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APPROXIMATION OF BOUNDED CONTINUOUS FUNCTIONS BY LINEAR COMBINATIONS OF PHILLIPS OPERATORS

Abstract. We study the approximation properties of linear combinations of the so-called Phillips operators, which can be considered as genuine Szász–Mirakjan–Durrmeyer operators. As main result, we prove a direct estimate for the rate of approximation of bounded continuous functions $f \in C[0,\infty)$, measured in $C[0,\infty)$-norm and thus generalizing the results, proved earlier by Gupta, Agrawal, and Gairola in [3]. Our estimates rely on the recent results, obtained in the joint works of M. Heilmann and the author—[10, 11].

1. Introduction

We consider linear combinations of a variant of Szász–Mirakjan operators, which are known as Phillips operators or genuine Szász–Mirakjan–Durrmeyer operators, which, for $n \in \Re, n > 0$, are given by

$$(1.1) \qquad (\widetilde{S}_n f)(x) := s_{n,0} f(0) + \sum_{k=1}^{\infty} s_{n,k}(x) n \int_0^{\infty} s_{n,k-1}(t) f(t) dt,$$

where

$$s_{n,k}(x) = \frac{(nx)^k}{k!} e^{-nx},\ k \in \mathbb{N}_0,\ n > \alpha, x \in [0,\infty),$$

for every function f, for which the right-hand side of (1.1) makes sense. For $n > \alpha$ this is the case for real valued continuous functions on $[0,\infty)$ satisfying an exponential growth condition, i.e.

$$f \in C_\alpha[0,\infty) = \{f \in C[0,\infty) : |f(t)| \le Me^{\alpha t},\ t \in [0,\infty)\}.$$

For $\alpha = 0$, we use the following notation for bounded continuous functions

$$f \in C_B[0,\infty) = \{f \in C[0,\infty) : |f(t)| \le M, t \in [0,\infty)\}.$$

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In this paper, we consider linear combinations $\widetilde{S}_{n,r}$ of order r of the operators $\widetilde{S}_{n_i}$, i.e.

$$\widetilde{S}_{n,r} = \sum_{i=0}^{r} \alpha_i(n)\widetilde{S}_{n_i}, \qquad n_i,\ i = 0, \ldots, r \text{ different positive numbers.} \tag{1.2}$$

In general, the coefficients α_i may depend on n. Up to our current knowledge linear combinations of the genuine Szász–Mirakjan–Durrmeyer operators were considered by:

- May, (1977) [12]

$$n_i = 2^i n, \quad \alpha_i = \prod_{\substack{k=0\\k\neq i}}^{r} \frac{2^i}{2^i - 2^k},$$

- Agrawal, Gupta, (1989) [6]

$$n_i = d_i n, \quad \alpha_i = \prod_{\substack{k=0\\k\neq i}}^{r} \frac{d_i}{d_i - d_k},$$

- Agrawal, Gupta, (1990–92) [1, 3, 7]

$$\text{iterative combinations} \quad I - (I - \widetilde{S}_n)^{r+1}.$$

We will show that all these combinations fit into the following general approach

$$\widetilde{S}_{n,r} = \sum_{i=0}^{r} \alpha_i(n)\widetilde{S}_{n_i}, \qquad \text{where } n_i,\ i = 0, \ldots, r \text{ are different positive numbers.}$$

Determine $\alpha_i(n)$ such that $\widetilde{S}_{n,r}p = p\ \forall p \in \mathcal{P}_{r+1}$. This seems to be natural as the operators preserve linear functions. The requirement that each polynomial of degree at most $r + 1$ should be reproduced, leads to a linear system of equations which has the unique solution

Unique solution

$$\alpha_i(n) = n_i^r \prod_{\substack{k=0\\k\neq i}}^{r} \frac{1}{n_i - n_k}.$$

The coefficients have the following properties:

Properties

$$\sum_{i=0}^{r} \alpha_i(n) = 1, \tag{1.3}$$

$$\sum_{i=0}^{r} n_i^{-l}\alpha_i(n) = 0, \quad 1 \le l \le r, \tag{1.4}$$

$$\sum_{i=0}^{r} n_i^{-(r+1)} \alpha_i(n) = (-1)^r \prod_{k=0}^{r} \frac{1}{n_k}. \tag{1.5}$$

The last identity is important for an explicit limit in a Voronovskaja-type result, established recently in [10]. For the proof of such results one needs two additional assumptions.

