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# APPROXIMATION PROPERTIES OF CERTAIN SUMMATION INTEGRAL TYPE OPERATORS

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**Abstract.** In the present paper, we study approximation properties of a family of linear positive operators and establish direct results, asymptotic formula, rate of convergence, weighted approximation theorem, inverse theorem and better approximation for this family of linear positive operators.

## 1. Introduction

In 2005, Finta [4] studied a new type of Baskakov–Durrmeyer operators as follows

$$B_n(f,x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t)dt + p_{n,0}(x) f(0), \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},$$

$$b_{n,k}(t) = \frac{1}{B(n+1,k)} \frac{t^{k-1}}{(1+t)^{n+k+1}}.$$

Gupta *et al.* [12] studied pointwise convergence, an asymptotic formula and error estimation for the operators $B_n(f,\cdot)$. Recently, Govil and Gupta [7] studied some approximation properties for the operators $B_n(f,\cdot)$ and estimated local results in terms of modulus of continuity.

Generalization of the operators (1.1) with parameter $\gamma$ was discussed by Gupta [9], defined as

$$B_{n,\gamma}(f,x) = \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^{\infty} b_{n,k,\gamma}(t) f(t)dt + p_{n,0,\gamma}(x) f(0), \tag{1.2}$$

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where

$$p_{n,k,\gamma}(x) = \frac{\Gamma(n/\gamma + k)}{\Gamma(k+1)\Gamma(n/\gamma)} \cdot \frac{(\gamma x)^k}{(1+\gamma x)^{(n/\gamma)+k}},$$

$$b_{n,k,\gamma}(t) = \frac{\gamma\Gamma(n/\gamma + k + 1)}{\Gamma(k)\Gamma(n/\gamma + 1)} \cdot \frac{(\gamma t)^{k-1}}{(1+\gamma t)^{(n/\gamma)+k+1}}.$$

The operators $B_{n,\gamma}(f, \cdot)$ are called a certain integral modification of the well known Baskakov operators with the weight function of Beta basis function. As a special case i.e. $\gamma = 1$, the operators $B_{n,\gamma}(f, \cdot)$ reduce to the operators (1.1).

Recently, in [13, 18] authors introduced generalization of Bernstein polynomials with two parameters $\alpha$ and $\beta$ with $0 \leq \alpha \leq \beta$ and investigated convergence and approximation properties of these operators. Many other researchers work in this direction and obtain different approximation properties of many operators [19, 20, 15, 16, 11, 1]. Motivated by such type operators, we introduced Stancu type generalization of integral modification of the well known Baskakov operators with the weight function of Beta basis function as:

For $x \in [0, \infty)$, $\gamma > 0$, $0 \leq \alpha \leq \beta$,

$$B_{n,\gamma}^{\alpha,\beta}(f(t), x) = \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^\infty b_{n,k,\gamma}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt + p_{n,0,\gamma}(x) f\left(\frac{\alpha}{n+\beta}\right), \tag{1.3}$$

where $p_{n,k,\gamma}(x)$ and $b_{n,k,\gamma}(t)$ as defined in (1.2).

This present note deals with operators (1.3). We estimate moments and recurrence relation for the moments for the operators $B_{n,\gamma}^{(\alpha,\beta)}$. Also, we study asymptotic formula, local approximation theorems, rate of convergence, weighted approximation theorem, inverse theorem and better approximation for these operators.

## 2. Estimation of moments

In this section, we shall obtain $B_{n,\gamma}^{\alpha,\beta}(t^i, x)$, $i = 0, 1, 2, \ldots$.

**Lemma 2.1.** [9] *Let the function $T_{n,m,\gamma}(x)$, $m \in \mathbb{N} \cup \{0\}$ be defined as*

$$\begin{aligned} T_{n,m,\gamma}(x) &= B_{n,\gamma}((t-x)^m, x) \\ &= \sum_{k=1}^{\infty} p_{n,k,\gamma}(x) \int_0^\infty b_{n,k,\gamma}(t)(t-x)^m dt + (1+\gamma x)^{-n/\gamma}(-x)^m. \end{aligned} \tag{2.1}$$

*Then* $T_{n,0,\gamma}(x) = 1$, $T_{n,1,\gamma}(x) = 0$ *and* $T_{n,2,\gamma}(x) = \dfrac{2x(1+\gamma x)}{n-\gamma}$, *and also the following recurrence relation holds:*

$$\begin{aligned}&(n-\gamma m)T_{n,m+1,\gamma}(x)\\ &\quad = x(1+\gamma x)[T^{(1)}_{n,m,\gamma}(x) + 2mT_{n,m-1,\gamma}(x)] + m(1+2\gamma x)T_{n,m,\gamma}(x).\end{aligned}$$

**Lemma 2.2.** *The following equalities hold:*

(1) $B^{\alpha,\beta}_{n,\gamma}(1,x) = 1$,

(2) $B^{\alpha,\beta}_{n,\gamma}(t,x) = \dfrac{nx+\alpha}{n+\beta}$,

(3) $B^{\alpha,\beta}_{n,\gamma}(t^2,x) = \dfrac{n^2(n+\gamma)}{(n+\beta)^2(n-\gamma)}x^2 + \dfrac{2n\,(n+\alpha(n-\gamma))}{(n+\beta)^2(n-\gamma)}x + \dfrac{\alpha^2}{(n+\beta)^2}$ *for* $n > \gamma$.

