DE GRUYTER
OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVII No 4 2014

Marek Lassak

BANACH–MAZUR DISTANCE BETWEEN CONVEX QUADRANGLES

Abstract. It is proved that the Banach–Mazur distance between arbitrary two convex quadrangles is at most 2. The distance equals 2 if and only if the pair of these quadrangles is a parallelogram and a triangle.

Denote by $\mathcal{C}^n$, the family of all convex bodies in Euclidean n-space E^n and by $\mathcal{M}^n$, the family of all centrally symmetric convex bodies in E^n. Recall that the *Banach–Mazur distance*, called also for shortness the BM-*distance*, of $C, D \in \mathcal{C}^n$ is the number

$$\rho(C,D) = \inf_{a,h_\lambda} \{\lambda;\ a(D) \subset C \subset h_\lambda(a(D))\},$$

where h_λ stands for a homothety with ratio λ and a for an affine transformation. Clearly, the Banach–Mazur distance is a multiplicative metric. In particular, $\rho(C,D) = \rho(D,C)$.

For more than eight decades, this notion for $C, D \in \mathcal{M}^n$ has been playing an important role in functional analysis, C and D are then unit balls of normed spaces. An extensive survey of results was given by Tomczak-Jaegermann [8]. The theorem on approximation of centrally symmetric convex bodies by ellipsoids by John [3] implies that $\rho(C,D) \leq n$ for every $C, D \in \mathcal{M}^n$. This is asymptotically exact, see the paper [1] by Gluskin. Stromquist [6] proved that $\rho(C,D) \leq \frac{3}{2}$ for arbitrary $C, D \in \mathcal{M}^2$, and that this inequality cannot be improved.

In the last two decades, also the general case when C and D are not obligatory centrally symmetric has been considered in many papers. The theorem on approximation of arbitrary convex bodies by ellipsoids by John [3] easily implies that $\rho(C,D) \leq n^2$ for every $C, D \in \mathcal{C}^n$.

2010 *Mathematics Subject Classification*: 52A10, 52A21.
Key words and phrases: Banach–Mazur distance, convex body, convex quadrangle, parallelogram, triangle.

DOI: 10.2478/dema-2014-0079

This estimate has been improved since, at least for large n; however the exact asymptotic order of the bound is unknown, see the survey article [7] by Szarek for a wider context and references. For arbitrary $C, D \in \mathcal{C}^2$, we have $\rho(C,D) \leq 3$ (see [5]). We conjecture that $\rho(C,D) \leq 1 + \frac{1}{2}\sqrt{5}$ (≈ 2.118) for every $C, D \in \mathcal{C}^2$, and the equality holds only for a triangle T and the regular pentagon P. For a position of $a(T)$ with respect to P such that $a(T) \subset P \subset \left(1 + \frac{1}{2}\sqrt{5}\right)a(T)$ see Fig. 1 in [4]. A related conjecture is presented at the bottom of page 209 in [4] (there $1 + \frac{1}{2}\sqrt{5}$ is written in the form $\cos^2 36^\circ / \sin 18^\circ$). In the present note, we prove that the Banach–Mazur distance between arbitrary two convex quadrangles is at most 2, with the value 2 if and only if one of these quadrangles is a parallelogram and the other is a triangle.

We say that convex bodies A and A' are *affinely equivalent*, and we write $A \sim A'$, if there is a non-singular affine transformation which transforms A into A'. Clearly, if $A \sim A'$ and $B \sim B'$ then

$$\rho(A,B) = \rho(A',B'). \tag{1}$$

As usual, the point with coordinates x and y in Cartesian coordinate system of E^2 is denoted by (x,y).

Lemma. *Every convex quadrangle is affinely equivalent to a quadrangle with successive vertices* $(0,1)$, $(0,0)$, $(1,0)$, *and* (p,q), *which belongs to the triangle* T *with vertices* $(1,1)$, $(\frac{1}{2},\frac{1}{2})$, $(1,0)$.