Additional assumptions

$$n = n_0 < n_1 < \cdots < n_r \leq An, \tag{1.6}$$

$$\sum_{i=0}^{r} |\alpha_i(n)| \leq C. \tag{1.7}$$

The first of these conditions guarantees that

$$\sum_{i=0}^{r} n_i^{-l} \alpha_i(n) = \mathcal{O}(n^{-l}), \quad l \geq r+1.$$

The other is that the sum of the absolute values of the coefficients should be bounded independent of n. This is due to the fact that the linear combinations are no longer positive operators. So one has to be a little bit careful. Let us now look at the special cases. Of course it is clear that the choice $n_i = d_i n$ is a special case. Now, we look at a special case of this special case. As usual, we consider in this paper the linear combinations

$$\widetilde{S}_{n,r} = \sum_{i=0}^{r} \alpha_i(n) \widetilde{S}_{n_i}, \quad \alpha_i(n) = n_i^r \prod_{\substack{k=0 \\ k \neq i}}^{r} \frac{1}{n_i - n_k}. \tag{1.8}$$

For the **Special case** $n_i = d_i n$ with $d_i = \frac{1}{i+1}$, we get

$$\alpha_i = \prod_{\substack{k=0 \\ k \neq i}}^{r} \frac{\frac{1}{i+1}}{\frac{1}{i+1} - \frac{1}{k+1}} = \prod_{\substack{k=0 \\ k \neq i}}^{r} \frac{k+1}{k-i} = (-1)^i \binom{r+1}{i+1}.$$

So for the corresponding linear combinations, we have

$$\widetilde{S}_{n,r} = \sum_{i=0}^{r} (-1)^i \binom{r+1}{i+1} \widetilde{S}_{\frac{n}{i+1}} = \sum_{i=0}^{r} (-1)^i \binom{r+1}{i+1} \widetilde{S}_n^{i+1} = I - (I - \widetilde{S}_n)^{r+1},$$

where we have used our nice representation for the iterates of the operator (see Corollary 3.1 in [10]). So the iterative combinations are a special case of linear combination. For the latter P. N. Agrawal, V. Gupta and A. R. Gairola proved in [3] the following direct theorem—see Theorem 3 in [3]:

Theorem A. *Let $f \in C_N[0,\infty)$. If $f^{(s)}$ exists and is continuous on $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$, $0 < a_1 < a_2 < b_2 < b_1 < \infty$ then for sufficiently large n*

$$\|\widetilde{S}_{n,k}^{(s)}(f(u);t) - f^{(s)}(t)\|_{C(I_2)} \leq C \left\{ n^{-k} \|f\|_{C_N} + \omega_{2k}\left(f^{(s)}; n^{-\frac{1}{2}}; I_1\right) \right\}, \tag{1.9}$$

where C is independent of f, n and $\|f\|_{C_N} = \sup_{0\le t<\infty} |f(t)|e^{-Nt}$, $f \in C_N[0,\infty)$.

