular, it generalizes Antonevich, Bakhtin, Lebedev's crossed product by an endomorphism, and is related to Exel's interactions. One of the main goals is the Isomorphism Theorem established in the case of actions by endomorphisms.

1. Introduction

Recently, in [1] A. B. Antonevich, V. I. Bakhtin and A. V. Lebedev introduced a new crossed product of a C^* -algebra by an endomorphism (for abbreviation we shall call it ABL-crossed product) which in a sense, see [1], generalizes all the previous approaches to constructions of that kind in the case of a single endomorphism [4], [5], [18], [7], [16], [8], [11]. Afterwards, see [12], the ABL-crossed product was adapted to the case of actions by a semigroup Γ^+ which is a positive cone of a totally ordered commutative group Γ . It is fundamental that the ABL-construction arose against a background of R. Exel's crossed product [8], which was adapted to the semigroup context by N. S. Larsen [13] and requires a unital C^* -algebra \mathcal{A} , a semigroup homomorphism $\alpha : \Gamma^+ \rightarrow \text{End}(\mathcal{A})$ where $\text{End}(\mathcal{A})$ is the set of endomorphisms of \mathcal{A} (with composition as a semigroup operation), and also it depends on a choice of *transfer action*, i.e. a semigroup homomorphism $L : \Gamma^+ \rightarrow \text{PosLin}(\mathcal{A})$ where $\text{PosLin}(\mathcal{A})$ is the set of all linear bounded positive maps on \mathcal{A} , such that

$$L_x(\alpha_x(a)b) = aL_x(b), \quad \text{for all } a, b \in \mathcal{A} \text{ and } x \in \Gamma^+.$$

2010 *Mathematics Subject Classification*: 46L55.

Key words and phrases: C^* -algebra, interactions, partial isometry, crossed product, finely representable action, transfer operator.

The work in part supported by National Science Centre grant number DEC-2011/01/B/ST1/03838.

In other words, Exel's crossed product is a certain C^* -algebra associated to the system $(\mathcal{A}, \Gamma^+, \alpha, L)$ consisting of four elements (cf. Example 3.1), whereas the ABL-crossed product [1], [12], depends only on the triple $(\mathcal{A}, \Gamma^+, \alpha)$. The price to pay (which eventually is not that high, see [1]) is that ABL-crossed product is defined only for a special class of *finely representable systems* $(\mathcal{A}, \Gamma^+, \alpha)$, see [12], [1].

A link between Exel's and ABL-crossed product is provided by the result of V. I. Bakhtin and A. V. Lebedev [3], which being stated in the semigroup language [12], says that $(\mathcal{A}, \Gamma^+, \alpha)$ is finely representable if and only if there exists a transfer action L for $(\mathcal{A}, \Gamma^+, \alpha)$, such that

$$(1.1) \quad \alpha_x(L_x(a)) = \alpha_x(1)a\alpha_x(1), \quad \text{for all } a \in \mathcal{A}, x \in \Gamma^+,$$

in which case L is called a *complete transfer action*. It is important that the complete transfer action, if it exists, is unique and α and L determine uniquely one another via the formulae

$$L_x(a) = \alpha_x^{-1}(\alpha_x(1)a\alpha_x(1)), \quad \alpha_x(a) = L_x^{-1}(L_x(1)a), \quad a \in \mathcal{A}, x \in \Gamma^+,$$

see [3, Thm. 2.8], [12, Thm. 2.4].

Let us note that, although one can not help feeling that in the above picture the action α is somewhat privileged, there is no particular reason to single out α since we have one-to-one correspondence $\alpha \longleftrightarrow L$ (in the ABL-context, of course). This simple observation is a starting point for the present article. We attempt to clarify here a number of questions which arise naturally:

- Why not carry out the ABL-construction starting with L rather than with α ?
- Is it necessary for one of the elements in the pair (α, L) to act by multiplicative mappings?
- What happens if we drop this multiplicativity condition, which of the results concerning ABL-crossed products can be carried over then?

Furthermore, we are not simply interested in generalizing ABL-crossed product. We also aim at a powerful tool to study crossed products the so-called Isomorphism Theorem [19], [2], [14], [11], which has not been studied in the ABL-context yet.

We have to mention one more important fact. In [9], a similar dissatisfaction of an asymmetry between actions and transfer actions in the construction of Exel's crossed product led R. Exel to an object which he called interaction. Simply, due to the author of [9], *interaction* is a pair $(\mathcal{V}, \mathcal{H})$ of two positive bounded linear maps on a C^* -algebra \mathcal{A} such that

$$\mathcal{V} \circ \mathcal{H} \circ \mathcal{V} = \mathcal{V}, \quad \mathcal{H} \circ \mathcal{V} \circ \mathcal{H} = \mathcal{H},$$

$\mathcal{V}|_{\mathcal{H}(\mathcal{A})}$ and $\mathcal{H}|_{\mathcal{V}(\mathcal{A})}$ are multiplicative.

It is quite striking that a connection of the article [9] with the present paper is completely analogous to that of Exel's crossed product with ABL-crossed product (which will become clear during the further reading).

The paper is organized as follows.

In Section 2, we convert Exel's notion of interaction to the semigroup case and present some of its properties. Then in Section 3, we define complete interactions, explain their connection with complete transfer actions, and give a few characteristics of this notion. Section 4 is devoted to finely representable actions and associated crossed products. Here, we define fine representability of an action \mathcal{V} and then show that it implies the existence of (necessarily unique) action \mathcal{H} such that the pair $(\mathcal{V}, \mathcal{H})$ is a complete interaction. We also develop some terminology and facts concerning the internal structure of the crossed product, which we use later in Section 5 to obtain a necessary and sufficient condition for a representation of the crossed product to be faithful. The final Section 6 is dedicated to the Isomorphism Theorem which holds for the so-called topologically free actions. We present here a definition of a topological freedom for complete interactions, which in fact is a verbatim of the corresponding definition for partial actions, see [14]. Though in the generality under consideration we failed to establish the Isomorphism Theorem, we managed to obtain a partial result, see Theorem 6.4, and we obtained a complete goal, see Theorem 6.5, in the case of ABL-crossed products, that is when one of the actions from the pair $(\mathcal{V}, \mathcal{H})$ acts by endomorphisms.

2. Interactions

Let us start with establishing notation and more accurate definitions of basic notions appearing in the text. Throughout the paper, we let \mathcal{A} denote a C^* -algebra with an identity 1, and Γ^+ be a positive cone of a totally ordered abelian group Γ with an identity 0:

$$\Gamma^+ = \{x \in \Gamma : 0 \leq x\}, \quad \Gamma = \Gamma^+ \cup (-\Gamma^+), \quad \Gamma^+ \cap (-\Gamma^+) = \{0\}.$$

2.1. We say that \mathcal{V} is an *action* of Γ^+ on \mathcal{A} if $\mathcal{V} : \Gamma^+ \rightarrow \text{PosLin}(\mathcal{A})$ is a semigroup homomorphism, and then for each $x \in \Gamma^+$, we denote by $\mathcal{V}_x : \mathcal{A} \rightarrow \mathcal{A}$, the corresponding positive linear map:

$$\mathcal{V}_0 = \text{Id}, \quad \mathcal{V}_x \circ \mathcal{V}_y = \mathcal{V}_{x+y}, \quad x, y \in \Gamma^+.$$

If \mathcal{V} acts not only by linear but also multiplicative maps then usually we shall denote it by α and call the triple $(\mathcal{A}, \Gamma^+, \alpha)$ a C^* -dynamical system, cf. [12].

The following is a simple modification of [12, Def. 3.1].

2.2. A pair $(\mathcal{V}, \mathcal{H})$ consisting of two actions \mathcal{V} and \mathcal{H} of Γ^+ on a C^* -algebra \mathcal{A} will be called *interaction* if for each $x \in \Gamma^+$ the following conditions are satisfied

- i) $\mathcal{V}_x \mathcal{H}_x \mathcal{V}_x = \mathcal{V}_x$,
- ii) $\mathcal{H}_x \mathcal{V}_x \mathcal{H}_x = \mathcal{H}_x$,
- iii) $\mathcal{V}_x(ab) = \mathcal{V}_x(a)\mathcal{V}_x(b)$, if either a or b belong to $\mathcal{H}_x(\mathcal{A})$,
- iv) $\mathcal{H}_x(ab) = \mathcal{H}_x(a)\mathcal{H}_x(b)$, if either a or b belong to $\mathcal{V}_x(\mathcal{A})$.

We stress that the preceding definition is not a generalization, in the strict sense, of the one given by R. Exel in [9] (and presented above in the introduction).

