DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 3 2014

Cemal Belen, Mustafa Yildirim

ON SOME NEW SEQUENCE SPACES AND STATISTICAL CONVERGENCE METHODS FOR DOUBLE SEQUENCES

Abstract. In this paper, we introduce some new double sequence spaces with respect to an Orlicz function and define two new convergence methods related to the concepts of statistical convergence and lacunary statistical convergence for double sequences. We also present some inclusion theorems for our newly defined sequence spaces and statistical convergence methods.

1. Introduction

A double sequence $x = (x_{jk})$ of real numbers ($j, k \in \mathbb{N}$, $\mathbb{N}$ is the set of all positive integers) is said to be convergent in the Pringsheim's sense (or P-convergent) if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - \ell| < \epsilon$ whenever $j, k \geq N$. We shall write this as

$$\lim_{j,k} x_{jk} = \ell,$$

where j and k tend to infinity independent of each other [15].

A double sequence x is bounded if

$$\|x\| = \sup_{j,k} |x_{jk}| < \infty.$$

Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. By ℓ_2^∞, we denote the space of double sequences which are bounded.

The statistical convergence of a sequence $x = (x_k)$ was first studied by Fast [6]. A sequence $x = (x_k)$ is said to be statistically convergent to the number ℓ if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k : k \leq n, |x_k - \ell| \geq \varepsilon\}| = 0,$$

where $|A|$ denotes the cardinality of the set A.

2010 *Mathematics Subject Classification*: 40A35, 40C05, 40G15.
Key words and phrases: statistical convergence, double sequence, lacunary sequence, Orlicz function.

DOI: 10.2478/dema-2014-0050

The idea of statistical convergence for double sequences was introduced by Mursaleen and Edely [11]. A double sequence $x = (x_{jk})$ of real numbers is statistically convergent to ℓ, if for each $\epsilon > 0$

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{(i,j) \in \mathbb{N}\times\mathbb{N} : j \leq m,\ k \leq n,\ |x_{jk} - \ell| \geq \epsilon\}| = 0.$$

In this case, we write $st_2 - \lim_{j,k} x_{j,k} = \ell$ and denote the set of all statistically convergent double sequences with S_2.

The notion of almost convergence for single sequences was introduced by Lorentz [9] and for double sequences by Moricz and Rhoades [10]. A double sequence $x = (x_{jk})$ is said to be almost convergent to ℓ if

$$\lim_{p,q\to\infty} \tau_{pqtv}(x) = \ell$$

uniformly in t, v, where

$$\tau_{pqtv}(x) = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{j+t,k+v}.$$

Quite recently, Mursaleen and Mohiuddine (see [12]) introduced the following double sequence spaces by using almost convergence, while such spaces for single sequences were studied by Das and Sahoo [4].

$$w_2 = \left\{ x = (x_{jk}) : \lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqtv}(x-\ell) = 0 \right.$$
$$\left. \text{uniformly in } t, v, \text{ for some } \ell \right\},$$

$$[w_2] = \left\{ x = (x_{jk}) : \lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |\tau_{pqtv}(x-\ell)| = 0 \right.$$
$$\left. \text{uniformly in } t, v, \text{ for some } \ell \right\}.$$

Let $\theta = \{k_r\}_{r=0}^{\infty}$ be an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$, $k_0 = 0$. Such a sequence is called a lacunary sequence. The ideas of lacunary sequence and lacunary statistical convergence for double sequences were introduced by Savaş and Patterson (see, [14, 19]) as follows:

The double sequence $\theta = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \text{ as } r \to \infty$$

and

$$l_0 = 0, \quad \overline{h}_s = l_s - l_{s-1} \text{ as } s \to \infty.$$

Let $k_{r,s} = k_r l_s$, $h_{r,s} = h_{r,s} = h_r \overline{h}_s$, $q_r = k_r/k_{r-1}$, $\overline{q}_s = l_s/l_{s-1}$ and $q_{r,s} = q_r \overline{q}_s$. Also, the intervals determined by θ will be denoted by $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \ \& \ l_{s-1} < l \leq l_s\}$.