The aim of this paper is to generalize Theorem A for the more general settings of linear combinations, defined in (1.8) and thus to include the case of iterative combinations, too. The second improvement concerns the fact, that instead of the usual moduli of continuity ω_{2k}, we use the Ditzian–Totik moduli of smoothness (see [4]). We choose the step-weight $\varphi(x) = \sqrt{x}$ and assume $t > 0$ sufficiently small to define for $1 \le p \le \infty$:

$$\omega_\varphi^r(f,t)_p = \sup_{0<h\le t} \|\Delta_{h\varphi}^r f\|_p,$$

where the symmetric difference is given by

$$\Delta_{h\varphi(x)}^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right) h\varphi(x)\right),$$

whenever the arguments of the function f are contained in the corresponding interval. In [4, Chapters 2.3, 6.1] Ditzian and Totik proved that these moduli are equivalent to the K-functional:

$$K_\varphi^r(f,t^r)_p = \inf\{\|f-g\|_p + t^r\|\varphi^r g^{(r)}\|_p;\ g, \varphi^r g^{(r)} \in L_p[0,\infty),\ 1 \le p \le \infty\}.$$

One of the main results in our joint work with M. Heilmann (see [11], Theorem 5.6) states the following:

Theorem B. *Let $f \in L_{p,0}[0,\infty)$, $1 \le p < \infty$. Then*

$$\|\widetilde{S}_{n,r} f - f\|_p \le C \cdot \omega_\varphi^{2(r+1)}\left(f, \frac{1}{\sqrt{n}}\right)_p, \tag{1.10}$$

where c denotes a constant independent of n.

We point out that the proof of Theorem B relies on the use of Hardy inequality, which is not fulfilled for the case $p = \infty$. The space $L_{p,0}$ consists of all functions

$$f \in L_p[0,\infty) : \lim_{x\to 0+} f(x) = f_0 \in \Re.$$

As usual, for the case $p = \infty$, we consider the bounded continuous functions $f \in C_B[0,\infty)$. Our main result states the following

Theorem 1. *With $\varphi(x) = \sqrt{x}$, $x \in [0,\infty)$—fixed, we have*

$$\begin{aligned} &|\widetilde{S}_{n,r}(f,x) - f(x)| \\ &\quad \le C \cdot A(r,x) \cdot \left\{ n^{-(r+1)}\|f\|_{C_B[0,\infty)} + \omega_\varphi^{2(r+1)}\left(f, \frac{1}{\sqrt{n}}\right)_\infty \right\}, \end{aligned} \tag{1.11}$$

where $A(r,x)$ is constant, dependent only on r, x and C is a constant from the second condition, imposed on $\alpha_i(n)$.

The paper is organized as follows: in Section 2, we give some auxiliary results. The proof of Theorem 1 is given in Section 3.

2. Auxiliary results

As mentioned above, we consider the following linear combinations of the genuine Szász–Mirakjan–Durrmeyer operators:

$$\widetilde{S}_{n,r} = \sum_{i=0}^{r} \alpha_i(n)\widetilde{S}_{n_i}, \quad \alpha_i(n) = n_i^r \prod_{\substack{k=0\\k\neq i}}^{r} \frac{1}{n_i - n_k}.$$

We have the following representations (see [10]):

Moments $f_{\mu,x}(t) = (t-x)^\mu$, $\mu \in \mathbb{N}_0$

$$(\widetilde{S}_{n,r} f_{0,x})(x) = 1, \quad (\widetilde{S}_{n,r} f_{\mu,x})(x) = 0, 1 \le \mu \le r+1, \tag{2.1}$$

$$(\widetilde{S}_{n,r} f_{\mu,x})(x) = \sum_{j=1}^{\mu-(r+1)} \binom{\mu-j-1}{j-1} \frac{\mu!}{j!} x^j \sum_{i=0}^{r} n_i^{j-\mu} \alpha_i(n), \tag{2.2}$$

for $r+2 \le \mu \le 2r+2$,

$$(\widetilde{S}_{n,r} f_{\mu,x})(x) = \sum_{j=1}^{\left[\frac{\mu}{2}\right]} \binom{\mu-j-1}{j-1} \frac{\mu!}{j!} x^j \sum_{i=0}^{r} n_i^{j-\mu} \alpha_i(n), \text{ for } \mu \ge 2r+2. \tag{2.3}$$