EXAMPLE 2.3. Let $\mathcal{A} = M_2(\mathbb{C})$ be the algebra of 2×2 complex matrices. We define two positive maps on \mathcal{A} by the formulae

$$\mathcal{V}((a_{ij})) = \frac{a_{11}}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathcal{H}((a_{ij})) = \frac{a_{11} + a_{12} + a_{21} + a_{22}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is a pleasant exercise to show that \mathcal{V} and \mathcal{H} satisfies the conditions i) - iv) from 2.2, and hence they form an interaction in the sense of [9]. But they do not yield an interaction in our sense because, for instance, $\mathcal{H}^2 \circ \mathcal{V}^2 \circ \mathcal{H}^2 \neq \mathcal{H}^2$. Actually, the obstacle here is that \mathcal{V} and \mathcal{H} are implemented by a partial isometry which is not a power partial isometry (in particular $\mathcal{V}(1)\mathcal{H}(1) \neq \mathcal{H}(1)\mathcal{V}(1)$, cf. Proposition 3.3 below).

However, thanks to [9, Prop. 2.6, 2.7], the following fundamental properties of interactions are true.

PROPOSITION 2.4. *Let $(\mathcal{V}, \mathcal{H})$ be an interaction, and let $x \in \Gamma^+$ be fixed. Then*

- i) $\mathcal{V}_x(\mathcal{A})$ and $\mathcal{H}_x(\mathcal{A})$ are C^* -subalgebras of \mathcal{A} ,
- ii) $E_{\mathcal{V}_x} = \mathcal{V}_x \circ \mathcal{H}_x$ is a conditional expectation onto $\mathcal{V}_x(\mathcal{A})$,
- iii) $E_{\mathcal{H}_x} = \mathcal{H}_x \circ \mathcal{V}_x$ is a conditional expectation onto $\mathcal{H}_x(\mathcal{A})$,
- iv) the mappings $\mathcal{V}_x : \mathcal{H}_x(\mathcal{A}) \rightarrow \mathcal{V}_x(\mathcal{A})$, $\mathcal{H}_x : \mathcal{V}_x(\mathcal{A}) \rightarrow \mathcal{H}_x(\mathcal{A})$ are $*$ -isomorphisms, each being the inverse of the other, and we have $\mathcal{V}_x = \mathcal{V}_x \circ E_{\mathcal{H}_x}$ and $\mathcal{H}_x = \mathcal{H}_x \circ E_{\mathcal{V}_x}$.

As the algebra \mathcal{A} considered here is unital we may (for any interaction $(\mathcal{V}, \mathcal{H})$) study the elements $\mathcal{V}_x(1)$, $\mathcal{H}_x(1)$, $x \in \Gamma^+$, which happen to have very useful properties.

PROPOSITION 2.5. *Let $(\mathcal{V}, \mathcal{H})$ be an interaction. Then*

i) *for any $a \in \mathcal{A}$ and any $x, y \in \Gamma^+$ such that $y \geq x$, we have*

$$\begin{aligned} \mathcal{V}_y(\mathcal{H}_x(1)a) &= \mathcal{V}_y(a\mathcal{H}_x(1)) = \mathcal{V}_y(a), \\ \mathcal{H}_y(\mathcal{V}_x(1)a) &= \mathcal{H}_y(a\mathcal{V}_x(1)) = \mathcal{H}_y(a), \end{aligned}$$

in particular $\mathcal{V}_y(\mathcal{H}_x(1)) = \mathcal{V}_y(1)$ and $\mathcal{H}_y(\mathcal{V}_x(1)) = \mathcal{H}_y(1)$,

ii) *$\{\mathcal{V}_x(1)\}_{x \in \Gamma^+}$ and $\{\mathcal{H}_x(1)\}_{x \in \Gamma^+}$ form decreasing families of orthogonal projections,*

iii) *for any $x, y \in \Gamma^+$ such that $y \geq x$ we have*

$$\begin{aligned} a &= \mathcal{V}_x(1)a = a\mathcal{V}_x(1), & \text{if } a \in \mathcal{V}_y(\mathcal{A}), \\ a &= \mathcal{H}_x(1)a = a\mathcal{H}_x(1), & \text{if } a \in \mathcal{H}_y(\mathcal{A}). \end{aligned}$$

Proof. By symmetry between \mathcal{V} and \mathcal{H} it is enough to prove only a half of each hypothesis in each item. We fix $a \in \mathcal{A}$ and $y \geq x$.

i). We have

$$\begin{aligned} \mathcal{V}_y(\mathcal{H}_x(1)a) &= \mathcal{V}_{y-x}(\mathcal{V}_x(\mathcal{H}_x(1)a)) = \mathcal{V}_{y-x}(\mathcal{V}_x(\mathcal{H}_x(1))\mathcal{V}_x(a)) \\ &= \mathcal{V}_{y-x}(\mathcal{V}_x(\mathcal{H}_x(1))\mathcal{V}_x \circ \mathcal{H}_x \circ \mathcal{V}_x(a)) \\ &= \mathcal{V}_{y-x}(\mathcal{V}_x(\mathcal{H}_x(1 \cdot \mathcal{V}_x(a)))) = \mathcal{V}_y(a). \end{aligned}$$

Taking adjoints one obtains $\mathcal{V}_y(a\mathcal{H}_x(1)) = \mathcal{V}_y(a)$.

ii). Since \mathcal{V}_x is positive, $\mathcal{V}_x(1)$ is self-adjoint and it is an idempotent because

$$\mathcal{V}_x(1) = \mathcal{V}_x(E_{\mathcal{H}_x(1)}) = \mathcal{V}_x(E_{\mathcal{H}_x(1)}1) = \mathcal{V}_x(E_{\mathcal{H}_x(1)})\mathcal{V}_x(1) = \mathcal{V}_x(1)\mathcal{V}_x(1).$$

Thus $\mathcal{V}_x(1)$ is an orthogonal projection. Now observe that $\mathcal{V}_x(1) \geq \mathcal{V}_y(1)$, $y \geq x$, because using i) twice, we get

$$\begin{aligned} \mathcal{V}_x(1)\mathcal{V}_y(1) &= \mathcal{V}_x(\mathcal{H}_x(1))\mathcal{V}_x(\mathcal{V}_{y-x}(1)) = \mathcal{V}_x(\mathcal{H}_x(1))\mathcal{V}_{y-x}(1) \\ &= \mathcal{V}_x(\mathcal{V}_{y-x}(1)) = \mathcal{V}_y(1). \end{aligned}$$

iii). Let $a = \mathcal{V}_y(b)$ for a certain $b \in \mathcal{A}$. Using i), we have

$$a = \mathcal{V}_y(b) = \mathcal{V}_y(\mathcal{H}_y(1)b) = \mathcal{V}_y(\mathcal{H}_y(1))\mathcal{V}_y(b) = \mathcal{V}_y(1)a$$

and hence by ii) all the more $a = \mathcal{V}_x(1)a$. Similarly one gets $a = a\mathcal{V}_x(1)$. ■

As one would like to think of interactions as of the natural generalization of C^* -dynamical systems, one may be disappointed to see that for a C^* -dynamical system $(\mathcal{A}, \Gamma^+, \alpha)$ and its transfer action L , the pair (α, L) might not be an interaction. However, if the transfer action L is complete, the pair (α, L) is always an interaction, see Proposition 3.4, and the class of transfer actions that yield interactions is even wider (for definitions of transfer and complete transfer actions see Introduction). As an example, we present here a simple corollary to [9, Prop. 3.4].

PROPOSITION 2.6. *Let L be a transfer action for a C^* -dynamical system $(\mathcal{A}, \Gamma^+, \alpha)$ such that $L_x(1) = 1$ for each $x \in \Gamma^*$. Then (α, L) is an interaction.*

3. Complete interactions

Here we introduce a notion of a complete interaction which is a generalization of the complete transfer action notion, see 3.4. Afterwards, for a given action \mathcal{V} we write down the necessary and sufficient conditions for existence of an action \mathcal{H} such that $(\mathcal{V}, \mathcal{H})$ is a complete interaction. Moreover, we show the uniqueness of such action \mathcal{H} , see Theorem 3.5. In order to show that in general an action does not determine uniquely an interaction, we adapt to our needs an example from [3].

