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ON DERIVATIONS OF OPERATOR ALGEBRAS WITH INVOLUTION

Abstract. The purpose of this paper is to prove the following result. Let X be a complex Hilbert space, let $\mathcal{L}(X)$ be an algebra of all bounded linear operators on X and let $\mathcal{A}(X) \subset \mathcal{L}(X)$ be a standard operator algebra, which is closed under the adjoint operation. Suppose there exists a linear mapping $D : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the relation $2D(AA^*A) = D(AA^*)A + AA^*D(A) + D(A)A^*A + AD(A^*A)$ for all $A \in \mathcal{A}(X)$. In this case, D is of the form $D(A) = [A, B]$ for all $A \in \mathcal{A}(X)$ and some fixed $B \in \mathcal{L}(X)$, which means that D is a derivation.

Throughout, R will represent an associative ring with center $Z(R)$. As usual, we write $[x, y]$ for $xy - yx$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ on a ring R is called involution in case $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or *-ring. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. Let Q be a subring of a ring R. An additive mapping $D : Q \to R$ is called a derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in Q$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in Q$. A derivation $D : R \to R$ is inner in case there exists $a \in R$, such that $D(x) = [x, a]$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [9] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [2]. Cusack [7] generalized Herstein's result to 2-torsion free semiprime rings (see also [3] for an alternative proof). Let us point out that Beidar, Brešar, Chebotar and

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Martindale [1] have considerably generalized Herstein theorem (see also [5] and [10]). Let X be a real or complex Banach space and let $\mathcal{L}(X)$ and $\mathcal{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $\mathcal{L}(X)$, respectively. An algebra $\mathcal{A}(X) \subset \mathcal{L}(X)$ is said to be standard in case $\mathcal{F}(X) \subset \mathcal{A}(X)$. Let us point out that any standard operator algebra is prime, which is a consequence of Hahn–Banach theorem.

Motivated by the work of Brešar [4], Vukman [20] brought up the following conjecture.

Conjecture 1. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be an additive mapping satisfying the relation*

$$2D(xyx) = D(xy)x + xyD(x) + D(x)yx + xD(yx), \tag{1}$$

for all pairs $x, y \in R$. In this case D is a derivation.

The conjecture above is still an open problem. Putting x for y in the relation (1) we obtain

$$2D(x^3) = D(x^2)x + x^2D(x) + D(x)x^2 + xD(x^2). \tag{2}$$

In case of a *-ring we obtain, after putting x^* for y in the relation (1), the relation

$$2D(xx^*x) = D(xx^*)x + xx^*D(x) + D(x)x^*x + xD(x^*x). \tag{3}$$

The relation (2) has been considered in [20]. It is our aim in this paper to prove the result below, which is related to (3).

Theorem 2. *Let X be a complex Hilbert space and let $\mathcal{A}(X)$ be a standard operator algebra, which is closed under the adjoint operation. Suppose there exists a linear mapping $D : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the relation*

$$2D(AA^*A) = D(AA^*)A + AA^*D(A) + D(A)A^*A + AD(A^*A), \tag{4}$$

for all $A \in \mathcal{A}(X)$. In this case D is of the form $D(A) = [A, B]$ for all $A \in \mathcal{A}(X)$ and some fixed $B \in \mathcal{L}(X)$, which means that D is a derivation.

In the proof of Theorem 2, we will use the result below first proved by Chernoff [6] (see also [12, 15, 16, 18]).

Theorem 3. *Let X be a real or complex Banach space, let $\mathcal{A}(X)$ be a standard operator algebra and let $D : \mathcal{A}(X) \to \mathcal{L}(X)$ be a linear derivation. In this case D is of the form $D(A) = [A, B]$ for all $A \in \mathcal{A}(X)$ and some fixed $B \in \mathcal{L}(X)$.*

Proof of Theorem 2. Let us first consider the restriction of D on $\mathcal{F}(X)$. Let A be from $\mathcal{F}(X)$. Of course, in this case we also have $A^* \in \mathcal{F}(X)$. Let $P \in \mathcal{F}(X)$ be a self-adjoint projection with the property $AP = PA = A$. In this case, we also have $A^*P = PA^* = A^*$. Putting P for A in the relation (4), we obtain

$$D(P) = D(P)P + PD(P).$$

Because of the above relation, the substitution $A + P$ for A in the relation (4) gives

$$\begin{aligned}
&2D(A^2 + AA^* + A^*A) + 4D(A) + 2D(A^*)\\
&= D(A)(2A + A^*) + (2A + A^*)D(A) + D(A^*)A + AD(A^*)\\
&\quad + D(P)(2A + A^*) + (2A + A^*)D(P) + 2PD(A) + D(AA^*)P + AA^*D(P)\\
&\quad + D(P)A^*A + PD(A^*A) + 2D(A)P + D(A^*)P + PD(A^*).
\end{aligned}$$