The next statement is proved in [11]—(see Lemma 5.1 there and also (2.16) in [10]) which we formulate as:

Lemma 1. *Let*

$$n = n_0 < n_1 < \cdots < n_r \le An.$$

We define

$$\alpha_i(n) = n_i^r \cdot \prod_{k=0,\, k\neq i}^{r} \frac{1}{n_i - n_k}.$$

It is known that

$$\sum_{i=0}^{r} \alpha_i(n) = 1, \tag{2.4}$$

$$\sum_{i=0}^{r} n_i^{-l} \cdot \alpha_i(n) = 0,\ 1 \le l \le r, \tag{2.5}$$

$$\sum_{i=0}^{r} n_i^{-(r+1)} \cdot \alpha_i(n) = (-1)^r \cdot \prod_{k=0}^{r} \frac{1}{n_k}. \tag{2.6}$$

The following holds true

$$\sum_{i=0}^{r} n_i^{-(r+l)} \cdot \alpha_i(n) = O\left(n^{-(r+l)}\right), \, l \geq 1, \tag{2.7}$$

$$\sum_{i=0}^{r} n_i^{L} \alpha_i(n) = O(n^L), \, L \geq 1. \tag{2.8}$$

Lemma 2. *Suppose $f \in L_p[0,\infty)$ and $1 \leq p \leq \infty$. Then*

$$M^{-1} \cdot \omega_\varphi^r(f,t)_p \leq K_\varphi^r(f,t^r)_p \leq M \cdot \omega_\varphi^r(f,t)_p, \tag{2.9}$$

for some constant $M > 0$.

The last statement expresses the equivalence between Ditzian–Totik moduli of smoothness and related K-functional—see [4, Chapters 2 and 3].

3. Direct estimates

Our goal in this section is to prove direct estimate for approximation by the linear combinations of genuine Szász–Mirakjan–Durrmeyer operators. Similar estimates are proved in [1, 2, 3, 5, 6, 7, 8, 9, 10, 11], but here we estimate the degree of approximation in terms of the Ditzian–Totik modulus of smoothness of order $2(r+1)$.

Proof of Theorem 1. We use the method developed by Ditzian–Totik—(see Theorem 9.3.2 in [4]) to prove direct theorem for approximation by linear combinations of a broad class of linear positive operators. First, we recall some estimates for the norm of intermediate derivatives (see Theorem 9.5.3 in [4]):

$$\|\varphi^{2r-2i} f^{(2r-i)}\|_{L_p[0,1]} \leq C\left\{\|\varphi^{2r} f^{(2r)}\|_{L_p[0,1]} + \|f\|_{L_p[0,1]}\right\}, \tag{3.1}$$

where $1 \leq p \leq \infty$, $\varphi^2(x) = x$ and $i < r$.

$$\|\varphi^{2r-2i} f^{(2r-i)}\|_{L_p[1,\infty)} \leq C\left\{\|\varphi^{2r} f^{(2r)}\|_{L_p(R+)} + \|f\|_{L_p(R+)}\right\}, \tag{3.2}$$

where $1 \leq p \leq \infty$, $\varphi^2(x) = x$ and $i < r$. We adopt the assumption that for $p = \infty$, we consider continuous bounded functions f, $f^{(2r-i)}$, $1 \leq i \leq 2r-1$ over $[0,\infty)$. We choose the best $[\sqrt{n}]$-th degree polynomial approximation of f in $C[0,2]$ and recall (see Theorem 7.2.1 and Theorem 7.3.1 in [4]), that we have

$$\|f - P_{[\sqrt{n}]}\|_{C[0,1]} \leq M \cdot K_{2(r+1),\varphi}\left(f, n^{-(r+1)}\right)_\infty, \tag{3.3}$$