EXAMPLE 3.1. (An example of a C^* -dynamical system $(\mathcal{A}, \Gamma^+, \alpha)$ which admits uncountably many transfer actions satisfying assumptions of Proposition 2.6.) Let $\mathcal{A} = C(X)$ where $X = \mathbb{R} \pmod{1}$ and let $\Gamma^+ = \mathbb{N}$. We define an action α by endomorphisms of \mathcal{A} by the formula

$$\alpha_n(a)(x) = a(2^n x \pmod{1}), \quad n \in \mathbb{N}.$$

We fix any continuous function ρ on X having the properties

$$0 \leq \rho(x) \leq 1, \quad \rho\left(\frac{x}{2} + \frac{1}{2}\right) + \rho\left(\frac{x}{2}\right) = 1, \quad x \in [0, 1).$$

Take the standard tent map: $T(x) = 1 - |1 - 2x|$, $x \in [0, 1]$, and associate with ρ a family of cocycles given by

$$\rho_0 \equiv 1 \quad \text{and} \quad \rho_n(x) = \rho(T^{n-1}(x)) \cdots \rho(T(x)) \cdot \rho(x), \quad \text{for } n > 0.$$

Then it is not hard to check that ρ_n satisfies the relations

$$0 \leq \rho_n(x) \leq 1, \quad \sum_{k=0}^{2^n-1} \rho_n\left(\frac{x}{2^n} + \frac{k}{2^n}\right) = 1, \quad x \in [0, 1),$$

and the following formula defines an action L on $C(X)$

$$L_n(a)(x) = \sum_{k=0}^{2^n-1} \rho_n\left(\frac{x}{2^n} + \frac{k}{2^n}\right) a\left(\frac{x}{2^n} + \frac{k}{2^n}\right), \quad x \in [0, 1), \quad n \in \mathbb{N}.$$

Clearly, for any ρ chosen L is a transfer action for α and since $L_n(1) = 1$ for each $n \in \mathbb{N}$, the pair (α, L) is an interaction by Proposition 2.6.

DEFINITION 3.2. The interaction $(\mathcal{V}, \mathcal{H})$ will be called *complete*, if the following conditions are satisfied

$$(3.1) \quad \mathcal{H}_x(\mathcal{V}_x(a)) = \mathcal{H}_x(1)a\mathcal{H}_x(1), \quad \mathcal{V}_x(\mathcal{H}_x(a)) = \mathcal{V}_x(1)a\mathcal{V}_x(1),$$

$$(3.2) \quad \mathcal{H}_y(1)\mathcal{V}_x(1) = \mathcal{V}_x(1)\mathcal{H}_y(1), \quad x, y \in \Gamma^+,$$

for all $x, y \in \Gamma^+$ and $a \in \mathcal{A}$.

The interaction in Example 3.1 is not complete because condition (3.1) is not fulfilled. The condition (3.2) is closely related to the following criterium for the product of partial isometries to be a partial isometry, cf. Example 2.3, and the proof of Proposition 4.2.

PROPOSITION 3.3. [10, Lem. 2] *Let S and T be partial isometries. Then ST is a partial isometry iff S^*S commutes with TT^* .*

Now we explain the relationship between complete interactions and the complete transfer actions for C^* -dynamical systems. We denote by $Z(\mathcal{A})$ the center of \mathcal{A} .

PROPOSITION 3.4. *If L is a complete transfer action for a C^* -dynamical system $(\mathcal{A}, \Gamma^+, \alpha)$, then the pair (α, L) is a complete interaction and*

$$L_x(1) \in Z(\mathcal{A}), \quad x \in \Gamma^+.$$

Conversly, if $(\mathcal{V}, \mathcal{H})$ is a complete interaction such that $\mathcal{H}_x(1) \in Z(\mathcal{A})$, $x \in \Gamma^+$, then $(\mathcal{A}, \Gamma^+, \mathcal{V})$ is a C^ -dynamical system and \mathcal{H} is its complete transfer action.*

Proof. Let us prove the first part of the proposition. Condition 2.2 i) follows from [12, 2.2], and 2.2 ii) follows from [12, 2.3], see also [3, (2.15)]. Since α_x is an endomorphism 2.2 iii) is trivial. We recall that $L_x(1)$ belongs to the center of \mathcal{A} and $L_x(\alpha_x(a)) = L_x(1)a$, cf. [12, Thm. 2.4]. Hence (3.1), (3.2) are valid and to show 2.2.iv), we notice that

$$L_x(\alpha_x(a)b) = aL_x(b) = aL_x(1)L_x(b) = L_x(\alpha_x(a))L_x(b).$$

By taking adjoints one obtains $L_x(b\alpha_x(a)) = L_x(b)L_x(\alpha_x(a))$.

To prove the remaining part of the statement it suffices to show that if $\mathcal{H}_x(1)$ belongs to $Z(\mathcal{A})$ then \mathcal{V}_x is multiplicative. By Proposition 2.5, formula (3.1) and the definition of interaction, we have

$$\begin{aligned} \mathcal{V}_x(ab) &= \mathcal{V}_x(\mathcal{H}(1)ab\mathcal{H}(1)) = \mathcal{V}_x(a\mathcal{H}(1)b(1)\mathcal{H}(1)) \\ &= \mathcal{V}_x(a)\mathcal{V}_x(\mathcal{H}(1)b(1)\mathcal{H}(1)) = \mathcal{V}_x(a)\mathcal{V}_x(b), \end{aligned}$$

for arbitrary $a, b \in \mathcal{A}$ and the proof is complete. ■

In view of the above proposition, the following statement is a generalization of [12, Thm. 2.4].

THEOREM 3.5. *Let \mathcal{V} be an action of Γ^+ on \mathcal{A} . The following conditions are equivalent:*

- 1) *there exists an action \mathcal{H} such that $(\mathcal{V}, \mathcal{H})$ is a complete interaction,*

- 2) (i) there exists an action \mathcal{H} such that $(\mathcal{V}, \mathcal{H})$ is an interaction,
- (ii) $\mathcal{V}_x(\mathcal{A}), \mathcal{H}_x(\mathcal{A})$ are hereditary subalgebras of \mathcal{A} for each $x \in \Gamma^+$,
- (iii) $\mathcal{V}_x(1)$ and $\mathcal{H}_y(1)$ commute for all $x, y \in \Gamma^+$,
- 3) (i) $\mathcal{V}_x(1)$ is an orthogonal projection and $\mathcal{V}_x(\mathcal{A}) = \mathcal{V}_x(1)\mathcal{A}\mathcal{V}_x(1)$ for each $x \in \Gamma^+$,
- (ii) there exists a decreasing family $\{P_x\}_{x \in \Gamma^+}$ of orthogonal projections such that
 - a) $\mathcal{V}_x(1)$ and P_y commute for all $x, y \in \Gamma^+$,
 - b) $\mathcal{V}_x(P_{x+y}) = \mathcal{V}_x(1)P_y$, for each $x, y \in \Gamma^+$,
 - c) the mappings $\mathcal{V}_x : P_x\mathcal{A}P_x \rightarrow \mathcal{V}_x(\mathcal{A})$ are $*$ -isomorphisms.

Moreover, the objects in 1) – 3) are defined in a unique way, i.e. the action \mathcal{H} in 1) and 2) is unique and the family of projections $\{P_x\}_{x \in \Gamma^+}$ in 3) is unique as well. These objects are combined by formulae

$$(3.3) \quad P_x = \mathcal{H}_x(1), \quad x \in \Gamma^+,$$

and

$$(3.4) \quad \mathcal{H}_x(a) = \mathcal{V}_x^{-1}(\mathcal{V}_x(1)a\mathcal{V}_x(1)), \quad a \in \mathcal{A},$$

where $\mathcal{V}_x^{-1} : \alpha_x(\mathcal{A}) \rightarrow P_x\mathcal{A}P_x$ is the inverse mapping to $\mathcal{V}_x : P_x\mathcal{A}P_x \rightarrow \mathcal{V}_x(\mathcal{A})$, $x \in \Gamma^+$.

Proof. 1) \Leftrightarrow 2). In view of (3.2) and Proposition 2.4, it is enough to show that 2) (ii) is equivalent to (3.1). It is straightforward that if (3.1) holds then

$$\mathcal{H}_x(\mathcal{A}) = \mathcal{H}_x(1)\mathcal{A}\mathcal{H}_x(1), \quad \mathcal{V}_x(\mathcal{A}) = \mathcal{V}_x(1)\mathcal{A}\mathcal{V}_x(1)$$

are hereditary subalgebras. Conversely, if $\mathcal{H}_x(\mathcal{A})$ and $\mathcal{V}_x(\mathcal{A})$ are hereditary subalgebras of \mathcal{A} then the argument used in the proof of [8, Prop. 4.1] shows that $\mathcal{V}_x(1)\mathcal{A}\mathcal{V}_x(1) \subset \mathcal{V}_x(\mathcal{A})$ and $\mathcal{H}_x(1)\mathcal{A}\mathcal{H}_x(1) \subset \mathcal{H}_x(\mathcal{A})$. By Proposition 2.5, we have $\mathcal{H}_x(\mathcal{A}) \subset \mathcal{H}_x(1)\mathcal{A}\mathcal{H}_x(1)$ and $\mathcal{V}_x(\mathcal{A}) \subset \mathcal{V}_x(1)\mathcal{A}\mathcal{V}_x(1)$, and hence (3.1) holds.

1), 2) \Rightarrow 3). Take $P_x = \mathcal{H}_x(1)$, $x \in \Gamma^+$. Item 3) then follows from Propositions 2.4 and 2.5.