Let θ be a double lacunary sequence. The double number sequence $x = (x_{jk})$ is $S_{\theta_{r,s}}$-convergent (or lacunary statistically convergent) to ℓ provided for every $\epsilon > 0$,

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} |\{(j,k) \in I_{r,s} : |x_{jk} - \ell| \geq \epsilon\}| = 0.$$

In this case, we write $S_{\theta_{r,s}} - \lim x = \ell$ or $x_{jk} \to \ell(S_{\theta_{r,s}})$. Note that this definition is a multidimensional analog of lacunary statistical convergence presented in [7].

Recall (see [8]) that an Orlicz function $M : [0,\infty) \to [0,\infty)$ is a continuous, convex, nondecreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function which is defined and characterized by Ruckle [16]. Subsequently Orlicz function was used to define sequence spaces by Parashar and Choudhary [13] and others. Furthermore, Savaş and Patterson introduced some double sequence spaces by using Orlicz function (see [17, 18, 20]).

An Orlicz function M is said to satisfy Δ_2-condition, if for all values of u, there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

2. Some new spaces of double sequences

In this section, we first introduce the concept of lacunary $[w_2]$-convergence, while such a method for single sequences was introduced by Basarir [1].

Definition 2.1. Let $\theta = \{(k_r, l_s)\}$ be a double lacunary sequence and $x = (x_{jk})$ be a double sequence. Then x is said to be lacunary $[w_2]$-convergent to ℓ if

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s}} |\tau_{pqtv}(x-\ell)| = 0, \text{ uniformly in } t, v.$$

By $[w_2]_\theta$ we denote the set of all lacunary $[w_2]$-convergent double sequences and we write $[w_2]_\theta$-$\lim x = \ell$, for $x \in [w_2]_\theta$.

Definition 2.2. Let M be an Orlicz function. We define the following sequence spaces:

$$[w_2(M)] = \Big\{ x = (x_{jk}) : \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} M\Big(\frac{|\tau_{pqtv}(x-\ell)|}{\rho}\Big) \to 0$$

$$\text{as } m, n \to \infty, \text{uniformly in } t, v, \text{ for some } \ell, \text{ some } \rho > 0 \Big\},$$

$$[w_2(M)]_\theta = \left\{ x = (x_{jk}) : \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s}} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho} \right) \to 0 \text{ as } r,s \to \infty, \right.$$

$$\left. \text{uniformly in } t,v, \text{ for some } \ell, \text{ some } \rho > 0 \right\}.$$

If $M(x) = x$ then $[w_2(M)] = [w_2]$ and $[w_2(M)]_\theta = [w_2]_\theta$.

We also note that the space $[w_2(M)]_\theta$ was introduced and examined for single sequences by Chishti in [2].

Theorem 2.3. *Let $\theta = \{(k_r, l_s)\}$ be a double lacunary sequence with $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$. Then for any Orlicz function M, $[w_2(M)] \subset [w_2(M)]_\theta$.*

Proof. Suppose $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$; then there exists $\delta > 0$ such that $q_r > 1+\delta$ and $\overline{q}_s > 1+\delta$ and hence $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\overline{h}_s}{l_s} \geq \frac{\delta}{1+\delta}$. Therefore, for $x \in [w_2(M)]$ we can write

$$\begin{aligned} \frac{1}{k_{r,s}} \sum_{p,q=0}^{k_r,l_s} M\left(\frac{|\tau_{pqst}(x-\ell)|}{\rho} \right) &\geq \frac{1}{k_{r,s}} \sum_{(p,q)\in I_{r,s}} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho} \right) \\ &= \frac{h_{r,s}}{k_{r,s}} \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s}} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho} \right) \\ &\geq \left(\frac{\delta}{1+\delta} \right)^2 \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s}} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho} \right), \end{aligned}$$

for each t and v. ∎

Theorem 2.4. *Let $\theta = \{(k_r, l_s)\}$ be a double lacunary sequence with $\limsup_r q_r < \infty$ and $\limsup_s \overline{q}_s < \infty$. Then for any Orlicz function M, $[w_2(M)]_\theta \subset [w_2(M)]$.*

