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A GENERAL COINCIDENCE AND COMMON FIXED POINT THEOREM FOR TWO HYBRID PAIRS OF MAPPINGS

Abstract. In this paper, a general coincidence and common fixed points theorem for two hybrid pairs of mappings satisfying property $(E.A)$ is proved, generalizing the main results from [3] and [9].

1. Introduction

Sessa [18] introduced the concept of weakly commuting mappings. Jungck [5] defined the notion of compatible mappings in order to generalize the concept of weak commutativity and showed that weak commuting mappings are compatible but the converse is not true. In recent years, a number of fixed point theorems and coincidence theorems have been obtained by various authors using this notion. Jungck further weakened the notion of weak compatibility in [6] and Jungck and Rhoades further extended weak compatibility for set-valued mappings. Pant [11], [12], [13] initiated the study of noncompatible mappings and Singh and Mishra [19] introduced the notion of (I, T)-commuting.

More recently, Aamri and Moutawakil [1] defined property $(E.A)$ for self mappings of a metric space.

Liu et al. [10] extended the notion of $(E.A)$ property for two pairs of mappings. The class of mappings satisfying $(E.A)$ property contains the class of noncompatible mappings. Kamran [8] extends the property $(E.A)$ for hybrid pairs of single and multi-valued mappings and generalize the notion of $(I.T)$-commutativity for hybrid pairs. Recently, Sintunavarat and Kumam [21] established new coincidence and common fixed point theorems for hybrid strict contraction maps by dropping the assumption "f is T weakly commut-

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ing for a hybrid pair (f,T) of single and multivalued maps" in Theorem 3.10 [8]. Quite recently, some fixed point theorems for hybrid pairs are obtained in [9] and [3].

The study of fixed points for mappings satisfying implicit relations is initiated in [14], [15] for single valued mappings. In [16], [17] and in other papers, the study of fixed points for hybrid pairs of mappings satisfying implicit relations is initiated.

Actually, the method is used in the study of fixed points and coincidence points in metric spaces, symmetric spaces, quasi-metric spaces, ultra metric spaces, reflexive spaces, in two or three metric spaces for single valued mappings, hybrid pairs of mappings and set valued mappings. Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces, intuitionistic metric spaces and G-metric spaces. The method unified some results from literature.

The purpose of this paper is to prove a general coincidence and fixed point theorem for two hybrid pair of mappings with common $(E.A)$-property satisfying an implicit relation generalizing main results by [9] and [3].

2. Preliminaries

Let (X,d) be a metric space. We denote by $CL(X)$ (respectively, $CB(X)$), the set of all nonempty closed (respectively, nonempty closed bounded) subsets of (X,d) and by H, the Hausdorff–Pompeiu metric, i.e.

$$H(A,B)=\max\Big\{\sup_{x\in A} d(x,B),\sup_{x\in B} d(x,A)\Big\},$$

where $A,B\in\mathfrak{P}(X)$ and

$$d(x,A)=\inf_{y\in A}\{d(x,y)\}.$$

Definition 2.1. Let $f:(X,d)\to(X,d)$ and $F:(X,d)\to CL(X)$.

(1) A point $x\in X$ is said to be a coincidence point of f and F if $fx\in Fx$. The set of all coincidence points of f and F is denoted by $C(f,F)$.

(2) A point $x\in X$ is a common fixed point of f and F if $x=fx\in Fx$.

Definition 2.2. Let $f:(X,d)\to(X,d)$ and $T:(X,d)\to CL(X)$. Then f is said to be T-weakly commuting at $x\in X$ [19] if $ffx\in Tfx$.

Definition 2.3. Let $f:(X,d)\to(X,d)$ and $T:(X,d)\to CB(X)$. f and T satisfy property $(E.A)$ [8] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n=t\in A=\lim_{n\to\infty} Tx_n$.

Liu et al. [10] extend Definition 2.3 for two hybrid pair of mappings.

Definition 2.4. Let $f, g : (X,d) \to (X,d)$ and $F, G : (X,d) \to CB(X)$. The pairs (f,F) and (g,G) are said to satisfy a common property $(E.A)$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X, some $t \in X$ and $A, B \in CB(X)$ such that $\lim_{n\to\infty} Fx_n = A, \lim Gy_n = B$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in A \cap B$.

Motivated by Berinde and Berinde [2], Kamran [9] proved the following theorem.