3) \Rightarrow 1). Fix $x \in \Gamma^+$. Let $\mathcal{V}_x^{-1} : \mathcal{V}_x(\mathcal{A}) = \mathcal{V}_x(1)\mathcal{A}\mathcal{V}_x(1) \rightarrow P_x\mathcal{A}P_x$ be the inverse mapping to $\mathcal{V}_x : P_x\mathcal{A}P_x \rightarrow \mathcal{V}_x(\mathcal{A})$. Define \mathcal{H}_x by the formula $\mathcal{H}_x(a) = \mathcal{V}_x^{-1}(\mathcal{V}_x(1)a\mathcal{V}_x(1))$. Clearly \mathcal{H}_x is linear and positive, and (3.1) is fulfilled. Furthermore, 2.2.i), ii) hold. To prove 2.2.iii) we note that

$$\begin{aligned} \mathcal{V}_x(\mathcal{H}_x(\mathcal{V}_x(a)b)) &= \mathcal{V}_x(1)\mathcal{V}_x(a)b\mathcal{V}_x(1) = \mathcal{V}_x(a)\mathcal{V}_x(1)b\mathcal{V}_x(1) = \mathcal{V}_x(a)\mathcal{V}_x(\mathcal{H}_x(b)) \\ &= \mathcal{V}_x(\mathcal{H}_x(\mathcal{V}_x(a)))\mathcal{V}_x(\mathcal{H}_x(b)) = \mathcal{V}_x(\mathcal{H}_x(\mathcal{V}_x(a))\mathcal{H}_x(b)), \end{aligned}$$

and as the elements $\mathcal{H}_x(\mathcal{V}_x(a)b)$ and $\mathcal{H}_x(\mathcal{V}_x(a))\mathcal{H}_x(b)$ belong to the subalgebra $P_x\mathcal{A}P_x$ where the mapping \mathcal{V}_x is injective, they coincide. Similarly one

proves that $\mathcal{H}_x(a\mathcal{V}_x(b)) = \mathcal{H}_x(a)\mathcal{H}_x(\mathcal{V}_x(b))$ and thus 2.2.iii) holds. The same argument proves 2.2.iv) and therefore to show that $(\mathcal{V}, \mathcal{H})$ is an interaction, we only need to prove that \mathcal{H} is an action of the semigroup Γ^+ . Using 3) (ii) and 2.2.iii), we have

$$\mathcal{V}_y(P_{x+y}\mathcal{A}P_{x+y}) = \mathcal{V}_y(P_{x+y})\mathcal{V}_y(\mathcal{A})\mathcal{V}_y(P_{x+y}) = P_x\mathcal{V}_y(\mathcal{A})P_x$$

and as $P_{x+y}\mathcal{A}P_{x+y} \subset P_y\mathcal{A}P_y$ we obtain that $\mathcal{V}_y : P_{x+y}\mathcal{A}P_{x+y} \rightarrow P_x\mathcal{V}_y(\mathcal{A})P_x$ is a *-isomorphism and the inverse is given by \mathcal{H}_y . Thus we have

$$\begin{aligned} \mathcal{H}_y(\mathcal{H}_x(\mathcal{A})) &= \mathcal{H}_y(P_x\mathcal{A}P_x) = \mathcal{H}_y(\mathcal{V}_y(1)P_x\mathcal{A}P_x\mathcal{V}_y(1)) \\ &= \mathcal{H}_y(P_x\mathcal{V}_y(1)\mathcal{A}\mathcal{V}_y(1)P_x) = \mathcal{H}_y(P_x\mathcal{V}_y(\mathcal{A})P_x) = P_{x+y}\mathcal{A}P_{x+y}. \end{aligned}$$

Hence $\mathcal{H}_y(\mathcal{H}_x(a))$ and $\mathcal{H}_{x+y}(a)$ belong to the subalgebra $P_{x+y}\mathcal{A}P_{x+y}$ where the map \mathcal{V}_{x+y} is injective, and as

$$\begin{aligned} \mathcal{V}_{x+y}(\mathcal{H}_y(\mathcal{H}_x(a))) &= \mathcal{V}_x(\mathcal{V}_y(\mathcal{H}_y(\mathcal{H}_x(a)))) = \mathcal{V}_x(\mathcal{V}_y(1)\mathcal{H}_x(a)\mathcal{V}_y(1)) \\ &= \mathcal{V}_x(\mathcal{V}_y(1)P_x\mathcal{H}_x(a)P_x\mathcal{V}_y(1)) \\ &= \mathcal{V}_x(P_x\mathcal{V}_y(1)P_x\mathcal{H}_x(a)P_x\mathcal{V}_y(1)P_x) \\ &= \mathcal{V}_x(P_x\mathcal{V}_y(1)P_x)\mathcal{V}_x(\mathcal{H}_x(a))\mathcal{V}_x(P_x\mathcal{V}_y(1)P_x) \\ &= \mathcal{V}_x(\mathcal{V}_y(1))\mathcal{V}_x(\mathcal{H}_x(a))\mathcal{V}_x(\mathcal{V}_y(1)) \\ &= \mathcal{V}_{x+y}(1)\mathcal{V}_x(1)a\mathcal{V}_x(1)\mathcal{V}_{x+y}(1) = \mathcal{V}_{x+y}(1)a\mathcal{V}_{x+y}(1) \\ &= \mathcal{V}_{x+y}(\mathcal{H}_{x+y}(a)) \end{aligned}$$

we have $\mathcal{V}_{x+y} = \mathcal{V}_y \circ \mathcal{V}_x$. The uniqueness of the objects in 1) - 3) is now straightforward. ■

4. Finely representable actions and their crossed products

In this section, we define finely representable actions as the ones possessing nondegenerated covariant representations, and thereby possessing nondegenerated crossed products. These actions are closely related to complete interactions. Namely, it is not very difficult to prove (see Proposition 4.2) that every finely representable action is a 'part' of a complete interaction, and although it might be difficult to prove, it is very likely that the opposite is also true, cf. [3], [12].

Furthermore, we investigate a dense *-subalgebra of the crossed product via quasi-monomials. In particular, we prove certain inequality which will be of primary importance in the forthcoming sections.

DEFINITION 4.1. Let \mathcal{V} be an action of Γ^+ on a C^* -algebra \mathcal{A} . We say that \mathcal{V} is *finely representable* if there exists a triple (C, σ, U) , called a *covariant representation* of \mathcal{V} , consisting of a unital C^* -algebra C , unital monomorphism $\sigma : \mathcal{A} \rightarrow C$ and a semigroup homomorphism $U : \Gamma^+ \rightarrow C$ such that for every $x \in \Gamma^+$, U_x is a partial isometry, and for every $a \in \mathcal{A}$, $x \in \Gamma^+$, the

following conditions are satisfied

$$(4.1) \quad \sigma(\mathcal{V}_x(a)) = U_x \sigma(a) U_x^*, \quad U_x^* \sigma(a) U_x \in \sigma(\mathcal{A}).$$

Let us clarify how the interaction notion is involved in the above definition.

PROPOSITION 4.2. *If \mathcal{V} is a finely representable action of Γ^+ on \mathcal{A} , then there exists a (necessarily unique) action \mathcal{H} such that $(\mathcal{V}, \mathcal{H})$ is a complete interaction. Moreover, for any covariant representation (C, σ, U) , the following formulae hold*

$$(4.2) \quad \sigma(\mathcal{V}_x(a)) = U_x \sigma(a) U_x^*, \quad \sigma(\mathcal{H}_x(a)) = U_x^* \sigma(a) U_x, \quad a \in \mathcal{A}, x \in \Gamma^+.$$

Proof. If conditions (4.1) are satisfied then (identifying \mathcal{A} with $\sigma(\mathcal{A})$) one can set

$$\mathcal{H}_x(\cdot) = U_x^*(\cdot)U_x, \quad x \in \Gamma^+.$$

Using fundamental properties of partial isometries one easily verifies that $(\mathcal{V}, \mathcal{H})$ is an interaction and that conditions (3.1) are satisfied. Condition (3.2) follows from the fact that $U_x U_y = U_{x+y}$ is a partial isometry, and Proposition 3.3. Thus $(\mathcal{V}, \mathcal{H})$ is a complete interaction. By Theorem 3.5, \mathcal{H} is unique, and hence (4.2) holds for any covariant representation of \mathcal{V} . ■

The following statement is partially converse to the above one.

THEOREM 4.3. *Let $(\mathcal{V}, \mathcal{H})$ be a complete interaction such that one of the equivalent conditions i), ii), iii) hold*

- i) each \mathcal{V}_x is an endomorphism,
- ii) $\mathcal{H}_x(1) \in Z(\mathcal{A})$, for all $x \in \Gamma^+$,
- iii) $(\mathcal{A}, \Gamma^+, \mathcal{V})$ is a C^* -dynamical system,

or a counter part of one of them with \mathcal{V} replaced by \mathcal{H} hold. Then both \mathcal{V} and \mathcal{H} are finely representable actions.