Proof. Since $\limsup_r q_r < \infty$ and $\limsup_s \overline{q}_s < \infty$, there exists $H > 0$ such that $q_r < H$ and $\overline{q}_s < H$ for all r and s. Let $x \in [w_2(M)]_\theta$ and $\epsilon > 0$. Then there exist positive integers r_0 and s_0 such that for every $r \geq r_0$ and $s \geq s_0$ and for all t and v,

$$A_{r,s} = \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s}} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho} \right) < \epsilon.$$

Let $M = \max\{A_{r,s} : 1 \leq r \leq r_0, 1 \leq s \leq s_0\}$ and m and n be such that $k_{r-1} < m < k_r$ and $l_{s-1} < n < l_s$. Thus we obtain the following:

$$\frac{1}{mn} \sum_{pq=1,1}^{m,n} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho} \right) \leq \frac{1}{k_{r-1} l_{s-1}} \sum_{pq=1,1}^{k_r,l_s} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho} \right)$$

$$\leq \frac{1}{k_{r-1}l_{s-1}} \sum_{\alpha,\beta=1,1}^{r,s} \left(\sum_{(p,q)\in I_{\alpha,\beta}} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho}\right)\right)$$

$$= \frac{1}{k_{r-1}l_{s-1}} \sum_{\alpha,\beta=1,1}^{r_0,s_0} h_{\alpha,\beta}A_{\alpha,\beta} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0<\alpha\leq r)\cup(s_0<\beta\leq s)} h_{\alpha,\beta}A_{\alpha,\beta}$$

$$\leq \frac{M}{k_{r-1}l_{s-1}} \sum_{\alpha,\beta=1,1}^{r_0,s_0} h_{\alpha,\beta} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0<\alpha\leq r)\cup(s_0<\beta\leq s)} h_{\alpha,\beta}A_{\alpha,\beta}$$

$$\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \left(\sup_{\alpha\geq r_o\cup\beta\geq s_0} A_{\alpha,\beta}\right)\frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0<\alpha\leq r)\cup(s_0<\beta\leq s)} h_{\alpha,\beta}$$

$$\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}}\epsilon \sum_{(r_0<\alpha\leq r)\cup(s_0<\beta\leq s)} h_{\alpha,\beta}$$

$$\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \epsilon H^2.$$

Since k_r and l_s both approach infinity as both m and n approach infinity, it follows that

$$\frac{1}{mn}\sum_{pq=1,1}^{m,n} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho}\right) \to 0, \text{ uniformly in } t,v.$$

Therefore $x \in [w_2(M)]$. ■

The following is an immediate consequence of Theorem 2.1 and Theorem 2.2.

Theorem 2.5. *Let $\theta = \{(k_r,l_s)\}$ be a double lacunary sequence with $1 < \liminf_r q_r < \limsup_r q_r < \infty$ and $1 < \liminf_s \overline{q}_s < \limsup_s \overline{q}_s < \infty$. Then for any Orlicz function M, $[w_2(M)]_\theta = [w_2(M)]$.*

From Theorem 2.3 and Theorem 2.4, we obtain the following corollary for any double lacunary sequence $\theta = \{(k_r,l_s)\}$.

Corollary 2.6. (i) *Let $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$. Then $[w_2] \subset [w_2]_\theta$.*
(ii) *Let $\limsup_r q_r < \infty$ and $\limsup_s \overline{q}_s < \infty$. Then $[w_2]_\theta \subset [w_2]$.*

We need the following trivial lemma to prove the next theorem.

