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ULAM STABILITIES FOR THE DARBOUX PROBLEM FOR PARTIAL FRACTIONAL DIFFERENTIAL INCLUSIONS

Abstract. In this article, we investigate some Ulam's type stability concepts for the Darboux problem of partial fractional differential inclusions with a nonconvex valued right hand side. Our results are based upon Covitz-Nadler fixed point theorem and fractional version of Gronwall's inequality.

1. Introduction

The fractional calculus deals with extensions of derivatives and integrals to noninteger orders. It represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [15, 25, 28, 32]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [6], Kilbas *et al.* [22], Miller and Ross [24], the papers of Abbas *et al.* [1, 2, 3, 4, 5, 7], Vityuk and Golushkov [34], and the references therein.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: Under what conditions does there exist an additive mapping near an approximately additive mapping? (for more details see [33]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces [17]. Thereafter, this type of stability is called the Ulam–Hyers stability. In 1978, Rassias [29] provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. The concept

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of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is how do the solutions of the inequality differ from those of the given functional equation?, or if for every solution of the perturbed equation there exists a solution of the equation that is close to it. Considerable attention has been given to the study of the Ulam–Hyers and Ulam–Hyers–Rassias stability of all kinds of functional equations; see the monographs [18, 19]. Bota–Boriceanu and Petrusel [9], Petru *et al.* [26, 27], and Rus [30, 31] discussed the Ulam–Hyers stability for operatorial equations and inclusions. Castro and Ramos [11], and Jung [21] considered the Hyers–Ulam–Rassias stability for a class of Volterra integral equations. Ulam stability for fractional differential equations with Caputo derivative are proposed by Wang *et al.* [35, 36]. Some stability results for fractional integral equation are obtained by Wei *et al.* [37]. More details from historical point of view, and recent developments of such stabilities are reported in [20, 30, 37].

Motivated by those papers, in this paper, we discuss the Ulam stabilities for the following fractional partial differential inclusion

$$(1) \qquad {}^cD^r_\theta u(x,y) \in F(x,y,u(x,y)); \quad \text{if } (x,y) \in J := [0,a] \times [0,b],$$

where $a, b > 0$, $\theta = (0,0)$, ${}^cD^r_\theta$ is the fractional Caputo derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $F : J \times E \to \mathcal{P}(E)$ is a set-valued function with nonempty values in a (real or complex) separable Banach space E, and $\mathcal{P}(E)$ is the family of all nonempty subsets of E.

This paper initiates the Ulam stabilities of the Darboux problem for hyperbolic fractional differential inclusions in Banach spaces when the right hand side is nonconvex valued. We prove that the inclusion (1) is generalized Ulam–Hyers–Rassias stable.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Denote $L^1(J)$ the space of Bochner-integrable functions $u : J \to E$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x,y)\|_E dydx,$$

where $\|.\|_E$ denotes a complete norm on E.

Let $L^\infty(J)$ be the Banach space of measurable functions $u : J \to E$ which are essentially bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : \|u(x,y)\|_E \leq c, \text{ a.e. } (x,y) \in J\}.$$

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into E, and $\mathcal{C} := C(J)$ is the Banach space of all continuous functions from J into E with the norm $\|.\|_\infty$ defined by

$$\|u\|_\infty = \sup_{(x,y)\in J} \|u(x,y)\|_E.$$

Let $\theta = (0,0),\ r_1,\ r_2 > 0$ and $r = (r_1, r_2)$. For $f \in L^1(J)$, the expression

$$(I_\theta^r f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)dtds$$

is called the left-sided mixed Riemann–Liouville integral of order r, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt;\ \xi > 0$.

In particular,

$$(I_\theta^\theta f)(x,y) = f(x,y),\ (I_\theta^\sigma f)(x,y) = \int_0^x \int_0^y f(s,t)dtds, \text{ for almost all } (x,y) \in J,$$

where $\sigma = (1,1)$.