Theorem 2.1. *Let $f : (X,d) \to (X,d)$ and $T : (X,d) \to CL(X)$ be such that*

(i) *f and T satisfy property $(E.A)$,*
(ii) *for all $x \neq y$,*

$$H(Tx,Ty) < \max\left\{d(fx,fy), \frac{d(fx,Tx)+d(fy,Ty)}{2}, \frac{d(fy,Tx)+d(fx,Ty)}{2}\right\} + Ld(fx,Ty),$$

where $L \geq 0$.

If $f(X)$ is a closed subset of X, then f and T have a coincidence point $a \in X$. Further, if f is T-weakly commuting at a and $fa = ffa$, then f and T have a common fixed point.

Quite recently, Hashim and Abbas [3] extended Theorem 2.1 for two pairs of hybrid mappings dropping the condition "f is T-weakly commuting".

Theorem 2.2. [3] *Let $f, g : (X,d) \to (X,d)$ and $F, G : (X,d) \to CL(X)$ such that*

(i) *(f,F) and (g,G) satisfy the common property $(E.A)$,*
(ii) *for all $x \neq y$,*

$$\begin{aligned} H(Fx,Gy) < \max\{&d(fx,gy), \alpha[d(fx,Fx)+d(gy,Gy)], \\ &\alpha[d(fx,Gy)+d(fy,Gx)]\} \\ &+ L\min\{d(fx,Fx), d(gy,Gy), d(fx,Gy), d(gy,Fx)\}, \end{aligned}$$

where $0 < \alpha < 1$ and $L \geq 0$.

If $f(X)$ and $g(X)$ are closed subsets of X then $C(f,F) \neq \varnothing$ and $C(g,G) \neq \varnothing$. Further,

(a) *if $fv = ffv$ for $v \in C(f,F)$ then f and F have a common fixed point,*
(b) *if $gv = ggv$ for $v \in C(g,G)$ then g and G have a common fixed point,*
(c) *if (a) and (b) are both true, then f, g, F and G have a common fixed point.*

This theorem generalizes Theorem 2.3 [10].

THEOREM 2.3. [3] *Let* $f, g : (X, d) \to (X, d)$ *and* $F, G : (X, d) \to CL(X)$ *such that*

(i) (f, F) *and* (g, G) *satisfy the common property* $(E.A)$,

(ii) *for all* $x \neq y$,

$$H(Fx, Gy) < \max\left\{d(fx, gy), \alpha d(fx, Fx), \alpha d(gy, Gy), \alpha\frac{d(fx, Gy) + d(gy, Fx)}{2}\right\} + L\min\{d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\},$$

where $0 < \alpha < 1$ *and* $L \geq 0$.

If $f(X)$ *and* $g(X)$ *are closed subsets of* X*, then the conclusion of Theorem 2.2 follows.*

This theorem extends and generalizes Theorem 1 [20].

3. Implicit relations

DEFINITION 3.1. Let $\mathfrak{F}_C$ be the set of all continuous functions $F(t_1, \ldots, t_6)$: $\mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) : F is nondecreasing in variable t_1,
(F_2) : $F(t, 0, 0, t, t, 0) > 0$, $\forall t > 0$,
(F_3) : $F(t, 0, t, 0, 0, t) > 0$, $\forall t > 0$.

EXAMPLE 3.1.

$$F(t_1, \ldots, t_6) = t_1 - \max\{t_2, \alpha(t_3 + t_4), \alpha(t_5 + t_6)\} - L\min\{t_2, t_4, t_5, t_6\},$$

where $\alpha \in (0, 1)$ and $L \geq 0$.

(F_1): Obviously,
(F_2) : $F(t, 0, 0, t, t, 0) = t(1 - \alpha) > 0$, $\forall t > 0$,
(F_3) : $F(t, 0, t, 0, 0, t) = t(1 - \alpha) > 0$, $\forall t > 0$.

EXAMPLE 3.2.

$$F(t_1, \ldots, t_6) = t_1 - \max\left\{t_2, \alpha t_3, \alpha t_4, \alpha\frac{t_5 + t_6}{2}\right\} - L\min\{t_2, t_4, t_5, t_6\},$$

where $\alpha \in [0, 1)$ and $L \geq 0$.

(F_1): Obviously,
(F_2) : $F(t, 0, 0, t, t, 0) = t(1 - \alpha) > 0$, $\forall t > 0$,
(F_3) : $F(t, 0, t, 0, 0, t) = t(1 - \alpha) > 0$, $\forall t > 0$.

EXAMPLE 3.3.

$$F(t_1, \ldots, t_6) = t_1 - k\max\{t_2, t_3, t_4, t_5, t_6\} - L\min\{t_3, t_4, t_5, t_6\},$$

where $k \in (0, 1)$ and $L \geq 0$.