Proof. It follows from Proposition 3.4 and [12, Thm. 3.2]. ■

Unfortunately the author was not able to answer the following general question:

PROBLEM 1. Let $(\mathcal{V}, \mathcal{H})$ be an arbitrary complete interaction. Are the actions \mathcal{V} and \mathcal{H} finely representable?

Fortunately, this obstacle does not really affect our further considerations.

Let us note that by Proposition 4.2, every finely representable action \mathcal{V} determines uniquely another finely representable action \mathcal{H} such that for every covariant representation (C, σ, U) of \mathcal{V} , the triple (C, σ, U^*) where $(U^*)_x = U_x^*$, is a covariant representation for \mathcal{H} and vice versa. In particular,

\mathcal{H} is finely representable and in view of the following definition the crossed products by \mathcal{V} and \mathcal{H} coincide.

DEFINITION 4.4. Let \mathcal{V} be a finely representable action and let $(\mathcal{V}, \mathcal{H})$ be the corresponding complete interaction. The *crossed product* (also called *covariance algebra*) of the C^* -algebra \mathcal{A} by the action \mathcal{V} , which we denote by $\mathcal{A} \times_{(\mathcal{V}, \mathcal{H})} \Gamma$ to indicate the role (and the symmetry) of the interaction $(\mathcal{V}, \mathcal{H})$, is the universal unital C^* -algebra generated by a copy of \mathcal{A} and a family $\{\hat{U}_x\}_{x \in \Gamma^+}$ of partial isometries subject to relations

$$(4.3) \quad \mathcal{V}_x(a) = \hat{U}_x a \hat{U}_x^*, \quad \mathcal{H}_x(a) = \hat{U}_x^* a \hat{U}_x, \quad a \in \mathcal{A}, \quad x \in \Gamma^+, \\ \hat{U}_x \hat{U}_y = \hat{U}_{x+y}, \quad x, y \in \Gamma^+.$$

If (C, σ, U) is a covariant representation of \mathcal{V} then we denote by $(\sigma \times U)$ the homomorphism of $\mathcal{A} \times_{(\mathcal{V}, \mathcal{H})} \Gamma$ into C established by

$$(\sigma \times U)(a) = \sigma(a), \quad (\sigma \times U)(\hat{U}_x) = U_x, \quad a \in \mathcal{A}, \quad x \in \Gamma^+.$$

In order to study covariance algebras, it is important to understand the structure of a $*$ -subalgebra C_0 of $\mathcal{A} \times_{(\mathcal{V}, \mathcal{H})} \Gamma$ generated by \mathcal{A} and a semigroup $\hat{U} = \{\hat{U}_x\}_{x \in \Gamma^+}$. Let us thus investigate C_0 .

The basic elements in C_0 are the ones of the form

$$(4.4) \quad \prod_{i=1}^n a_i \hat{U}_{x_i}^* = a_1 \hat{U}_{x_1}^* a_2 \dots a_n \hat{U}_{x_n}^*, \quad \prod_{i=1}^n a_i \hat{U}_{x_i} = a_1 \hat{U}_{x_1} a_2 \dots a_n \hat{U}_{x_n},$$

$x_1, \dots, x_n \in \Gamma^+$, $a_1, \dots, a_n \in \mathcal{A}$. We shall call them *monomials* of negative and positive type, respectively. In this context, the element $x_1 + \dots + x_n$ is a *degree* of both of these monomials, and any finite sum of monomials of the same type and the same degree will be called a *quasi-monomial*. Namely quasi-monomials of degree x are the elements of the form

$$(4.5) \quad q_{-x} = \sum_{\substack{y=(y_1, \dots, y_n) \in Q \\ y_1 + \dots + y_n = x}} \prod_{i=1}^n a_i^{-y} \hat{U}_{y_i}^*, \quad q_x = \sum_{\substack{y=(y_1, \dots, y_n) \in Q \\ y_1 + \dots + y_n = x}} \prod_{i=1}^n a_i^y \hat{U}_{y_i},$$

where Q is a finite set consisting of finite sequences with entries in Γ^+ (presumably with different lengths). In particular, every quasi-monomial q_0 of degree 0 is in fact a monomial and $q_0 \in \mathcal{A}$. We claim that

PROPOSITION 4.5. C_0 consists of finite sums of monomials (4.4), and a fortiori of sums of quasi-monomials.

Proof. It is clear that the finite sums of monomials form a self-adjoint linear space (containing \mathcal{A} and $\{\hat{U}_x\}_{x \in \Gamma}$). In fact, they form an algebra because every "mixed monomial" $a_1 \hat{U}_{x_1} b_1 \hat{U}_{y_1}^* a_2 \hat{U}_{x_2} \dots a_n \hat{U}_{x_n} b_n \hat{U}_{y_n}^*$ equals to a "non-

mixed monomial" in one of the forms

$$c_1 \hat{U}_{z_1}^* c_2 \dots c_m \hat{U}_{z_m}^* \quad \text{or} \quad c_1 \hat{U}_{z_1} c_2 \dots c_m \hat{U}_{z_m}$$

depending on whether $x_1 + \dots + x_n \leq y_1 + \dots + y_n$ or $y_1 + \dots + y_n \leq x_1 + \dots + x_n$ (this is an easy fact due to the total ordering of Γ and (4.3)). ■

Consequently, for any $a \in C_0$ there exists a finite set $F \subset \Gamma^+ \setminus \{0\}$, and a family of quasi-monomials $q_{\pm x}$ of degree $x \in F$ and $a_0 \in \mathcal{A}$, such that

$$(4.6) \quad a = \sum_{x \in F} q_{-x} + a_0 + \sum_{x \in F} q_x.$$

Moreover, as the next proposition shows, the quasi-monomial a_0 of degree 0 is uniquely determined by a .

PROPOSITION 4.6. *For any $a \in C_0$, and any presentation of a in the form (4.6), the following inequality holds*

$$(4.7) \quad \|a_0\| \leq \|a\|.$$

Proof. Take any faithful non-degenerate representation $\pi : \mathcal{A} \times_{(\mathcal{V}, \mathcal{H})} \Gamma \rightarrow L(H)$, and consider the Hilbert space $\tilde{H} = \bigoplus_{g \in \Gamma} H_g$ where $H_g = H$, for all $g \in \Gamma$, and the representation $\nu : \mathcal{A} \times_{(\mathcal{V}, \mathcal{H})} \Gamma \rightarrow L(\tilde{H})$ given by the formulae

$$\begin{aligned} (\nu(a)\xi)_g &= \pi(a)(\xi_g), & \text{where } a \in \mathcal{A}, \quad \tilde{H} \ni \xi = \{\xi_g\}_{g \in \Gamma}; \\ (\nu(\hat{U}_x)\xi)_g &= \pi(\hat{U}_x)(\xi_{g-x}), & (\nu(\hat{U}_x^*)\xi)_g &= \pi(\hat{U}_x^*)(\xi_{g+x}). \end{aligned}$$

Routine verification shows that $\nu(\mathcal{A})$ and $\nu(\hat{U}_x)$ satisfy all the conditions of a covariant representation and thus ν is well defined.

Now take any $a \in \mathcal{A} \times_{(\mathcal{V}, \mathcal{H})} \Gamma$ given by (4.6) and for a given $\varepsilon > 0$ chose a vector $\eta \in H$ such that

$$(4.8) \quad \|\eta\| = 1 \quad \text{and} \quad \|\pi(a_0)\eta\| > \|\pi(a_0)\| - \varepsilon.$$

Set $\xi \in \tilde{H}$ by $\xi_g = \delta_{(0,g)}\eta$, where $\delta_{(i,j)}$ is the Kronecker symbol. Then we have $\|\xi\| = 1$ and the explicit form of $\nu(a)\xi$ and (4.8) imply

$$\|\nu(a)\xi\| \geq \|\pi(a_0)\eta\| > \|\pi(a_0)\| - \varepsilon,$$

which by the arbitrariness of ε proves the desired inequality. Namely

$$\|a\| \geq \|\nu(a)\| \geq \|\pi(a_0)\| = \|a_0\|. \quad \blacksquare$$

REMARK 4.7. It is clear that the form (4.6) of $a \in C_0$ is far from being unique in general. However, if $(\mathcal{V}, \mathcal{H})$ comes from a C^* -dynamical system, i.e. one of the conditions i)–iii) from Theorem 4.3 holds, then every monomial and every quasi-monomial of degree $x \in \Gamma^+$ can be presented in one of the forms $q_{-x} = \hat{U}_x^* a_{-x}$ or $q_x = a_x \hat{U}_x$, cf. [15]. Consequently, any element

$a \in C_0$ can be presented in the form

$$a = \sum_{x \in F} \hat{U}_x^* a_{-x} + a_0 + \sum_{x \in F} a_x \hat{U}_x, \quad \text{where } a_{-x} \in \mathcal{A} \hat{U} \hat{U}^* \text{ and } a_x \in \hat{U} \hat{U}^* \mathcal{A}.$$

Moreover, see [12], [15], the coefficients $a_{\pm x}$ in the above formula are uniquely determined by a .