Lemma 2.7. *Let M be an Orlicz function which satisfies Δ_2-condition and let $0 < \delta < 1$. Then for each $x \geq \delta$, we have $M(x) < K\delta^{-1}M(2)$ for some constant $K > 0$.*

Theorem 2.8. *For any Orlicz function M which satisfies Δ_2-condition, we have $[w_2]_\theta \subset [w_2(M)]_\theta$ and $[w_2] \subset [w_2(M)]$.*

Proof. It is sufficient to show that $[w_2]_\theta \subset [w_2(M)]_\theta$. The other inclusion can be proved by using similar method. Let $x \in [w_2]_\theta$. Then

$$B_{rs} = \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s}} |\tau_{pqtv}(x-\ell)| \to 0$$

as $r, s \to \infty$ uniformly in t, v. Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \epsilon$ for $0 \le t \le \delta$. Then, we have

$$\begin{aligned}
\frac{1}{h_{r,s}} &\sum_{(p,q)\in I_{r,s}} M\left(|\tau_{pqtv}(x-\ell)|\right) \\
&= \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s}\ \&\ |\tau_{pqtv}(x-\ell)|\le\delta} M\left(|\tau_{pqtv}(x-\ell)|\right) \\
&\quad + \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s}\ \&\ |\tau_{pqtv}(x-\ell)|>\delta} M\left(|\tau_{pqtv}(x-\ell)|\right) \\
&< \frac{1}{h_{r,s}} h_{r,s}\epsilon + \frac{1}{h_{r,s}} K\delta^{-1} M(2) h_{r,s} B_{rs},
\end{aligned}$$

by Lemma 2.7. Hence

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s}} M\left(|\tau_{pqtv}(x-\ell)|\right) = 0$$

uniformly in t, v. That is $x \in [w_2(M)]_\theta$. ■

Now we extend the spaces $[w_2(M)]$ and $[w_2(M)]_\theta$ to more general spaces denoted by $[w_2(M,\alpha)]$ and $[w_2(M,\alpha)]_\theta$, respectively. Let $\alpha = (\alpha_{pq})$ be a sequence of real numbers such that $\alpha_{pq} > 0$ for all p, q and $\sup_{p,q} \alpha_{pq} = H < \infty$. Define

$$\begin{aligned}
[w_2(M,\alpha)] = \Bigg\{ & x = (x_{jk}) : \\
& \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m}\sum_{q=0}^{n} \left\{ M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho} \right) \right\}^{\alpha_{pq}} \to 0 \\
& \text{as } m, n \to \infty, \text{uniformly in } t, v, \text{ for some } \ell, \text{ some } \rho > 0 \Bigg\},
\end{aligned}$$

$$\begin{aligned}
[w_2(M,\alpha)]_\theta = \Bigg\{ & x = (x_{jk}) : \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s}} \left\{ M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho} \right) \right\}^{\alpha_{pq}} \to 0 \\
& \text{as } r, s \to \infty, \text{uniformly in } t, v, \text{ for some } \ell, \text{ some } \rho > 0 \Bigg\}.
\end{aligned}$$

Theorem 2.9. (i) *If* $0 < \inf \alpha_{pq} < \alpha_{pq} < 1$ *then* $[w_2(M,\alpha)] \subset [w_2(M)]$ *and* $[w_2(M,\alpha)]_\theta \subset [w_2(M)]_\theta$.

(ii) *If* $1 \leq \alpha_{pq} \leq \sup \alpha_{pq} < \infty$ *then* $[w_2(M)] \subset [w_2(M,\alpha)]$ *and* $[w_2(M)]_\theta \subset [w_2(M,\alpha)]_\theta$.

Proof. (i) Let $x \in [w_2(M,\alpha)]$. Since $0 < \inf \alpha_{pq} < \alpha_{pq} < 1$, we have

$$\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho}\right) \leq \frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left\{M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho}\right)\right\}^{\alpha_{pq}},$$

whence $x \in [w_2(M)]$.

(ii) Let $1 \leq \alpha_{pq} \leq \sup \alpha_{pq} < \infty$ and $x \in [w_2(M)]$. Then for each $0 < \epsilon < 1$, there exists a positive integer N such that

$$\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho}\right) \leq \epsilon < 1,$$

for all $m,n \geq N$ and for all t and v. Hence, we have

$$\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left\{M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho}\right)\right\}^{\alpha_{pq}} \leq \frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho}\right).$$

Thus, we obtain $x \in [w_2(M,\alpha)]$. Second parts of (i) and (ii) can be proved similarly. ∎

3. Double statistical convergence methods

In this section, we extend some statistical convergence methods for single sequences to double sequences. We also establish the relations between our newly defined spaces.