and

$$\|\varphi^{2(r+1)} P_{[\sqrt{n}]}^{(2(r+1))}\|_{C[0,1]} \leq M \cdot n^{r+1} \cdot K_{2(r+1),\varphi}\left(f, n^{-(r+1)}\right)_\infty, \tag{3.4}$$

as in $[0,1]$, $\varphi(x) \approx (x(2-x))^{\frac{1}{2}}$. We now define the function $g_n^*(x)$ by

$$g_n^*(x) = P_{[\sqrt{n}]}(x) \cdot \psi(x) + g_n(x)(1-\psi(x)), \tag{3.5}$$

where $\psi(x)$ is decreasing, $\psi(x) \in C^\infty$, $\psi(x) = 1$ for $x \le \frac{1}{4}$ and $\psi(x) = 0$ for $x \ge \frac{3}{4}$ and $g_n \in C_B[0,\infty)$ is an arbitrary auxiliary sufficiently smooth function. The standart technique implies (see the proof of Theorem 9.3.2 in [4])

$$\|f - g_n^*\|_{C[0,\infty)} \le M_* K_{2(r+1),\varphi}\left(f, n^{-(r+1)}\right)_\infty, \tag{3.6}$$

and

$$\|\varphi^{2(r+1)} g_n^{*(2(r+1))}\|_{C[0,\infty)} \le M_* \cdot n^{-(r+1)} K_{2(r+1),\varphi}\left(f, n^{-(r+1)}\right)_\infty . \tag{3.7}$$

For a fixed $x \in [0,\infty)$, we now write

$$\begin{aligned} &|\widetilde{S}_{n,r}(f,x) - f(x)|_{C[0,\infty)} \\ &\le |\widetilde{S}_{n,r}(f - g_n^*, x) - (f(x) - g_n^*(x)|_{C[0,\infty)} + |\widetilde{S}_{n,r}(g_n^*, x) - g_n^*(x)|_{C[0,\infty)} \\ &\le 2\|f - g_n^*\|_{C[0,\infty)} + |\widetilde{S}_{n,r}(g_n^*, x) - g_n^*(x)|_{C[0,\infty)} \\ &\le 2M_* K_{2(r+1),\varphi}\left(f, n^{-(r+1)}\right)_\infty + |\widetilde{S}_{n,r}(g_n^*, x) - g_n^*(x)|_{C[0,\infty)}. \end{aligned} \tag{3.8}$$

To estimate the second summand, we write for the function g_n^* its Taylor serie:

$$g_n^*(t) = \sum_{\mu=0}^{2r+1} \frac{(t-x)^\mu}{\mu!} g_n^{*(\mu)}(x) + \frac{1}{(2r+2)!} \int_x^t g_n^{*(2r+2)}(s)(t-s)^{2r+1} ds, \tag{3.9}$$

where $t \in [0,\infty)$. We apply the operator $\widetilde{S}_{n,r}$ to the both sides of (3.9) and obtain

$$\begin{aligned} &\widetilde{S}_{n,r}(g_n^*, x) - g_n^*(x) \\ &\quad = \frac{1}{(2r+2)!} \sum_{i=0}^{r} \alpha_i(n) \cdot \widetilde{S}_{n_i}\left(\int_x^t g_n^{*(2r+2)}(s)(t-s)^{2r+1} ds, x\right) \\ &\quad + \sum_{\mu=r+2}^{2r+1} \widetilde{S}_{n,r}(f_{\mu,x}, x) \cdot g_n^{*(\mu)}(x) = S_1 + S_2, \end{aligned} \tag{3.10}$$

where we have used (2.1). Hence

$$S_1 \le \frac{1}{(2r+1)!} \cdot \left|\widetilde{S}_{n,r}\left(\left|\int_x^t g_n^{*(2r+2)}(s)(t-s)^{2r+1} ds\right|, x\right)\right|. \tag{3.11}$$