**Proof.** The operators $B^{\alpha,\beta}_{n,\gamma}$ are well defined on the function $1, t, t^2$, we obtain

$$B^{\alpha,\beta}_{n,\gamma}(1,x) = 1,$$

$$B^{\alpha,\beta}_{n,\gamma}(t,x) = \frac{n}{n+\beta}B_{n,\gamma}(t,x) + \frac{\alpha}{n+\beta}B_{n,\gamma}(1,x) = \frac{nx+\alpha}{n+\beta}.$$

For $n > \gamma$, we have

$$\begin{aligned}B^{\alpha,\beta}_{n,\gamma}(t^2,x) &= \left(\frac{n}{n+\beta}\right)^2 B_{n,\gamma}(t^2,x) + \frac{2n\alpha}{(n+\beta)^2}B_{n,\gamma}(t,x) + \frac{\alpha^2}{(n+\beta)^2}B_{n,\gamma}(1,x)\\ &= \left(\frac{n}{n+\beta}\right)^2\left(\frac{2x}{n-\gamma} + \frac{x^2(n+\gamma)}{(n-\gamma)}\right) + \frac{2n\alpha x}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}\\ &= \frac{n^3x^2 - \alpha^2\gamma + n\alpha(\alpha - 2x\gamma) + n^2x(2+2\alpha+x\gamma)}{(n+\beta)^2(n-\gamma)}\\ &= \frac{n^2(n+\gamma)}{(n+\beta)^2(n-\gamma)}x^2 + \frac{2n\,(n+\alpha(n-\gamma))}{(n+\beta)^2(n-\gamma)}x + \frac{\alpha^2}{(n+\beta)^2}. \ \blacksquare\end{aligned}$$

**Lemma 2.3.** *If we define the central moments as* $\mu^{\alpha,\beta}_{n,m,\gamma}(x) = B^{\alpha,\beta}_{n,\gamma}((t-x)^m,x)$, $m \in \mathbb{N}$. *Then*

$$\mu^{\alpha,\beta}_{n,1,\gamma}(x) = B^{\alpha,\beta}_{n,\gamma}(t-x,x) = \frac{\alpha-\beta x}{n+\beta},$$

$$\begin{aligned}\mu^{\alpha,\beta}_{n,2,\gamma}(x) &= B^{\alpha,\beta}_{n,\gamma}((t-x)^2,x) = \frac{\alpha^2}{(n+\beta)^2} + x\frac{\left(2n^2 - 2n\alpha\beta + 2\alpha\beta\gamma\right)}{(n+\beta)^2(n-\gamma)}\\ &\quad + x^2\frac{\left(n\beta^2 + 2n^2\gamma - \beta^2\gamma\right)}{(n+\beta)^2(n-\gamma)}.\end{aligned}$$

**Remark 2.4.** For all $m \in \mathbb{N}$, $0 \le \alpha \le \beta$; we have the following recursive relation for the images of the monomials $t^m$ under $B^{\alpha,\beta}_{n,\gamma}(t^m,x)$ in terms of

$B_{n,\gamma}(t^j,x)$; $j=0,1,2,\dots,m$ as

$$B_{n,\gamma}^{\alpha,\beta}(t^m,x)=\sum_{j=0}^{m}\binom{m}{j}\frac{n^j\alpha^{m-j}}{(n+\beta)^m}B_{n,\gamma}(t^j,x).$$

Also,

$$B_{n,\gamma}^{\alpha,\beta}((t-x)^m,x)=\sum_{k=0}^{m}\binom{m}{k}(-x)^{m-k}B_{n,\gamma}^{\alpha,\beta}(t^k,x).$$

It is easily verified that for each $x\in(0,\infty)$

$$\begin{aligned}B_{n,\gamma}^{\alpha,\beta}(t^m,x)=&\frac{n^m\Gamma(n/\gamma+m)\Gamma(n/\gamma-m+1)}{(n+\beta)^m\Gamma(n/\gamma+1)\Gamma(n/\gamma)}x^m\\&+\frac{mn^{m-1}\Gamma(n/\gamma+m-1)\Gamma(n/\gamma-m+1)}{(n+\beta)^m\Gamma(n/\gamma+1)\Gamma(n/\gamma)}\\&\times\left[n(m-1)+\alpha(n/\gamma-m+1)\right]x^{m-1}\\&+\frac{\alpha m(m-1)n^{m-2}\Gamma(n/\gamma+m-2)\Gamma(n/\gamma-m+2)}{(n+\beta)^m\Gamma(n/\gamma+1)\Gamma(n/\gamma)}\\&\times\left[n(m-2)+\frac{\alpha(n/\gamma-m+2)}{2}\right]x^{m-2}+O(n^{-2}).\end{aligned}$$