5. Conditional expectation and faithful representations of crossed products

From now on, we fix a finely representable action \mathcal{V} and hence by Proposition 4.2, we also fix a complete interaction $(\mathcal{V}, \mathcal{H})$. Here we use Proposition 4.6 to define a conditional expectation from $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma$ onto \mathcal{A} , for which certain 'spectral' formula holds, see (5.1), and to give a criterion for a representation of $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma$ to be faithful. In the literature such necessary and sufficient condition plays important role and is usually called property (*) (for different versions and a history of property (*) see in particular [15], [14], [1], [11]).

The first advantage of inequality (4.7) is that it implies that the mapping $E_0 : C_0 \rightarrow \mathcal{A}$ given by

$$E_0(a) = a_0,$$

where a is of the form (4.6), is well defined and can be extended to the conditional expectation acting on the whole of $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma$. We shall show that using E_0 one may express (by the formula generalizing the C^* -equality $\|a\|^2 = \|aa^*\|$, $a \in \mathcal{A}$) the norm of elements from $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma$ by the norms of elements from \mathcal{A} , see Theorem 5.2. But first, we need to estimate the growth rate of number of quasi-monomials appearing in the powers of an element $a \in C_0$.

PROPOSITION 5.1. *For any $a \in C_0$ there exists a family $\{F_k\}_{k \in \mathbb{N}}$ of finite subsets of $\Gamma^+ \setminus \{0\}$ such that*

$$a^k = \sum_{x \in F_k} q_{-x}(k) + q_0(k) + \sum_{x \in F_k} q_x(k),$$

where $q_{\pm x}(k)$ are quasi-monomials of degree x , $x \in \Gamma^+$, $k \in \mathbb{N}$, and

$$\lim_{k \rightarrow \infty} |F_k|^{\frac{1}{k}} = 1,$$

where $|F|$ denotes the number of elements in a set F . In other words, the growth rate of number of quasi-monomials appearing in the k -th power of a , is subexponential.

Proof. Let a be given by (4.6) where $F = \{x_1, \dots, x_n\}$, then the quasi-monomials in (4.6) are numbered by the elements of $F_0 = \{0, \pm x_1, \dots, \pm x_n\}$, and it is clear that the quasi-monomials appearing in a^k may be numbered

by the set $F_0^k = \{y_1 y_2 \dots y_k : y_i \in F_0\}$. Thus putting $F_k = F_0^k \cap (\Gamma^+ \setminus \{0\})$ and recalling that abelian groups are subexponential, one obtains the hypothesis. ■

THEOREM 5.2. *Let $a \in C_0 \subset \mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma$. Then we have*

$$(5.1) \quad \|a\| = \lim_{k \rightarrow \infty} \sqrt[4k]{\|E_0 [(a \cdot a^*)^{2k}]\|}.$$

Proof. Let a be of the form (4.6). Applying to a the known equality $\|\sum_{i=1}^m d_i\|^2 \leq m \|\sum_{i=1}^m d_i a_i^*\|$ (which holds for any elements d_1, \dots, d_m in an arbitrary C^* -algebra), where $m = 2|F| + 1$ and $d_k, k = 1, \dots, m$ are appropriate quasi-monomials, we obtain that

$$\|a\|^2 \leq (2|F| + 1) \left\| a_0 a_0^* + \sum_{x \in F} (q_{-x} q_{-x}^* + q_x q_x^*) \right\| = (2|F| + 1) \|E_0(aa^*)\|.$$

On the other hand, as E_0 is contractive, we have $\|a\|^2 = \|aa^*\| \geq \|E_0(aa^*)\|$ and thus

$$(5.2) \quad \|E_0(aa^*)\| \leq \|aa^*\| = \|a\|^2 \leq (2|F| + 1) \|E_0(aa^*)\|.$$

Applying (5.2) to $(aa^*)^k$ and having in mind that $(aa^*)^k = (aa^*)^{k*}$ and $\|(aa^*)^{2k}\| = \|a\|^{4k}$, one has

$$\left\| E_0 \left[(aa^*)^{2k} \right] \right\| \leq \|(aa^*)^k \cdot (aa^*)^{k*}\| = \|a\|^{4k} \leq (2|F_k| + 1) \left\| E_0 \left[(aa^*)^{2k} \right] \right\|,$$

where $F_k \subset \Gamma^+ \setminus \{0\}$ is the set of all degrees of non-zero quasi-monomials appearing in a^k . By Proposition 5.1, we have $\lim_{k \rightarrow \infty} (2|F_k| + 1)^{\frac{1}{k}} = 1$, and thus

$$\sqrt[4k]{\|E_0 [(aa^*)^{2k}]\|} \leq \|a\| \leq \sqrt[4k]{2|F_k| + 1} \cdot \sqrt[4k]{\|E_0 [(aa^*)^{2k}]\|}$$

implies that $\|a\| = \lim_{k \rightarrow \infty} \sqrt[4k]{\|E_0 [(aa^*)^{2k}]\|}$. ■

One would perceive the origin of the following definition in Proposition 4.6.

DEFINITION 5.3. Let (C, σ, U) be a covariant representation of \mathcal{V} . We shall say that (C, σ, U) possesses *property (*)* if for any element $a \in C_0$ (that is for a of the form (4.6)), we have

$$(*) \quad \|E_0(a)\| \leq \|(\sigma \times U)(a)\|$$

or in other words $\|a_0\|$ is not greater than the norm of

$$\sum_{x \in F} \left(\sum_{\substack{y=(y_1, \dots, y_n) \in Q \\ y_1 + \dots + y_n = x}} \prod_{i=1}^n \sigma(a_i^{-y}) U_{y_i}^* \right) + \sigma(a_0) + \sum_{x \in F} \left(\sum_{\substack{y=(y_1, \dots, y_n) \in Q \\ y_1 + \dots + y_n = x}} \prod_{i=1}^n \sigma(a_i^{-y}) U_{y_i} \right).$$

We are ready to formulate and prove the main result of this section.

THEOREM 5.4. *Let (C, σ, U) be a covariant representation of $(\mathcal{V}, \mathcal{H})$. The homomorphism $(\sigma \times U) : \mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma \rightarrow C$ is a monomorphism if and only if (C, σ, U) possesses property $(*)$.*

Proof. Necessity follows from Proposition 4.6. In order to show the sufficiency take any $a \in C_0$. By Theorem 5.2 and the definition of property $(*)$, we have

$$\begin{aligned} \|a\| &= \lim_{k \rightarrow \infty} \sqrt[4k]{\|E_0 [(aa^*)^{2k}]\|} \leq \lim_{k \rightarrow \infty} \sqrt[4k]{\|(\sigma \times U)(aa^*)^{2k}\|} \\ &= \lim_{k \rightarrow \infty} \sqrt[4k]{\|(\sigma \times U)(aa^*)^k(\sigma \times U)(aa^*)^k\|} = \lim_{k \rightarrow \infty} \sqrt[4k]{\|(\sigma \times U)(a)\|^{4k}} \\ &= \|(\sigma \times U)(a)\|. \end{aligned}$$

Hence $\|a\| = \|(\sigma \times U)(a)\|$ on a dense subset of $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma$. ■

COROLLARY 5.5. *There is the action of the dual group $\hat{\Gamma}$ by the automorphisms of $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma$ given by*

$$\lambda a := a, \quad a \in \mathcal{A}, \quad \lambda \hat{U}_x := \lambda_x \hat{U}_x,$$

for $x \in \Gamma^+$, $\lambda \in \hat{\Gamma}$, $\lambda_x = \lambda(x)$ (here we consider Γ as a discrete group).

Proof. Suppose that $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma$ is faithfully and nondegenerately represented on a Hilbert space H . Then for each $\lambda \in \hat{\Gamma}$ the triple $(id, \lambda \hat{U}, H)$, where $\lambda \hat{U} = \{\lambda \hat{U}_x\}_{x \in \Gamma^+}$, is a covariant representation possessing property $(*)$, whence $(id \times \lambda \hat{U})$ is an automorphism of $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma$. ■

6. Topologically free interactions

In this section, we rely heavily on the paper [14] where A. V. Lebedev defined topological freedom for partial actions of groups and obtained the Isomorphism Theorem for partial crossed products. Roughly speaking, the contribution of the author of the present paper to the current section reduces nearly only to an observation that the definition of topological freedom given by A. V. Lebedev also makes sense in the context of complete interactions. In particular, Lemma 6.3 and its proof is an almost faithful verbatim of [14, Lem. 2.7].