Proof. From the four triangles whose vertices are at the vertices of a convex quadrangle take a triangle K whose area is the largest. Of course, there is an affine transformation which transforms K into the triangle M with successive vertices $(0,1)$, $(0,0)$, $(1,0)$ such that the fourth vertex (p,q) of the quadrangle after the transformation is in the first quadrant. Since affine transformations do not change the ratio of areas, the area of M is not smaller than the area of any of the remaining three triangles whose vertices are at $(0,1)$, $(0,0)$, $(1,0)$ and (p,q). This implies that (p,q) is in the triangle N with vertices $(0,1)$, $(1,0)$ and $(1,1)$, as in the opposite case one of the three remaining triangles would have area larger than that of M. Dissecting the triangle N with the line $y = x$ into two symmetric triangles, we may take one of them, say T, and assume that every (p,q) belongs to it, which ends the proof. ■

Observe that the quadrangle from the Lemma is degenerated to a triangle if and only if (p,q) is in the segment with end-points $(\frac{1}{2},\frac{1}{2})$ and $(1,0)$, and that it is a parallelogram if and only if $d = (1,1)$.

Recall the observation of Grünbaum (see [2], p. 259) that the BM-distance between any planar centrally symmetric convex body and a triangle is 2. In particular, the BM-distance between a parallelogram and a triangle is 2.

Theorem. *The Banach–Mazur distance between two convex quadrangles is at most* 2. *The distance* 2 *is attained if and only if one of the quadrangles is a parallelogram and the other is a triangle.*

Proof. Thanks to the Lemma and (1), it is sufficient to consider quadrangles $abcd$ and $abce$, where $a = (0,1)$, $b = (0,0)$, $c = (1,0)$, and both d and e are in the triangle T described in the Lemma.

For any point g and number $\omega > 0$, denote by g_ω the homothetical image of g with respect to $(0,0)$ and ratio ω.

Case 1. One of the quadrangles is contained in the other, say $abce \subset abcd$.

Of course, $e \in abcd$. Since $e \in T$, we have $d \in a_2bc_2e_2$. Hence $abcd \subset a_2bc_2e_2$. This and $abce \subset abcd$ imply that the BM-distance between $abcd$ and $abce$ is at most 2.

Moreover, the BM-distance is below 2 unless e is in the segment with endpoints $(\frac{1}{2}, \frac{1}{2})$ and $(0,1)$ (i.e., unless $abce$ is a triangle), and unless $d = (1,1)$ (i.e., unless $abcd$ is a parallelogram). This confirms the second thesis of Theorem in Case 1.

Case 2. None of the quadrangles is contained in the other.

Of course, $d \notin abcd$ and $e \notin abce$. Let $e = (\alpha, \beta)$ and $d = (\gamma, \delta)$. Without loss of generality, we may assume that the slope of the vector be is not larger than that of bd, i.e., that $\frac{\beta}{\alpha} \leq \frac{\delta}{\gamma}$.

We find μ such that $d \in a_\mu e_\mu$ (see Figure). Namely, since the straight line through $a_\mu = (0,\mu)$ and $e_\mu = (\mu\alpha, \mu\beta)$ has equation $y - \mu\beta = \frac{\beta-1}{\alpha}(x - \mu\alpha)$, from $(\gamma, \delta) \in a_\mu e_\mu$, we obtain

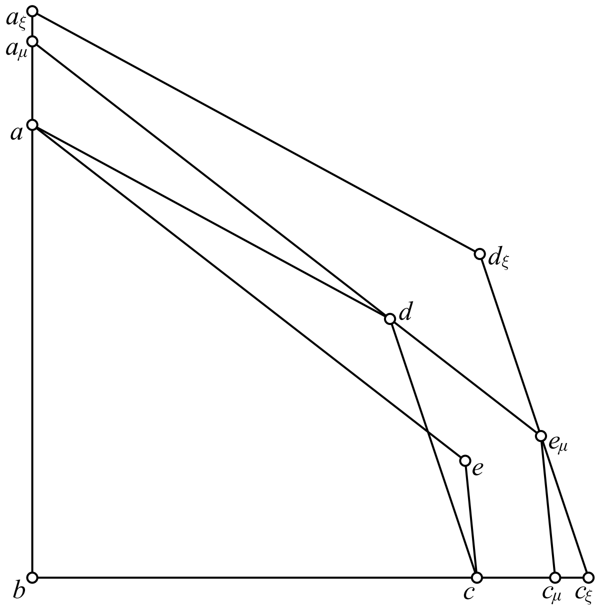
$$\mu = \delta + \frac{\gamma}{\alpha}(1-\beta). \tag{2}$$