For instance, $I_\theta^r f$ exists for all $r_1, r_2 \in (0,\infty)$, when $f \in L^1(J)$. Note also that when $f \in C(J)$, then $(I_\theta^r f) \in C(J)$, moreover

$$(I_\theta^r f)(x,0) = (I_\theta^r f)(0,y) = 0;\ x \in [0,a],\ y \in [0,b].$$

Example 2.1. Let $\lambda, \omega \in (0,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$ then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2}, \text{ for almost all } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

Definition 2.2. [34] Let $r \in (0,1] \times (0,1]$ and $f \in L^1(J)$. The Caputo fractional-order derivative of order r of f is defined by the expression

$${}^cD_\theta^r f(x,y) = (I_\theta^{1-r} D_{xy}^2 f)(x,y).$$

The case $\sigma = (1,1)$ is included and we have

$$({}^cD_\theta^\sigma f)(x,y) = (D_{xy}^2 f)(x,y);\ \text{for almost all } (x,y) \in J.$$

Example 2.3. Let $\lambda, \omega \in (0,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$ then

$${}^cD_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2}, \text{ for almost all } (x,y) \in J.$$

Consider the following Darboux problem of partial differential equations

$$\begin{cases} {}^cD_\theta^r u(x,y) = f(x,y), & \text{if } (x,y) \in J, \\ u(x,0) = \varphi(x),\ x \in [0,a], \\ u(0,y) = \psi(y),\ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{2}$$

where $\varphi : [0,a] \to E$, $\psi : [0,b] \to E$ are given absolutely continuous functions.

In the sequel, we need the following lemma given by the authors [2].

Lemma 2.4. [2] *Let* $0 < r_1, r_2 \leq 1$, *and* $\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0)$. *A function* $u \in \mathcal{C}$ *is a solution of the fractional integral equation*

$$u(x,y) = \mu(x,y) + I_\theta^r f(x,y), \tag{3}$$

if and only if u *is a solution of the problem* (2).

Let $(X, \|\cdot\|)$ be a Banach space. Denote $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

Definition 2.5. A multivalued map $T : X \to \mathcal{P}(X)$ is convex (closed) valued if $T(x)$ is convex (closed) for all $x \in X$. T is bounded on bounded sets if $T(B) = \cup_{x \in B} T(x)$ is bounded in X for all $B \in \mathcal{P}_{bd}(X)$ (*i.e.* $\sup_{x \in B} \sup_{y \in T(x)} \|y\| < \infty$). T is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $T(x_0)$, there exists an open neighborhood N_0 of x_0 such that $T(N_0) \subseteq N$. T is said to be completely continuous if $T(B)$ is relatively compact for every $B \in \mathcal{P}_{bd}(X)$. T has a fixed point if there is $x \in X$ such that $x \in T(x)$. The fixed point set of the multivalued operator T will be denoted by $FixT$. A multivalued map $G : X \to P_{cl}(E)$ is said to be measurable if for every $v \in E$, the function $x \longmapsto d(v, G(x)) = \inf\{\|v - z\|_E : z \in G(x)\}$ is measurable.

For more details on multivalued maps see the books of Aubin and Cellina [8], Górniewicz [10], Hu and Papageorgiou [16] and Kisielewicz [23].

For each $u \in \mathcal{C}$, define the set of selections of F by

$$S_{F,u} = \{w \in L^1(J) : w(x,y) \in F(x,y,u(x,y));\ (x,y) \in J\}.$$

Let (X,d) be a metric space induced from the normed space $(X, \|\cdot\|)$.

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\Big\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(A,b)\Big\},$$

where $d(A,b) = \inf_{a \in A} d(a,b)$, $d(a,B) = \inf_{b \in B} d(a,b)$. Then $(\mathcal{P}_{bd,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [23]).