To start with let us note that a complete interaction defines in a natural way partial dynamical systems (the actions of a group Γ by partial homeomorphisms) on the primitive ideal space $\text{Prim } \mathcal{A}$ and the spectrum $\hat{\mathcal{A}}$ of \mathcal{A} considered here as topological spaces equipped with the Jacobson topology.

Let us give the description of these partial dynamical systems. For any $x \in \Gamma^+$, we set

$$\mathcal{A}_x = \mathcal{V}_x(1)\mathcal{A}\mathcal{V}_x(1), \quad \mathcal{A}_{-x} = \mathcal{H}_x(1)\mathcal{A}\mathcal{H}_x(1),$$

and thus we have a family $\{\mathcal{A}_g\}_{g \in \Gamma}$ of hereditary subalgebras of \mathcal{A} .

We recall that for any subset $S \subset \mathcal{A}$, the set $\text{supp } S = \{I \in \text{Prim } \mathcal{A} : I \not\perp S\}$ is open in $\text{Prim } \mathcal{A}$ (see [6, Prop. 3.1.2]), and for any hereditary C^* -subalgebra \mathcal{B} of \mathcal{A} , the mapping $I \rightarrow I \cap \mathcal{B}$ establishes a homeomorphism $\text{supp } \mathcal{B} \longleftrightarrow \text{Prim } \mathcal{B}$ (see [17, Thm. 5.5.5]). Analogously, the set $\hat{\mathcal{A}}^S = \{\pi \in \hat{\mathcal{A}} : \pi(S) \neq 0\}$ is open in $\hat{\mathcal{A}}$ and for any hereditary C^* -subalgebra \mathcal{B} of \mathcal{A} , the mapping $\pi \rightarrow \pi|_{\mathcal{B}}$ establishes a homeomorphism $\hat{\mathcal{A}}^{\mathcal{B}} \longleftrightarrow \hat{\mathcal{B}}$ (see [6, 3.2.1]). Thus we may and we shall identify the family $\{\text{Prim } \mathcal{A}_g\}_{g \in \Gamma}$ with the family $\{\text{supp } \mathcal{A}_g\}_{g \in \Gamma}$ of open sets in $\text{Prim } \mathcal{A}$, and family $\{\hat{\mathcal{A}}_g\}_{g \in \Gamma}$ with the family $\{\hat{\mathcal{A}}^{\mathcal{A}_g}\}_{g \in \Gamma}$ of open sets in $\hat{\mathcal{A}}$.

Let us define the mappings $\tau_x : \hat{\mathcal{A}}_{-x} \rightarrow \hat{\mathcal{A}}_x$, $\tau_{-x} : \hat{\mathcal{A}}_x \rightarrow \hat{\mathcal{A}}_{-x}$, for $x \in \Gamma^+$, in the following way:

$$\begin{aligned} \tau_x(\pi)(a) &= \pi(\mathcal{H}_x(a)), & \pi \in \hat{\mathcal{A}}_{-x}, \quad a \in \mathcal{A}_x, \\ \tau_{-x}(\pi)(a) &= \pi(\mathcal{V}_x(a)), & \pi \in \hat{\mathcal{A}}_x, \quad a \in \mathcal{A}_{-x}. \end{aligned}$$

By [17, Thm. 5.5.7], τ_x and τ_{-x} are homeomorphisms. Let us also define the mapping $t_g : \text{Prim } \mathcal{A}_{-g} \rightarrow \text{Prim } \mathcal{A}_g$, for $g \in \Gamma$ in the following way: for any point $I \in \text{Prim } \mathcal{A}_{-g}$ such that $I = \ker \pi$ where $\pi \in \hat{\mathcal{A}}_{-g}$, we set

$$t_g(I) = \ker \tau_g(\pi).$$

Clearly t_g is a homeomorphism.

Concluding, for τ_g and t_g defined in the above described way, $\{\tau_g\}_{g \in \Gamma}$ defines an action of Γ by *partial homeomorphisms* of $\hat{\mathcal{A}}$ and $\{t_g\}_{g \in \Gamma}$ defines an action of Γ by *partial homeomorphisms* of $\text{Prim } \mathcal{A}$.

DEFINITION 6.1. We say that the interaction $(\mathcal{V}, \mathcal{H})$ is *topologically free* if one of the following equivalent conditions holds

- i) for any finite set $\{x_1, \dots, x_k\} \subset \Gamma^+$ and any nonempty open set $U \subset \text{Prim } \mathcal{A}_{-x_1} \cap \dots \cap \text{Prim } \mathcal{A}_{-x_k}$ there exists a point $I \in U$ such that all the points $t_{x_i}(I)$, $i = 1, \dots, k$, are distinct.
- ii) For any finite set $\{x_1, \dots, x_k\} \subset \Gamma^+$ and any nonempty open set U , there exists a point $I \in U$ such that all the points $t_{x_i}(I)$, $i = 1, \dots, k$ that are defined, are distinct.
- iii) If we denote by G_x the set

$$G_x = \{I \in \text{Prim } \mathcal{A}_{-x} : t_x(I) = I\},$$

for any finite set $\{x_1, \dots, x_n\} \subset \Gamma^+ \setminus \{0\}$, the interior of the set $\bigcup_{i=1}^n G_{x_i}$ is empty.

The main statements of this section are Theorems 6.4 and 6.5 and the most important technical result is Lemma 6.3. Among the technical instruments of the proof of this lemma is the next simple Lemma 6.2.

LEMMA 6.2. ([2], Lem. 12.15.) *Let B be a C^* -subalgebra of the algebra $L(H)$ of linear bounded operators in a Hilbert space H . If $P_1, P_2 \in L(H)$ are two orthogonal projections lying in the commutant B' of B such that the restrictions*

$$B|_{H_{P_1}} \quad \text{and} \quad B|_{H_{P_2}}$$

(where $H_{P_1} = P_1(H), H_{P_2} = P_2(H)$) are both irreducible and these restrictions are distinct representations then $H_{P_1} \perp H_{P_2}$.

LEMMA 6.3. *Let \mathcal{V} be a finely representable action such that the corresponding complete interaction $(\mathcal{V}, \mathcal{H})$ be topologically free. Let (C, σ, U) is a covariant representation of \mathcal{V} , and let b be an operator of the form*

$$(6.1) \quad b = \sum_{x \in F} \sigma(a_l^{-x}) U_x^* \sigma(a_r^{-x}) + \sigma(a_0) + \sum_{x \in F} \sigma(a_l^x) U_x \sigma(a_r^x),$$

where F is a finite subset of $\Gamma^+ \setminus \{0\}$. Then for every $\varepsilon > 0$ there exists an irreducible representation $\pi : \sigma(\mathcal{A}) \rightarrow L(H_\pi)$ such that for any irreducible representation $\nu : (\sigma \times U)(\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma) \rightarrow L(H_\nu)$, which is an extension of π ($H_\pi \subset H_\nu$), we have

- (i) $\|\pi[\sigma(a_0)]\| \geq \|a_0\| - \varepsilon,$
- (ii) $P_\pi \pi[\sigma(a_0)] P_\pi = P_\pi \nu(b) P_\pi,$

where $P_\pi \in L(H_\nu)$ is the orthogonal projection onto H_π .

Proof. As σ is faithful, we may and we shall identify throughout the proof $\sigma(\mathcal{A})$ and \mathcal{A} . For any $a \in \mathcal{A}$ and $I \in \text{Prim } \mathcal{A}$, we denote by $\check{a}(I)$ the number

$$(6.2) \quad \check{a}(I) = \inf_{j \in I} \|a + j\|.$$

For every $a \in \mathcal{A}$, the function $\check{a}(\cdot)$ is lower semicontinuous on $\text{Prim } \mathcal{A}$ and attains its upper bound equal to $\|a\|$, see for instance [6, 3.3.2 and 3.3.6].

Let $I_0 \in \text{Prim } \mathcal{A}$ be a point at which $\check{a}_0(I_0) = \|a_0\|$ and π_0 be an irreducible representation of \mathcal{A} such that $I_0 = \ker \pi_0$ (thus $\|\pi_0(a_0)\| = \|a_0\|$). Since the function $\check{a}_0(\cdot)$ is lower semicontinuous, it follows that for any $\varepsilon > 0$ there exists an open set $U \subset \text{Prim } \mathcal{A}$ such that

$$(6.3) \quad \check{a}_0(I) > \|a_0\| - \varepsilon, \quad \text{for every } I \in U.$$

As $F = \{x_1, \dots, x_k\}$ is finite and the interaction $(\mathcal{V}, \mathcal{H})$ is topologically free, there exists a point $I \in U$ such that all the points $t_{x_i}(I), i = 1, \dots, k$ are distinct (if they are defined, i.e. if $I \in \text{Prim } \mathcal{A}_{-x_i}$).