The following definition is a multidimensional analog of the concept of $\overline{S}$-statistical convergence introduced by Esi in [5].

Definition 3.1. A double sequence $x = (x_{jk})$ is said to be $\overline{S}_2$ -statistically convergent to ℓ if for every $\epsilon > 0$

$$\lim_{m,n\to\infty} \frac{1}{mn}\left|\{p \leq m, q \leq n : |\tau_{pqtv}(x-\ell)| \geq \epsilon\}\right| = 0, \text{ uniformly in } t \text{ and } v.$$

In this case, we write $\overline{S}_2$-$\lim x = \ell$ or $x_{jk} \to \ell\left(\overline{S}_2\right)$ and denote by $\overline{S}_2$, the space of all $\overline{S}_2$-statistically convergent double sequences.

Definition 3.2. Let $\theta = \{(k_r, l_s)\}$ be a double lacunary sequence. A double sequence $x = (x_{jk})$ is said to be $\overline{S}_{\theta_{r,s}}$-statistically convergent to ℓ if for every $\epsilon > 0$

$$\lim_{r,s\to\infty}\frac{1}{h_{r,s}}\left|\left\{(p,q)\in I_{r,s}:\left|\tau_{pqtv}(x-\ell)\right|\geq\epsilon\right\}\right|=0,\text{ uniformly in } t \text{ and } v.$$

In this case we write $\overline{S}_{\theta_{r,s}}$-$\lim x=\ell$ or $x_{jk}\to\ell\left(\overline{S}_{\theta_{r,s}}\right)$ and denote by $\overline{S}_{\theta_{r,s}}$, the space of all $\overline{S}_{\theta_{r,s}}$-statistically convergent double sequences.

By using the same idea in Theorem 2.2 and Theorem 2.3 of [19], we can easily obtain the following relation between the spaces $\overline{S}_2$ and $\overline{S}_{\theta_{r,s}}$.

Theorem 3.3. (i) $\overline{S}_2$-$\lim x=\ell$ *implies* $\overline{S}_{\theta_{r,s}}$- $\lim x=\ell$ *if and only if* $\liminf_r q_r>1$ *and* $\liminf_s \overline{q}_s>1$.

(ii) $\overline{S}_{\theta_{r,s}}$-$\lim x=\ell$ *implies* $\overline{S}_2$ -$\lim x=\ell$ *if and only if* $\limsup_r q_r<\infty$ *and* $\limsup_s \overline{q}_s<\infty$.

Now we prove the following relations between the spaces $[w_2], [w_2]_\theta$, $\overline{S}_2$ and $\overline{S}_{\theta_{r,s}}$.

Theorem 3.4.

(i) $x_{jk}\to\ell\,[w_2]$ *implies* $x_{jk}\to\ell\left(\overline{S}_2\right)$.

(ii) *If* $x\in\ell_2^\infty$ *and* $x_{jk}\to\ell\left(\overline{S}_2\right)$ *then* $x_{jk}\to\ell\,[w_2]$.

(iii) $\overline{S}_2\cap\ell_2^\infty=[w_2]\cap\ell_2^\infty$.

(iv) $x_{jk}\to\ell\,[w_2]_\theta$ *implies* $x_{jk}\to\ell\left(\overline{S}_{\theta_{r,s}}\right)$.

(v) *If* $x\in\ell_2^\infty$ *and* $x_{jk}\to\ell\left(\overline{S}_{\theta_{r,s}}\right)$ *then* $x_{jk}\to\ell\,[w_2]_\theta$.

(vi) $\overline{S}_{\theta_{r,s}}\cap\ell_2^\infty=[w_2]_\theta\cap\ell_2^\infty$.