We find ξ such that $e_\mu \in d_\xi c_\xi$ (see Figure). From the fact that the equation of the straight line through $d_\xi = (\xi\gamma, \xi\delta)$ and $c_\xi = (\xi, 0)$ is $y = \frac{\delta}{\gamma-1}(x - \xi)$, we get

$$\xi = \mu\left[\alpha + \frac{\beta}{\delta}(1-\gamma)\right]. \tag{3}$$

From $abcd \subset a_\mu bc_\mu e_\mu \subset a_\xi bc_\xi d_\xi$ (see Figure), (3) and (2), we conclude that the BM-distance between $abcd$ and $a_\mu bc_\mu e_\mu$ (and hence between $abcd$ and $abce$ by (1)) is at most

$$\begin{aligned}\xi &= \left[\delta + \frac{\gamma}{\alpha}(1-\beta)\right]\left[\alpha + \frac{\beta}{\delta}(1-\gamma)\right] \\ &= \alpha\delta + \gamma(1-\beta) + \beta(1-\gamma) + \frac{\beta\gamma}{\delta\alpha}(1-\beta)(1-\gamma).\end{aligned}$$



Since $\frac{\gamma}{\alpha}\frac{\beta}{\delta} \leq 1$ by hypothesis, the above is at most

$$\gamma(1-\beta)+\beta(1-\gamma)+\alpha\delta+(1-\beta)(1-\gamma)=1+\alpha\delta-\beta\gamma.$$

By $\alpha \leq 1$ and $\delta \leq 1$ our ξ is always at most 2.

Below we show that $\xi \neq 2$, which means that the BM-distance between *abcd* and *abce* never attains 2 in Case 2.

Assume that $\xi = 2$. Then $\alpha\delta - \beta\gamma = 1$, which is possible only if $\alpha = \delta = 1$ and $\beta\gamma = 0$. Since $d \in T$ from $\delta = 1$, we get $\gamma = 1$ and thus $d = (1, 1)$. From $\gamma = 1$ and $\beta\gamma = 0$, we see that $\beta = 0$. This and $e \in T$ imply that $e = (1, 0)$. Since e belongs to *abcd*, we get a contradiction with the assumption of Case 2. ■

Acknowledgement. The author wishes to thank the Referee whose comments simplified the proof of Theorem.

References

[1] E. D. Gluskin, *Diameter of the Minkowski compactum is approximately equal to n*, Funct. Anal. Appl. 15 (1981), 57–58.

[2] B. Grünbaum, *Measures of symmetry for convex sets*, 1963 Proc. Sympos. Pure Math., Vol. VII, 233–270, Amer. Math. Soc., Providence, R.I.

[3] F. John, *Extremum problems with inequalities as subsidiary conditions*, Courant Anniversary Volume, 1948, 187–204.

[4] M. Lassak, *Approximation of convex bodies by triangles*, Proc. Amer. Math. Soc. 115 (1992), 207–210.

[5] M. Lassak, *Banach–Mazur distance of planar convex bodies*, Aequationes Math. 74 (2007), 282–286.

[6] W. Stromquist, *The maximum distance between two-dimensional Banach spaces*, Math. Scand. 48 (1981), 205–225.

[7] S. J. Szarek, *Convexity, complexity and high dimensions*, Proceedings of the International Congress of Mathematicians, Madrid, Spain, August 22–30, 2006, Volume II: Invited Lectures, Zürich: European Mathematical Society (EMS), 1599–1621.

[8] N. Tomczak-Jaegermann, *Banach–Mazur Distances and Finite-Dimensional Operator Ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics 38, Longman Scientifical and Technical, New York, 1989.

M. Lassak
INSTITUTE OF MATHEMATICS AND PHYSICS
UNIVERSITY OF TECHNOLOGY AND LIFE SCIENCES
85-789 BYDGOSZCZ, POLAND
E-mail: lassak@utp.edu.pl

Received August 12, 2013; revised version November 20, 2013.

Communicated by W. Domitrz.