Let π be an irreducible representation of \mathcal{A} such that $\ker \pi = I$ and let ν be any extension of π up to an irreducible representation of $(\pi \times U)(\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma)$. For this representation ν , we have

$$H_\pi \subset H_\nu,$$

where H_π is the representation space for π and H_ν is that for ν . Furthermore, for the orthogonal projection $P_\pi : H_\nu \rightarrow H_\pi$, we have $P_\pi \in \nu(\mathcal{A})'$.

By the choice of π and (6.3), we conclude that there exists a vector $\xi \in H_\pi$ such that $\|\xi\| = 1$ and

$$(6.4) \quad \|\pi(a_0)\xi\| > \|a_0\| - \varepsilon.$$

Thus (i) is proved.

To prove (ii) it is enough to show that for any monomials $\sigma(a_l^x)U_x\sigma(a_r^x)$, $\sigma(a_l^{-x})U_x^*\sigma(a_r^{-x})$, which are elements of the sum (6.3), we have

$$(6.5) \quad P_\pi \nu\left(\sigma(a_l^x)U_x\sigma(a_r^x)\right) P_\pi = 0, \quad P_\pi \nu\left(\sigma(a_l^{-x})U_x^*\sigma(a_r^{-x})\right) P_\pi = 0.$$

We will only prove the former relation as the proof for the latter one is completely analogous. We fix an element x in the set $F = \{x_1, \dots, x_k\}$ and consider the following possible positions of I .

If $I \notin \text{Prim } \mathcal{A}_x$ then we have $\nu(U_x U_x^*)P_\pi = \pi(\mathcal{V}_x(1)) = 0$ and thus

$$\begin{aligned} P_\pi \nu\left(\sigma(a_l^x)U_x\sigma(a_r^x)\right) P_\pi &= P_\pi \nu\left(\sigma(a_l^x)U_x U_x^* U_x \sigma(a_r^x)\right) P_\pi \\ &= \nu\left(\sigma(a_l^x)\right) \nu(U_x U_x^*) P_\pi \nu(U_x \sigma(a_r^x)) P_\pi = 0. \end{aligned}$$

If $I \notin \text{Prim } \mathcal{A}_{-x}$ then observing that $\nu(U_x^* U_x)P_\pi = \pi(\mathcal{H}_x(1)) = 0$, we have

$$P_\pi \nu\left(\sigma(a_l^x)U_x\sigma(a_r^x)\right) P_\pi = P_\pi \nu\left(\sigma(a_l^x)U_x\right) \nu(U_x^* U_x) P_\pi \nu\left(\sigma(a_r^x)\right) = 0.$$

Finally, let $I \in \text{Prim } \mathcal{A}_x \cap \text{Prim } \mathcal{A}_{-x}$.

In this case π is an irreducible representation as for \mathcal{A}_x so also for \mathcal{A}_{-x} and $t_{\pm x}(I) \in \text{Prim } \mathcal{A}_x$ (according to the definition of t_g). Moreover, we have

$$(6.6) \quad \nu(U_x^* U_x)\eta = \eta, \quad \nu(U_x U_x^*)\eta = \eta \quad \text{for any } \eta \in H_\pi.$$

In other words, H_π belongs as to the initial and final subspaces of $\nu(U_x)$ so also to the initial and final subspaces of $\nu(U_x^*)$.

We will use Lemma 6.2 where $P_1 = P_\pi$ and $P_2 = \nu(U_x)P_\pi\nu(U_x^*)$. By the definition of ν , we have $P_1 \in \nu(\mathcal{A})'$ and (6.6) means that $P_1 = P_1 \nu(U_x^* U_x) = P_1 \nu(U_x U_x^*)$. Moreover

$$\nu(U_x) : P_1(H_\nu) \rightarrow P_2(H_\nu)$$

is an isomorphism. Observe also that

$$(6.7) \quad P_2 \in \nu(\mathcal{A}_x)'.$$

Indeed, for any $a \in \mathcal{A}_x$ we have

$$\begin{aligned} P_2\nu(a) &= \nu(U_x)P_1\nu(U_x^*)\nu(a) = \nu(U_x)P_1\nu(U_x^*)\nu(U_xU_x^*)\nu(a) \\ &= \nu(U_x)P_1\left(\nu(U_x^*)\nu(a)\nu(U_x)\right)\nu(U_x^*) = \nu(U_x)P_1\nu(\mathcal{H}_x(a))\nu(U_x^*) \\ &= \nu(U_x)\nu(\mathcal{H}_x(a))\nu(U_x^*U_x)P_1\nu(U_x^*) = \nu(\mathcal{V}_x(\mathcal{H}_x(a)))\nu(U_x)P_1\nu(U_x^*) \\ &= \nu(a)P_2. \end{aligned}$$

Thus (6.7) is true. In addition, the irreducibility of $\nu(\mathcal{A}_x)|_{H_{P_1}}$ implies the irreducibility of $\nu(\mathcal{A}_x)|_{H_{P_2}}$ (here $H_{P_1} = P_1(H_\nu) = H_\pi$ and $H_{P_2} = P_2(H_\nu)$).

Now observe that for $a \in \mathcal{A}_x$, we have

$$P_1\nu(a) = 0 \iff \pi(a)P_1 = 0 \iff \check{a}(I) = 0$$

and

$$\begin{aligned} P_2\nu(a) = 0 &\iff \nu(U_x)P_1\nu(U_x^*)\nu(a\mathcal{V}_x(1)) = 0 \\ &\iff \nu(U_x)P_1\nu(\mathcal{H}_x(a))\nu(U_x^*) = 0 \\ &\iff \nu(U_x)P_1\nu(\mathcal{H}_x(a))P_1\nu(U_x^*) = 0 \\ &\iff P_1\nu(\mathcal{H}_x(a)) = 0 \iff \pi(\mathcal{H}_x(a)) = 0 \\ &\iff \check{\mathcal{H}}_x(a)(I) = 0 \iff \check{a}(t_g(I)) = 0. \end{aligned}$$

Thus, since the points I and $t_x(I)$ are distinct, we conclude that the representations $\nu(\mathcal{A}_x)|_{H_{P_1}}$ and $\nu(\mathcal{A}_x)|_{H_{P_2}}$ are distinct. Applying Lemma 6.2, we find that

$$P_1 \cdot P_2 = 0,$$

from which we have

$$P_\pi\nu(U_x)P_\pi = P_\pi\nu(U_x)\nu(U_x^*U_x)P_\pi = P_\pi\nu(U_x)P_\pi\nu(U_x^*U_x) = P_1P_2\nu(U_x) = 0.$$

Thus

$$P_\pi\nu\left(\sigma(a_l^x)U_x\sigma(a_r^x)\right)P_\pi = \nu(\sigma(a_l^x))(P_\pi\nu(U_x)P_\pi)\nu(\sigma(a_r^x)) = 0,$$

which finishes the proof of (6.5) and therefore the proof of the lemma as well. ■

As a consequence, in the presence of topological freedom, we get that all covariant representations satisfy a 'weaker version of (*) property'.

THEOREM 6.4. *Let \mathcal{V} be a finely representable action such that the corresponding complete interaction $(\mathcal{V}, \mathcal{H})$ be topologically free. Then for every element $a \in C_0$ of the form*

$$(6.8) \quad a = \sum_{x \in F} a_l^{-x} \hat{U}_x^* a_r^{-x} + a_0 + \sum_{x \in F} a_l^x \hat{U}_x a_r^x,$$

where F is a finite subset of $\Gamma^+ \setminus \{0\}$ and for every (C, σ, U) covariant representation of \mathcal{V} , the operator $(\sigma \times U)(a)$ determines uniquely the coefficient a_0 . Namely for every a of the form (6.8), the following inequality holds

$$\|E_0(a)\| \leq \|(\sigma \times U)(a)\|.$$

Since in the case of covariance algebras for C^* -dynamical systems, see Remark 4.7, each finite sum of quasi-monomials may be presented in the form (6.8), we immediately obtain the following statement, cf. also Proposition 3.4.

THEOREM 6.5. (Isomorphism Theorem for C^* -dynamical systems)

Let $(\mathcal{V}, \mathcal{H})$ be a topologically free complete interaction such that the hypotheses of Theorem 4.3 hold. Then \mathcal{V} and \mathcal{H} are finely representable and for any covariant representation (C, σ, U) the formulae

$$(\sigma \times U)(a) = \sigma(a), \quad (\sigma \times U)(\hat{U}_x) = U_x, \quad a \in \mathcal{A}, \quad x \in \Gamma^+,$$

determines the isomorphism $(\sigma \times U)$ from $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma$ onto $(\sigma \times U)(\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \Gamma)$.

In view of the foregoing statement, Theorem 6.4 may be regarded as a part of Isomorphism Theorem for interactions. However, the following problem still remains open.

PROBLEM 2. Can the Isomorphism Theorem be extended to the general finely representable actions case?

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Received November 11, 2011; revised version October 2, 2012.

Communicated by V. Gupta.