(F_1): Obviously,
$(F_2): F(t,0,0,t,t,0) = t(1-k) > 0, \forall t > 0,$
$(F_3): F(t,0,t,0,0,t) = t(1-k) > 0, \forall t > 0.$

Example 3.4. $F(t_1,\ldots,t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a,b,c,d,e \geq 0$, $c+d<1$ and $b+e<1$.

(F_1): Obviously,
$(F_2): F(t,0,0,t,t,0) = t[1-(c+d)] > 0, \forall t > 0,$
$(F_3): F(t,0,t,0,0,t) = t[1-(b+e)] > 0, \forall t > 0.$

Example 3.5. $\mathrm{F}(\mathrm{t}_1,\ldots,\mathrm{t}_6) = \mathrm{t}_1^2 - \mathrm{t}_2^2 - \mathrm{a}\frac{\mathrm{t}_3^2+\mathrm{t}_4^2}{1-\min\{\mathrm{t}_5,\mathrm{t}_6\}}$, *where* $a \in (0,1)$.

(F_1): Obviously,
$(F_2): F(t,0,0,t,t,0) = t^2(1-a) > 0, \forall t > 0,$
$(F_3): F(t,0,t,0,0,t) = t^2(1-a) > 0, \forall t > 0.$

Example 3.6. $F(t_1,\ldots,t_6) = t_1 - \alpha\max\{t_2,t_3,t_4\} - (1-\alpha)(at_5+bt_6)$, where $\alpha \in (0,1)$, $a,b \geq 0$ and $\max\{a,b\} < 1$.

(F_1): Obviously,
$(F_2): F(t,0,0,t,t,0) = t(1-\alpha)(1-a) > 0, \forall t > 0,$
$(F_3): F(t,0,t,0,0,t) = t(1-\alpha)(1-b) > 0, \forall t > 0.$

Example 3.7. $F(t_1,\ldots,t_6) = t_1 - \max\{ct_2,ct_3,ct_4,at_5+bt_6\}$, where $a,b,c \geq 0$ and $\max\{a,b,c\} < 1$.

(F_1): Obviously,
$(F_2): F(t,0,0,t,t,0) = t[1-\max\{a,c\}] > 0, \forall t > 0,$
$(F_3): F(t,0,t,0,0,t) = t[1-\max\{b,c\}] > 0, \forall t > 0.$

Example 3.8. $F(t_1,\ldots,t_6) = t_1^2 - \frac{t_3^2t_4^2+t_5^2t_6^2}{1+t_2}$.

(F_1): Obviously,
$(F_2): F(t,0,0,t,t,0) = t^2 > 0, \forall t > 0,$
$(F_3): F(t,0,t,0,0,t) = t^2 > 0, \forall t > 0.$

4. Main results

Theorem 4.1. *Let* (X,d) *be a metric space,* $f,g: X \to X$ *and* $F,G: X \to CL(X)$ *such that*

(4.1) (f,F) *and* (g,G) *satisfy the common property* $(E.A)$,

(4.2) $$\phi(H(Fx,Gy), d(fx,gy), d(fx,Fx), d(gy,Gy), d(fx,Gy), d(gy,Fx)) < 0,$$

for all $x,y \in X$, *where* $\phi \in \mathfrak{F}_C$.

If $f(X)$ *and* $g(X)$ *are closed subsets of* X *then* $C(f,F) \neq \varnothing$ *and* $C(g,G) \neq \varnothing$.

Further,

(a) *if $fv = ffv$ for $v \in C(f,F)$ then f and F have a common fixed point,*
(b) *if $gw = ggw$ for $w \in C(g,G)$ then g and G have a common fixed point,*
(c) *if $fv = ffv$ for $v \in C(f,F)$ and $gw = ggw$ for $w \in C(g,G)$ then f, g, F and G have a common fixed point.*

Proof. Since (f,F) and (g,G) satisfy the common property $(E.A)$ $A, B \in CL(X)$ such that

$$\lim_{n\to\infty} Fx_n = A, \quad \lim_{n\to\infty} Gy_n = B \text{ and } \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A \cap B.$$

Since $f(X)$ and $g(X)$ are closed, we have $u = fv$ and $u = gw$ for some $v, w \in X$. First we prove that $gw \in Gw$.