It is known that for s between t and x, we have

$$\frac{|t-s|}{\varphi^2(s)} \le \frac{|t-x|}{\varphi^2(x)}.$$

Therefore

$$\frac{|t-s|^{r+1}}{\varphi^{2r+2}(s)} \le \frac{|t-x|^{r+1}}{\varphi^{2r+2}(x)}.$$

To estimate from above the integral term in (3.11), we proceed as follows

$$\begin{aligned}(3.12)\qquad S_1 &\le \frac{1}{(2r+1)!} \cdot \|g_n^{*(2r+2)}\varphi^{2r+2}\|_{C[0,\infty)} \cdot \sum_{i=0}^{r} |\alpha_i(n)| \\ &\quad \cdot \widetilde{S}_{n_i}\left(\frac{|t-x|^{r+1}}{\varphi^{2r+2}(x)} \cdot \left|\int_x^t |t-s|^r ds, x\right.\right) \\ &\le \frac{1}{(2r+1)!} \cdot \|g_n^{*(2r+2)}\varphi^{2r+2}\|_{C[0,\infty)} \cdot \sum_{i=0}^{r} |\alpha_i(n)| \cdot \widetilde{S}_{n_i}\left(\frac{|t-x|^{2r+2}}{\varphi^{2r+2}(x)}, x\right).\end{aligned}$$

We consider two subcases according to the position of the point x:

I case: Let $x \ge \frac{1}{4}$. From the assumptions (1.6), (1.7) and the estimates of the moments of the Phillips operator (see for example Lemma 2.1 in [10] for $\mu = 2r+2$), we get for $x \in [\frac{1}{4}, \infty)$

$$(3.13)\qquad S_1 \le \frac{C(r)}{(2r+1)!} \cdot \|g_n^{*(2r+2)}\varphi^{2r+2}\|_{C[0,\infty)} \cdot n^{-(r+1)}.$$

From (2.1), (2.2) and (2.3), we get for $\mu \ge r+2$ the following upper estimates for the moments of the linear combinations of the Pfillips operators (see Corollary 5.3 in [11]):

$$(3.14)\quad |(\widetilde{S}_{n,r} f_{\mu,x})(x)| \le C \begin{cases} n^{-\mu}, & x \in [0, \frac{1}{n}], \\ n^{-(r+1)} x^{\mu-r-1}, & x \in [\frac{1}{n}, \infty),\ r+2 \le \mu \le 2r+2, \\ n^{-[\frac{\mu+1}{2}]} x^{[\frac{\mu}{2}]}, & x \in [\frac{1}{n}, \infty),\ 2r+2 \le \mu. \end{cases}$$

Hence for $x \in [\frac{1}{4}, \infty)$, the second line of (3.14) implies

$$(3.15)\qquad S_2 \le C \cdot n^{-(r+1)} \sum_{\mu=r+2}^{2r+1} \|\varphi^{2\mu-2(r+1)} g_n^{*(\mu)}\|_{C[\frac{1}{4},\infty)}.$$

If we set $\mu = 2(r+1) - i$, $i < r+1$ then

$$2\mu - 2(r+1) = 4(r+1) - 2i - 2(r+1) = 2(r+1) - 2i.$$

Now the estimates (3.1) and (3.2) imply

$$(3.16)\qquad S_2 \le C(r) \cdot n^{-(r+1)} \left\{\|\varphi^{2(r+1)} g_n^{*(2r+2)}\|_{C[\frac{1}{4},\infty)} + \|g_n^*\|_{C[\frac{1}{4},\infty)}\right\}.$$

The proof of Theorem 1 for $x \in [\frac{1}{4}, \infty)$ follows from (3.13), (3.16) and (3.6), (3.7).