Putting $-A$ for A in the above relation and comparing the relation so obtained with the above relation gives

$$\begin{aligned}
4D(A) + 2D(A^*) = {}& D(P)(2A + A^*) + (2A + A^*)D(P)\\
&+ 2PD(A) + 2D(A)P + D(A^*)P + PD(A^*)
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
&2D(A^2 + AA^* + A^*A)\\
&\qquad = D(A)(2A + A^*) + (2A + A^*)D(A) + D(A^*)A + AD(A^*)\\
&\qquad\quad + D(AA^*)P + AA^*D(P) + D(P)A^*A + PD(A^*A).
\end{aligned} \tag{6}$$

So far we have not used the assumption of the theorem that X is a complex Hilbert space. Since X is complex, one can write iA for A in the relations (5) and (6), which gives

$$\begin{aligned}
4D(A) - 2D(A^*) = {}& D(P)(2A - A^*) + (2A - A^*)D(P)\\
&+ 2PD(A) + 2D(A)P - D(A^*)P - PD(A^*)
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
&2D(A^2 - AA^* - A^*A)\\
&\qquad = D(A)(2A - A^*) + (2A - A^*)D(A) - D(A^*)A - AD(A^*)\\
&\qquad\quad - D(AA^*)P - AA^*D(P) - D(P)A^*A - PD(A^*A).
\end{aligned} \tag{8}$$

Comparing (5) with (7) and (6) with (8), we obtain

$$2D(A) = D(P)A + PD(A) + D(A)P + AD(P) \tag{9}$$

and

$$D(A^2) = D(A)A + AD(A). \tag{10}$$

Since the relation (9) holds for all $A \in \mathcal{F}(X)$, one can conclude that D maps $\mathcal{F}(X)$ into itself. We therefore have a linear mapping D, which maps $\mathcal{F}(X)$ into itself and satisfies the relation (10) for all $A \in \mathcal{F}(X)$. In other words, D

is a Jordan derivation on $\mathcal{F}(X)$ and since $\mathcal{F}(X)$ is prime, it follows, according to Herstein theorem, that D is a derivation on $\mathcal{F}(X)$. Applying Theorem 3, one can conclude that D is of the form

$$D(A) = [A, B], \tag{11}$$

for all $A \in \mathcal{F}(X)$ and some fixed $B \in \mathcal{L}(X)$. It remains to prove that (11) holds for all $A \in \mathcal{A}(X)$ as well. For this purpose we introduce $D_1 : \mathcal{A}(X) \to \mathcal{L}(X)$ by $D_1(A) = [A, B]$, where B is from the relation (11) and consider mapping $D_0 = D - D_1$. The mapping D_0 is, obviously, linear and satisfies the relation (4). Besides, D_0 vanishes on $\mathcal{F}(X)$. It is our aim to prove that D_0 vanishes on $\mathcal{A}(X)$, as well. Let A be from $\mathcal{A}(X)$, let P be a one-dimensional self-adjoint projection and let us introduce $S \in \mathcal{A}(X)$ by $S = A + PAP - (AP + PA)$. Since $S - A \in \mathcal{F}(X)$, we have $D_0(S) = D_0(A)$. Besides, $SP = PS = 0$. By the relation (4), we have

$$\begin{aligned}
D_0(SS^*)S + SS^*D_0(S) + D_0(S)S^*S + SD_0(S^*S) &= 2D_0(SS^*S)\\
&= 2D_0((S+P)(S+P)^*(S+P))\\
&= D_0((S+P)(S+P)^*)(S+P) + (S+P)(S+P)^*D_0(S)\\
&\quad + D_0(S)(S+P)^*(S+P) + (S+P)D_0((S+P)^*(S+P))\\
&= D_0(SS^*)S + D_0(SS^*)P + SS^*D_0(S) + PD_0(S)\\
&\quad + D_0(S)S^*S + D_0(S)P + SD_0(S^*S) + PD_0(S^*S).
\end{aligned}$$

We therefore have

$$D_0(SS^*)P + PD_0(S) + D_0(S)P + PD_0(S^*S) = 0.$$

Putting $-A$ for A in the above relation (in this case S becomes $-S$), comparing the relation so obtained with the above relation and considering that $D_0(S) = D_0(A)$, we obtain

$$PD_0(A) + D_0(A)P = 0. \tag{12}$$

Multiplying the above relation from both sides by P, we obtain

$$PD_0(A)P = 0.$$

Right multiplication of the relation (12) by P and applying the above relation leads to

$$D_0(A)P = 0.$$

Since P is an arbitrary one-dimensional self-adjoint projection, it follows from the above relation that $D_0(A) = 0$ for all $A \in \mathcal{A}(X)$, which completes the proof of the theorem. ∎

We proceed with the following conjecture.