By (4.2), we have

$$\phi(H(Fx_n, Gw), d(fx_n, gw), d(fx_n, Fx_n), \\ d(gw, Gw), d(fx_n, Gw), d(gw, Fx_n)) < 0.$$

Letting n tend to infinity, we have successively

$$\phi(H(A, Gw), d(fv, gw), d(fv, A), d(gw, Gw), d(fv, Gw), d(gw, A)) \leq 0,$$
$$\phi(H(A, Gw), 0, 0, d(gw, Gw), d(gw, Gw), 0) \leq 0.$$

Since $gw \in A$ then $d(gw, Gw) \leq H(A, Gw)$. Then by (F_1), it follows that

$$\phi(d(gw, Gw), 0, 0, d(gw, Gw), d(gw, Gw), 0) \leq 0,$$

a contradiction with (F_2) if $d(gw, Gw) > 0$. Hence, $d(gw, Gw) = 0$ which implies that $gw \in Gw$. Hence $C(g,G) \neq \varnothing$.

Now, we prove that $fv \in Fv$. By (4.2), we have

$$\phi(H(Fv, Gy_n), d(fv, gy_n), d(fv, Fv), \\ d(gy_n, Gy_n), d(fv, Gy_n), d(gy_n, Fv)) < 0.$$

Letting n tend to infinity, we have successively

$$\phi(H(Fv, B), d(fv, gw), d(fv, Fv), d(gw, B), d(fv, B), d(gw, Fv)) \leq 0.$$

Since $fv \in B$ then $d(fv, Fv) \leq H(Fv, B)$. Then by (F_1), it follows that

$$\phi(d(fv, Fv), 0, d(fv, Fv), 0, 0, d(fv, Fv)) \leq 0,$$

a contradiction with (F_3) if $d(fv, Fv) > 0$. Hence, $d(fv, Fv) = 0$ which implies $fv \in Gv$. Therefore $C(f,F) \neq \varnothing$ and $C(g,G) \neq \varnothing$. Further, let $fv = ffv$ and $z = fv \in Fv$. Then $fz = ffv = fv = z$ and z is a fixed point of f.

Again, by (4.2), we have

$$\phi(H(Fz, Gw), d(fz, gw), d(fz, Fz), d(gw, Gw), d(fz, Gw), d(gw, Fz)) < 0.$$

Since $d(fz, Fz) = d(fv, Fz) = d(gw, Fz) \leq H(Gw, Fz)$ then by (F_1), we obtain

$$\phi(d(fz, Fz), 0, 0, d(fz, Fz), 0, 0, d(fz, Fz)) < 0,$$

a contradiction with (F_3) if $d(fz, Fz) > 0$. Hence $d(fz, Fz) = 0$, which implies $z = fz \in Fz$. Therefore, z is a common fixed point of f and F.

Further, let $gw = ggw$ and $z = fv = gw \in Gw$. Then $gz = ggw = gw = z$ and z is a fixed point of g.

Again, by (4.2), we have

$$\phi(H(Fv, Gz), d(fv, gz), d(fv, Fv), d(gz, Gz), d(fv, Gz), d(gz, Fv)) < 0.$$

Since $d(gz, Gz) = d(gw, Gz) = d(fv, Gz) \leq H(Fv, Gz)$ then by (F_1), we obtain

$$\phi(d(gz, Gz), 0, 0, d(gz, Gz), d(gz, Gz), 0) < 0,$$

a contradiction with (F_2) if $d(gz, Gz) > 0$. Hence $d(gz, Gz) = 0$, which implies $z = gz \in Gz$. Therefore, z is a common fixed point of g and G.

If $ffv = fv$ for $v \in C(f, F)$ and $ggw = gw$ for $w \in C(g, G)$, then f, g, F and G have a common fixed point. ∎

Remark 4.1. 1. By Theorem 4.1 and Example 3.1, we obtain Theorem 2.2.
2. By Theorem 4.1 and Example 3.2, we obtain Theorem 2.3.
3. By Theorem 4.1 and Examples 3.3–3.8, we obtain new particular results.

If $f = g$ and $F = G$, we obtain

Theorem 4.2. *Let (X, d) be a metric space, $f : X \to X$ and $F : X \to CL(X)$ such that*

(4.3) (f, F) *satisfy property* $(E.A)$,

(4.4) $$\phi(H(Fx, Fy), d(fx, fy), d(fx, Fx), d(fy, Fy), d(fx, Fy), d(fy, Fx)) < 0$$

for all $x, y \in X$, where ϕ satisfies (F_1) and (F_2).

If $f(X)$ is a closed subset of X then $C(f, F) \neq \varnothing$. Further, if $fv = ffv$ for $v \in C(f, F)$, then f and F have a common fixed point.

Proof. The proof follows by the first part of the proof of Theorem 4.1. ∎

Remark 4.2. By Theorem 4.2 and Example 3.1, we obtain a generalization of Theorem 2.1.

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