II case: Let $x \in [0, \frac{1}{4}]$. In this case, we observe that our auxiliary function g_n^* coincides with the algebraic polynomial $P_{[\sqrt{n}]}$. To overcome the difficulty, appeared close to the point 0, we observe that $\widetilde{S}_{n,r}P_{[\sqrt{n}]} - P_{[\sqrt{n}]}$ is an algebraic polynomial of degree not greater than $[\sqrt{n}]$ and we apply the wellknown inequality, originally proved by Bernstein (see Timan [13], or Theorem 8.4.8 in [4]) which states that

$$\|\widetilde{S}_{n,r}P_{[\sqrt{n}]} - P_{[\sqrt{n}]}\|_{C[0,1]} \leq C\|\widetilde{S}_{n,r}P_{[\sqrt{n}]} - P_{[\sqrt{n}]}\|_{C[\frac{1}{n},1]}. \tag{3.17}$$

We proceed in a similar way as in the first case $x \geq \frac{1}{4}$. With Taylor expansion for the polynomial $P_{[\sqrt{n}]}$, we infer

$$P_{[\sqrt{n}]}(t) = \sum_{\mu=0}^{[\sqrt{n}]} \frac{(t-x)^\mu}{\mu!} \cdot P_{[\sqrt{n}]}^{(\mu)}(x). \tag{3.18}$$

We apply the operator $\widetilde{S}_{n,r}$ to the both sides of (3.17) to get

$$\begin{aligned} \widetilde{S}_{n,r}(P_{[\sqrt{n}]}, x) - P_{[\sqrt{n}]}(x) &= \sum_{\mu=r+2}^{2r+2} \widetilde{S}_{n,r}(f_{\mu,x}, x) \cdot \frac{P_{[\sqrt{n}]}^{(\mu)}(x)}{\mu!} \\ &+ \sum_{\mu=2r+3}^{[\sqrt{n}]} \widetilde{S}_{n,r}(f_{\mu,x}, x) \cdot \frac{P_{[\sqrt{n}]}^{(\mu)}(x)}{\mu!} = T_1 + T_2. \end{aligned} \tag{3.19}$$

To estimate T_1, we proceed in the same way as in the first case. Consequently

$$|T_1| \leq C(r) \cdot n^{-(r+1)} \left\{ \|\varphi^{2(r+1)} P_{[\sqrt{n}]}^{(2r+2)}\|_{C[\frac{1}{n},1]} + \|P_{[\sqrt{n}]}\|_{C[\frac{1}{n},1]} \right\}. \tag{3.20}$$

To estimate from above the term T_2, we apply the third line of (3.14) and the wellknown Markov–Bernstein type inequality for algebraic polynomials

$$\|\varphi^{j+1} P_{[\sqrt{n}]}^{(j+1)}\|_{C[0,1]} \leq M \cdot [\sqrt{n}] \cdot \|\varphi^j P_{[\sqrt{n}]}^{(j)}\|_{C[0,1]}, \tag{3.21}$$

where M is an absolute positive constant independent of n. Let $\mu = 2r+3$. Then the absolute value of the first summand in T_2 can be estimated by

$$\begin{aligned} &Cn^{-(r+2)} \cdot |x^{r+1} P_{[\sqrt{n}]}^{(\mu)}(x)| = Cn^{-(r+2)} \cdot |\varphi^{2r+2}(x) P_{[\sqrt{n}]}^{(2r+3)}(x)| \\ &\leq Cn^{-r-\frac{3}{2}} \cdot |\varphi^{2r+3}(x) P_{[\sqrt{n}]}^{(2r+3)}(x)| \\ &\leq Cn^{-r-\frac{3}{2}} \cdot n^{\frac{1}{2}} \| |\varphi^{2r+2} P_{[\sqrt{n}]}^{(2r+2)}\|_{C[0,1]}. \end{aligned} \tag{3.22}$$

In a similar way we proceed also for $\mu = 2r+4, 2r+5, \ldots$, using that $x \geq \frac{1}{n}$. Lastly (3.3), (3.4), (3.20) imply the proof for the case 2. Thus the proof of Theorem 1 is completed. ■

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