Conjecture 4. *Let R be a semiprime *-ring with suitable torsion restrictions and let $D : R \to R$ be an additive mapping satisfying the relation*

$$2D(xx^*x) = D(xx^*)x + xx^*D(x) + D(x)x^*x + xD(x^*x),$$

for all $x \in R$. In this case D is a derivation.

Our next result is the following.

Theorem 5. *Let X be a complex Hilbert space, let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on X and let $\mathcal{A}(X) \subset \mathcal{L}(X)$ be a standard operator algebra, which is closed under the adjoint operation. Suppose there exists a linear operator $D : \mathcal{A}(X) \to \mathcal{L}(X)$, satisfying the relation*

$$D(AA^* + A^*A) = D(A)A^* + AD(A^*) + D(A^*)A + A^*D(A) \tag{13}$$

for all $A \subset \mathcal{A}(X)$. In this case D is of the form $D(A) = [A, B]$ for all $A \in \mathcal{A}(X)$ and some fixed $B \in \mathcal{L}(X)$, which means that D is a derivation.

In the proof of Theorem 5 we will use the result below proved by Vukman [19].

Theorem 6. *Let X be a real or complex Banach space and let $\mathcal{A}(X)$ be a standard operator algebra on X. Suppose there exists a linear mapping $D : \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the relation*

$$D(A^2) = D(A)A + AD(A)$$

for all $A \in \mathcal{A}(X)$. In this case D is of the form $D(A) = [A, B]$ for all $A \in \mathcal{A}(X)$ and some fixed $B \in \mathcal{L}(X)$.

Proof of Theorem 5. Putting $A + A^*$ for A in the relation (13) gives

$$\begin{aligned}&2D(A^2) + 2D((A^*)^2) + 2D(AA^* + A^*A) = 2D(A)A^* + 2D(A)A\\ &+ 2D(A^*)A^* + 2D(A^*)A + 2AD(A^*) + 2AD(A) + 2A^*D(A^*) + 2A^*D(A).\end{aligned}$$

Using the assumption (13) of the theorem reduces the above relation to

$$D(A^2) + D((A^*)^2) = D(A)A + AD(A) + D(A^*)A^* + A^*D(A^*). \tag{14}$$

Putting $A + B$ for A in the the relation (14), we obtain

$$\begin{aligned}&D(AB + BA) + D(A^*B^* + B^*A^*) = D(A)B + AD(B)\\ &+ D(B)A + BD(A) + D(A^*)B^* + A^*D(B^*) + D(B^*)A^* + B^*D(A^*).\end{aligned} \tag{15}$$

So far we have not used the assumption of the theorem that X is a complex Hilbert space. Since X is complex, one can write iA for A in the above relation, which gives

$$\begin{aligned}&D(AB + BA) - D(A^*B^* + B^*A^*) = D(A)B + AD(B)\\ &+ D(B)A + BD(A) - D(A^*)B^* - A^*D(B^*) - D(B^*)A^* - B^*D(A^*).\end{aligned} \tag{16}$$

Comparing the relation (15) with (16), we obtain

$$D(AB + BA) = D(A)B + AD(B) + D(B)A + BD(A). \tag{17}$$

Putting B for A in the above relation, we obtain

$$D(A^2) = D(A)A + AD(A). \tag{18}$$

We have proved that D is a Jordan derivation on $\mathcal{A}(X)$. Now Theorem 6 completes the proof. ∎

It should be mentioned that in the proofs of Theorem 2 and Theorem 5, we used some ideas similar to those used by Molnár in [13] and Vukman in [17]. Let us point out that in Theorem 2 and in Theorem 5, we obtain as a result the continuity of the mappings under purely algebraic assumptions concerning these mappings, which means that these results might be of some interest from the automatic continuity point of view. For results concerning automatic continuity, we refer the reader to [8] and [14].

We conclude with the following conjecture.

Conjecture 7. *Let R be a semiprime *-ring with suitable torsion restrictions and let $D : R \to R$ be an additive mapping satisfying the relation*

$$D(xx^* + x^*x) = D(x)x^* + xD(x^*) + D(x^*)x + x^*D(x),$$

for all $x \in R$. In this case D is a derivation.

Vukman and Kosi-Ulbl [11] considered an additive mapping $D : R \to R$, where R is a *-ring, satisfying the relation

$$D(x^*x) = D(x^*)x + x^*D(x)$$

for all $x \in R$. They proved that in case R is either a 2-torsion free semiprime ring with $Z(R) = \{0\}$ or a 2-torsion free prime ring, D is a derivation.

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