Definition 2.6. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(u), N(v)) \leq \gamma d(u,v) \text{ for each } u,\ v \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Now, we consider the Ulam stability of fractional differential inclusion (1). Let ϵ be a positive real number and $\Phi : J \to [0, \infty)$ be a continuous function. We consider the following inequalities

$$H_d({}^cD^r_\theta u(x,y), F(x,y,u(x,y))) \leq \epsilon, \quad \text{if } (x,y) \in J. \tag{4}$$

$$H_d({}^cD^r_\theta u(x,y), F(x,y,u(x,y))) \leq \Phi(x,y), \quad \text{if } (x,y) \in J. \tag{5}$$

$$H_d({}^cD^r_\theta u(x,y), F(x,y,u(x,y))) \leq \epsilon\Phi(x,y), \quad \text{if } (x,y) \in J. \tag{6}$$

Definition 2.7. Problem (1) is Ulam–Hyers stable if there exists a real number $c_F > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (4), there exists a solution $v \in \mathcal{C}$ of problem (1) with

$$\|u(x,y) - v(x,y)\|_E \leq \epsilon c_F; \ (x,y) \in J.$$

Definition 2.8. Problem (1) is generalized Ulam–Hyers stable if there exists $\theta_F \in C([0,\infty), [0,\infty))$, $\theta_F(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (4), there exists a solution $v \in \mathcal{C}$ of problem (1) with

$$\|u(x,y) - v(x,y)\|_E \leq \theta_F(\epsilon); \ (x,y) \in J.$$

Definition 2.9. Problem (1) is Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{F,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (6), there exists a solution $v \in \mathcal{C}$ of problem (1) with

$$\|u(x,y) - v(x,y)\|_E \leq \epsilon c_{F,\Phi}\Phi(x,y); \ (x,y) \in J.$$

Definition 2.10. Problem (1) is generalized Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each solution $u \in \mathcal{C}$ of the inequality (5), there exists a solution $v \in \mathcal{C}$ of problem (1) with

$$\|u(x,y) - v(x,y)\|_E \leq c_{F,\Phi}\Phi(x,y); \ (x,y) \in J.$$

Remark 2.11. It is clear that

(i) Definition 2.7 $\Rightarrow$ Definition 2.8,
(ii) Definition 2.9 $\Rightarrow$ Definition 2.10,
(iii) Definition 2.9 for $\Phi(x,y) = 1$ $\Rightarrow$ Definition 2.7.

Remark 2.12. A function $u \in \mathcal{C}$ is a solution of the inequality (4) if and only if there exists a function $g \in \mathcal{C}$ (which depends on u) such that

(i) $\|g(x,y)\|_E \leq \epsilon$,
(ii) ${}^cD^r_\theta u(x,y) - g(x,y) \in F(x,y,u(x,y))$; if $(x,y) \in J$.

One can have similar remarks for the inequalities (5) and (6). So, the Ulam stabilities of the fractional differential equations are some special types of data dependence of the solutions of fractional differential equations.

We need the following lemma.

Lemma 2.13. (Covitz–Nadler) [13] *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then N has fixed points.*

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 2.14. (Gronwall lemma) [14] *Let $\upsilon : J \to [0, \infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$\upsilon(x, y) \leq \omega(x, y) + c\int_0^x \int_0^y \frac{\upsilon(s, t)}{(x - s)^{r_1}(y - t)^{r_2}} dtds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\upsilon(x, y) \leq \omega(x, y) + \delta c\int_0^x \int_0^y \frac{\omega(s, t)}{(x - s)^{r_1}(y - t)^{r_2}} dtds,$$

for every $(x, y) \in J$.

3. Main results

In this section, we present conditions for the Ulam stability of problem (1).

Lemma 3.1. *If $u \in \mathcal{C}$ is a solution of the inequality (4) then there exists $f \in S_{F,u}$ such that u is a solution of the following integral inequality*

$$\|u(x, y) - \mu(x, y) - I_\theta^r f(x, y)\|_E \leq \frac{\epsilon a^{r_1} b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)}; \text{ if } (x, y) \in J. \tag{7}$$