Proof. (i) Let $x_{jk}\to\ell\,[w_2]$. For all t and v, we have

$$\begin{aligned}&\left|\left\{p\leq m,q\leq n:\left|\tau_{pqtv}(x-\ell)\right|\geq\epsilon\right\}\right|\\&\qquad\leq\sum_{\substack{p,q=1,1\\ \left|\tau_{pqtv}(x-\ell)\right|\geq\epsilon}}^{m,n}\left|\tau_{pqtv}(x-\ell)\right|\leq\sum_{p,q=1,1}^{m,n}\left|\tau_{pqtv}(x-\ell)\right|.\end{aligned}$$

Since $x_{jk}\to\ell\,[w_2]$, from the last inequality we conclude that

$$\lim_{m,n\to\infty}\frac{1}{mn}\left|\left\{p\leq m,q\leq n:\left|\tau_{pqtv}(x-\ell)\right|\geq\epsilon\right\}\right|=0,$$

for all t and v. This completes the proof of (i).

(ii) Suppose $x\in\ell_2^\infty$ then for all p,q and t,v, $\left|\tau_{pqtv}(x-\ell)\right|\leq M$. Also for given $\epsilon>0$ and for all t and v, we have

$$\frac{1}{mn}\sum_{p,q=1,1}^{m,n}|\tau_{pqtv}(x-\ell)| = \frac{1}{mn}\sum_{\substack{p,q=1,1\\|\tau_{pqtv}(x-\ell)|\geq\epsilon}}^{m,n}|\tau_{pqtv}(x-\ell)|$$
$$+\frac{1}{mn}\sum_{\substack{p,q=1,1\\|\tau_{pqtv}(x-\ell)|\geq\epsilon}}^{m,n}|\tau_{pqtv}(x-\ell)|$$
$$\leq \frac{M}{mn}\left|\{p\leq m, q\leq n : |\tau_{pqtv}(x-\ell)|\geq\epsilon\}\right| + \epsilon.$$

Hence $x \in \ell_2^\infty$ and $x_{jk} \to \ell\left(\overline{S}_2\right)$ implies $x_{jk} \to \ell\,[w_2]$.

(iii) This follows directly from (i) and (ii).

The proofs of (iv), (v) and (vi) are obviously similar to those of (i), (ii) and (iii). ■

We shall now establish an inclusion theorem by using Orlicz function.

Theorem 3.5. *For any Orlicz function M, $[w_2\,(M)]_\theta \subset \overline{S}_{\theta_{r,s}}$ and $[w_2\,(M)] \subset \overline{S}_2$.*

Proof. Let $x \in [w_2\,(M)]_\theta$ and $\epsilon > 0$. Then for all t and v,

$$\frac{1}{h_{r,s}}\sum_{(p,q)\in I_{r,s}} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho}\right)$$
$$\geq \frac{1}{h_{r,s}}\sum_{(p,q)\in I_{r,s}\ \&\ |\tau_{pqtv}(x-\ell)|\geq\epsilon} M\left(\frac{|\tau_{pqtv}(x-\ell)|}{\rho}\right)$$
$$> \frac{1}{h_{r,s}}M\left(\frac{\epsilon}{\rho}\right)\left|\{(p,q)\in I_{r,s} : |\tau_{pqtv}(x-\ell)|\geq\epsilon\}\right|.$$

This implies that $x \in \overline{S}_{\theta_{r,s}}$. ■

Finally, by considering Theorem 3.6 of [3], we prove the reverse inclusion stated in Theorem 3.5.

Theorem 3.6. *If M is a bounded function, which satisfies all conditions of the definition of Orlicz function except $M(x) \to \infty$ as $x \to \infty$, then $\overline{S}_{\theta_{r,s}} \subset [w_2\,(M)]_\theta$ and $\overline{S}_2 \subset [w_2\,(M)]$.*

Proof. We only proof the inclusion $\overline{S}_{\theta_{r,s}} \subset [w_2\,(M)]_\theta$. Suppose that $M(x) \leq K$ for some positive constant K and for all $x \geq 0$. Let $\epsilon > 0$. Choose δ with $0 < \delta < 1$ such that $M(t) < \epsilon$ for $0 \leq t \leq \delta$. Then, we have