Proof. By Remark 2.12, for $(x, y) \in J$, there exists $g \in \mathcal{C}$ such that $\|g(x, y)\|_E \leq \epsilon$ and

$$^cD_\theta^r u(x, y) - g(x, y) \in F(x, y, u(x, y)).$$

Then, there exists $f \in S_{F,u}$ such that for $(x, y) \in J$, we get

$$u(x, y) = \mu(x, y) + I_\theta^r[f(x, y) + g(x, y)].$$

Thus, for $(x, y) \in J$ we obtain

$$\begin{aligned} \|u(x, y) - \mu(x, y) - I_\theta^r f(x, y)\|_E &= \|I_\theta^r g(x, y)\|_E \\ &\leq I_\theta^r \|g(x, y)\|_E \\ &\leq \frac{\epsilon a^{r_1} b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)}. \end{aligned}$$

Hence, we obtain (7). ■

Remark 3.2. One can obtain similar results for the solutions of the inequalities (5) and (6).

Consider the following Darboux problem of partial differential inclusions

$$
\begin{cases}
{}^cD_\theta^r u(x,y) \in F(x,y,u(x,y)), & \text{if } (x,y) \in J,\\
u(x,0) = \varphi(x), \ x \in [0,a], \\
u(0,y) = \psi(y), \ y \in [0,b], \\
\varphi(0) = \psi(0),
\end{cases} \tag{8}
$$

where $\varphi : [0,a] \to E$, $\psi : [0,b] \to E$ are given absolutely continuous functions.

In the sequel, we present a result for the problem (8) with a nonconvex valued right hand side. Our considerations are based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler [13].

Theorem 3.3. *Assume that the following hypotheses hold*

(H_1) *There exists* $l \in L^\infty(J,[0,\infty))$ *such that*

$$H_d(F(x,y,u),F(x,y,\overline{u})) \le l(x,y)\|u-\overline{u}\|_E \ \ \textit{for every } u,\overline{u} \in E,$$

and

$$d(0,F(x,y,0)) \le l(x,y), \ \textit{a.e. } (x,y) \in J,$$

(H_2) $F : J \times E \longrightarrow \mathcal{P}_{cp}(E)$ *is such that* $F(.,.,u) : J \to \mathcal{P}_{cp}(E)$ *is measurable for each* $u \in E$.

If

$$\frac{l^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{9}$$

where $l^* = \|l\|_{L^\infty}$*, then problem* (8) *has at least one solution on* J.

Remark 3.4. For each $u \in \mathcal{C}$, the set $S_{F,u}$ is nonempty since by (H_2), F has a measurable selection (see [12], Theorem III.6).

Proof. Transform the problem (8) into a fixed point problem. Consider the multivalued operator $N : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ defined by

$$N(u) = \{h \in \mathcal{C} : h(x,y) = \mu(x,y) + I_\theta^r f(x,y);\ (x,y) \in J\},$$

where $f \in S_{F,u}$. Clearly, from Lemma 2.4, the fixed points of N are solutions to (8).

We shall show that N satisfies the assumptions of Lemma 2.13. The proof will be given in two steps.

Step 1. $N(u) \in \mathcal{P}_{cl}(\mathcal{C})$ for each $u \in \mathcal{C}$.

Indeed, let $(u_n)_{n\ge 0} \in N(u)$ such that $u_n \longrightarrow \tilde{u}$ in $\mathcal{C}$. Then, $\tilde{u} \in \mathcal{C}$ and there exists $f_n(.,.) \in S_{F,u}$ such that, for each $(x,y) \in J$,

$$u_n(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f_n(s,t)dtds.$$

Using (H_1) and the fact that F has compact values, we may pass to a subsequence if necessary to get that $f_n(.,.)$ converges to f in $L^1(J,E)$, and hence $f \in S_{F,u}$. Then, for each $(x,y) \in J$,

$$\begin{aligned} u_n(x,y) &\longrightarrow \tilde{u}(x,y) \\ &= \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)dtds. \end{aligned}$$

So, $\tilde{u} \in N(u)$.

Step 2. There exists $\gamma < 1$ such that

$$H_d(N(u), N(\overline{u})) \leq \gamma \|u - \overline{u}\|_\infty \text{ for each } u, \overline{u} \in \mathcal{C}.$$

Let $u, \overline{u} \in \mathcal{C}$ and $h \in N(u)$. Then, there exists $f(x,y) \in F(x,y,u(x,y))$ such that for each $(x,y) \in J$

$$h(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)dtds.$$

From (H_1) it follows that

$$H_d(F(x,y,u(x,y)), F(x,y,\overline{u}(x,y))) \leq l(x,y)\|u(x,y) - \overline{u}(x,y)\|_E.$$

Hence, there exists $w \in F(x,y,\overline{u}(x,y))$ such that

$$\|f(x,y) - w(x,y)\|_E \leq l(x,y)\|u(x,y) - \overline{u}(x,y)\|_E; \ (x,y) \in J.$$

Consider $U : J \to \mathcal{P}(E)$ given by

$$U(x,y) = \{w(x,y) \in E : \|f(x,y) - w(x,y)\|_E \leq l(x,y)\|u(x,y) - \overline{u}(x,y)\|_E\}.$$

Since the multivalued operator $u(x,y) = U(x,y) \cap F(x,y,\overline{u}(x,y))$ is measurable (see Proposition III.4 in [12]), there exists a function $\overline{f}(x,y)$ which is a measurable selection for $\overline{u}$. So, $\overline{f}(x,y) \in F(x,y,\overline{u}(x,y))$, and for each $(x,y) \in J$,

$$\|f(x,y) - \overline{f}(x,y)\|_E \leq l(x,y)\|u(x,y) - \overline{u}(x,y)\|_E.$$

Let us define for each $(x,y) \in J$

$$\overline{h}(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \overline{f}(s,t)dtds.$$

Then for each $(x,y)\in J$, we have

$$\begin{aligned}\|h(x,y)-\overline{h}(x,y)\|_E &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ &\quad \times \|f(s,t)-\overline{f}(s,t)\|_E dtds \\ &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ &\quad \times l(s,t)\|u(s,t)-\overline{u}(s,t)\|_E dtds \\ &\le \frac{l^*\|u-\overline{u}\|_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds.\end{aligned}$$

Thus, for each $(x,y)\in J$ we obtain

$$\|h-\overline{h}\|_\infty \le \frac{l^* a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|u-\overline{u}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of u and $\overline{u}$, it follows that

$$H_d(N(u),N(\overline{u})) \le \frac{l^* a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|u-\overline{u}\|_\infty.$$

So by (9), N is a contraction and thus, by Lemma 2.13, N has a fixed point u, which is a solution to problem (8). ■

Theorem 3.5. *Assume that the assumptions* $(H_1),(H_2)$ *and the following hypothesis hold*

(H_3) $\Phi\in L^1(J,[0,\infty))$ *and there exists* $\lambda_\Phi>0$ *such that, for each* $(x,y)\in J$ *we have*

$$(I_\theta^r\Phi)(x,y)\le \lambda_\Phi\Phi(x,y).$$

If the condition (9) *holds, then* (1) *is generalized Ulam–Hyers–Rassias stable.*

Proof. Let $u\in\mathcal{C}$ be a solution of the inequality (5). By Theorem 3.3, there exists v which is a solution of the problem (8). Hence, for each $(x,y)\in J$, we have

$$v(x,y)=\mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f_v(s,t)dtds,$$

where $f_v\in S_{F,v}$. By differential inequality (5), for each $(x,y)\in J$, we have

$$\|u(x,y)-\mu(x,y)-I_\theta^r f_u(x,y)\|_E\le I_\theta^r\Phi(x,y),$$

where $f_u\in S_{F,u}$. Thus, by (H_3) for each $(x,y)\in J$, we get

$$\|u(x,y)-\mu(x,y)-I_\theta^r f_u(x,y)\|_E\le \lambda_\Phi\Phi(x,y).$$

Hence for each $(x,y) \in J$, it follows that

$$\begin{aligned}\|u(x,y)-v(x,y)\|_E &\leq \|u(x,y)-\mu(x,y)-I_\theta^r f_u(x,y)\|_E \\ &\quad + \|I_\theta^r[f_u(x,y)-f_v(x,y)]\|_E \\ &\leq \lambda_\Phi \Phi(x,y) \\ &\quad + \frac{l^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ &\quad \times \|u(s,t)-v(s,t)\|_E dtds.\end{aligned}$$