$$\begin{aligned}
&\frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s}} M\left(\left|\tau_{pqtv}(x-\ell)\right|\right) \\
&\qquad = \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s} \ \& \ |\tau_{pqtv}(x-\ell)|\geq\delta} M\left(\left|\tau_{pqtv}(x-\ell)\right|\right) \\
&\qquad\quad + \frac{1}{h_{r,s}} \sum_{(p,q)\in I_{r,s} \ \& \ |\tau_{pqtv}(x-\ell)|<\delta} M\left(\left|\tau_{pqtv}(x-\ell)\right|\right) \\
&\qquad \leq \frac{K}{h_{r,s}} \left|\left\{(p,q)\in I_{r,s} : \left|\tau_{pqtv}(x-\ell)\right| \geq \delta\right\}\right| + M(\delta) \\
&\qquad \leq \frac{K}{h_{r,s}} \left|\left\{(p,q)\in I_{r,s} : \left|\tau_{pqtv}(x-\ell)\right| \geq \delta\right\}\right| + \epsilon.
\end{aligned}$$

Hence $x \in \overline{S}_{\theta_{r,s}}$ implies $x \in [w_2(M)]_\theta$. ■

References

[1] M. Başarir, *On some new sequence spaces*, Riv. Mat. Univ. Parma (5) 1 (1992), 339–347.

[2] T. A. Chishti, *Some spaces of lacunary convergent sequences defined by Orlicz functions*, Novi Sad J. Math. 35(2) (2005), 19–25.

[3] R. Çolak, M. Et, E. Malkowsky, *Strongly almost (w,λ)-summable sequences defined by Orlicz functions*, Hokkaido Math. J. 34 (2005), 265–276.

[4] G. Das, S. K. Sahoo, *On some sequence spaces*, J. Math. Anal. Appl. 164 (1992), 381–398.

[5] A. Esi, *Some new sequence space defined by modulus function and strongly almost statistical convergence*, J. Indian Math. Soc. 66 (1999), 81–87.

[6] H. Fast, *Sur la convergence statistique*, Colloq. Math. 2 (1951), 241–244.

[7] J. A. Fridy, C. Orhan, *Lacunary statistical convergence*, Pacific J. Math. 160(1) (1993), 43–51.

[8] M. A. Krasnoselskii, Y. B. Rutisky, *Convex Function and Orlicz Spaces*, 1961, Groningen, Netherlands.

[9] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. 80 (1948), 167–190.

[10] F. Moricz, B. E. Rhoades, *Almost convrgence of double sequences and strong regularity of summability matrices*, Math. Proc. Cambridge Philos Soc. 104 (1988), 283–294.

[11] M. Mursaleen, O. H. H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl. 288 (2003), 223–231.

[12] M. Mursaleen, S. A. Mohiuddine, *Banach limit and some new spaces of double sequences*, Turkish J. Math. 35 (2011), 1–10.

[13] S. D. Parashar, B. Choudhary, *Sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math. 25(4) (1994), 419–428.

[14] R. F. Patterson, E. Savaş, *Lacunary statistical convergence of double sequences*, Math. Commun. 10 (2005), 55–61.

[15] A. Pringsheim, *Zur theorie der zweifach unendlichen Zahlenfolgen*, Math. Ann. 53 (1900), 289–321.

[16] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math. 25 (1973), 973–978.
[17] E. Savaş, R. F. Patterson, *On some double almost lacunary sequence spaces defined by Orlicz functions*, Filomat 19 (2005), 35–44.
[18] E. Savaş, R. F. Patterson, *Some σ-double sequence spaces defined by Orlicz function*, J. Math. Anal. Appl. 324 (2006), 525–531.
[19] E. Savaş, R. F. Patterson, *Lacunary statistical convergence of multiple sequences*, Appl. Math. Lett. 19 (2006), 527–534.
[20] E. Savaş, R. F. Patterson, *Double sequence spaces defined by Orlicz functions*, Iran. J. Sci. Technol. Trans. A Sci. 31 (2007), 183–188.

C. Belen
DEPARTMENT OF MATHEMATICS EDUCATION
ORDU UNIVERSITY
ORDU, TURKEY
E-mail: cbelen52@gmail.com

M. Yildirim
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
CUMHURIYET UNIVERSITY
SIVAS, TURKEY
E-mail: yildirim@cumhuriyet.edu.tr

Received December 4, 2011.
Communicated by J. Wesołowski.