From Lemma 2.14, there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\begin{aligned}\|u(x,y)-v(x,y)\|_E &\leq \lambda_\Phi \Phi(x,y) \\ &\quad + \frac{\delta l^* \lambda_\Phi}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\Phi(s,t)dtds \\ &\leq (1+\delta l^* \lambda_\Phi)\lambda_\Phi \Phi(x,y) \\ &:= c_{F,\Phi}\Phi(x,y).\end{aligned}$$

Finally, the inclusion (1) is generalized Ulam–Hyers–Rassias stable. ∎

4. An example

Let

$$E = l^1 = \left\{ w = (w_1, w_2, \ldots, w_n, \ldots) : \sum_{n=1}^{\infty} |w_n| < \infty \right\}$$

be the Banach space with norm

$$\|w\|_E = \sum_{n=1}^{\infty} |w_n|.$$

Consider the following infinite system of partial hyperbolic fractional differential inclusions of the form

$$(10) \qquad {}^cD_\theta^r u(x,y) \in F(x,y,u(x,y)); \text{ a.e. } (x,y) \in J = [0,1]\times[0,1],$$

where $(r_1, r_2) \in (0,1]\times(0,1]$,

$$u = (u_1, u_2, \ldots, u_n, \ldots), \quad {}^cD_\theta^r u = ({}^cD_\theta^r u_1, {}^cD_\theta^r u_2, \ldots, {}^cD_\theta^r u_n, \ldots),$$

$$\begin{aligned}&F(x,y,u(x,y)) \\ &\quad = \{v \in E : \|f_1(x,y,u(x,y))\|_E \leq \|v\|_E \leq \|f_2(x,y,u(x,y))\|_E\};\end{aligned}$$

$(x,y) \in [0,1]\times[0,1]$, and $f_1, f_2 : [0,1]\times[0,1]\times E \to E$,

$$f_k = (f_{k,1}, f_{k,2}, \ldots, f_{k,n}, \ldots); \ k \in \{1,2\}, \ n \in \mathbb{N},$$

$$f_{1,n}(x,y,u_n(x,y)) = \frac{xy^2 u_n}{e^{10+x+y}(1+\|u_n\|_E)}; \ n \in \mathbb{N},$$

and

$$f_{2,n}(x,y,u_n(x,y)) = \frac{xy^2u_n}{e^{10+x+y}};\ n \in \mathbb{N}.$$

We assume that F is compact valued. For each $(x,y) \in J$ and all $z_1, z_2 \in E$, we have

$$\|f_2(x,y,z_2) - f_1(x,y,z_1)\|_E \leq xy^2e^{-10-x-y}\|z_2 - z_1\|_E,$$

then the hypotheses (H_1) and (H_2) are satisfied with $l(x,y) = \frac{xy^2}{e^{10+x+y}}$. We shall show that condition (9) holds with $a = b = 1$. Indeed $l^* = \dfrac{1}{e^{10}}$, and

$$\frac{l^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{e^{10}\Gamma(1+r_1)\Gamma(1+r_2)} < \frac{4}{e^{10}} < 1,$$

which is satisfied for each $(r_1, r_2) \in (0,1] \times (0,1]$. The hypothesis (H_3) is satisfied with $\Phi(x,y) = xy^2$ and $\lambda_\Phi = 8$. Indeed, for each $(x,y) \in [0,1] \times [0,1]$, we get

$$(I_\theta^r\Phi)(x,y) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+r_1)\Gamma(3+r_2)}x^{1+r_1}y^{2+r_2} \leq 8xy^2 = \lambda_\Phi\Phi(x,y).$$

Consequently, Theorem 3.5 implies that the inclusion (10) is generalized Ulam–Hyers–Rassias stable.

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