## 3. Direct result and asymptotic formula

Let the space $C_B[0,\infty)$ of all continuous and bounded functions be endowed with the norm $\|f\|=\sup\{|f(x)|:x\in[0,\infty)\}$. Further, let us consider the following $K$-functional:

$$K_2(f,\delta)=\inf_{g\in W^2}\{\|f-g\|+\delta\|g''\|\},\tag{3.1}$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$. By the method given in [2, p. 177, Theorem 2.4], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\le C\omega_2(f,\sqrt{\delta}),\tag{3.2}$$

where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h<\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|\tag{3.3}$$

is the second order modulus of smoothness of $f\in C_B[0,\infty)$. Also, we set

$$\omega(f,\sqrt{\delta})=\sup_{0<h<\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+h)-f(x)|.\tag{3.4}$$

**Theorem 3.1.** *For $f \in C_B[0,\infty)$, we have*

$$|B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)| \leq \omega_1\left(f, \frac{|\alpha-\beta x|}{n+\beta}\right)+C\omega_2\left(f, \sqrt{\mu_{n,2,\gamma}^{\alpha,\beta}(x)+\left(\frac{\alpha-\beta x}{n+\beta}\right)^2}\right),$$

*where $C$ is positive constant.*

**Proof.** We are introducing the auxiliary operators as follows

$$\hat{B}_{n,\gamma}^{\alpha,\beta}(f,x) = B_{n,\gamma}^{\alpha,\beta}(f,x) - f\left(\frac{nx+\alpha}{n+\beta}\right) + f(x),$$

for every $x \in [0,\infty)$. The operators $\hat{B}_{n,\gamma}^{\alpha,\beta}$ are linear and preserves the linear functions:

$$\hat{B}_{n,\gamma}^{\alpha,\beta}(t-x,x) = 0. \tag{3.5}$$

Let $g \in W_\infty^2$ and $x,t \in [0,\infty)$. By Taylor's expansion, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Applying $\hat{B}_{n,\gamma}^{\alpha,\beta}$, we get

$$\hat{B}_{n,\gamma}^{\alpha,\beta}(g,x) - g(x) = g'(x)\hat{B}_{n,\gamma}^{\alpha,\beta}(t-x,x) + \hat{B}_{n,\gamma}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du, x\right),$$

and applying (3.5), we get

$$\hat{B}_{n,\gamma}^{\alpha,\beta}(g,x) - g(x) = \hat{B}_{n,\gamma}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du, x\right).$$

Hence,

$$\begin{aligned}
|\hat{B}_{n,\gamma}^{\alpha,\beta}(g,x) - g(x)| &\leq \left|B_{n,\gamma}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du, x\right)\right| \\
&\quad + \left|\int_x^{\frac{nx+\alpha}{n+\beta}} \left(\frac{nx+\alpha}{n+\beta} - u\right) g''(u)du\right| \\
&\leq B_{n,\gamma}^{\alpha,\beta}((t-x)^2,x)\|g''\| + \int_x^{\frac{nx+\alpha}{n+\beta}} \left|\left(\frac{nx+\alpha}{n+\beta} - u\right)g''(u)\right| du \\
&\leq \left[\mu_{n,2,\gamma}^{\alpha,\beta}(x) + \left(\frac{\alpha-\beta x}{n+\beta}\right)^2\right]\|g''\|.
\end{aligned}$$

Since

$$\begin{aligned}
&|B_{n,\gamma}^{\alpha,\beta}(f,x)| \\
&\quad \leq \sum_{k=1}^{\infty} p_{n,k,\gamma}(x)\int_0^\infty b_{n,k,\gamma}(t)\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right|dt + p_{n,0,\gamma}\left|f\left(\frac{\alpha}{n+\beta}\right)\right| \leq \|f\|,
\end{aligned}$$

$$
\begin{aligned}
&|B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)| \\
&\le |\hat{B}_{n,\gamma}^{\alpha,\beta}(f-g,x)-(f-g)(x)| + |\hat{B}_{n,\gamma}^{\alpha,\beta}(g,x)-g(x)| + \left| \left(\frac{nx+\alpha}{n+\beta}\right) - f(x) \right| \\
&\le 4\|f-g\| + \left[\mu_{n,2,\gamma}^{\alpha,\beta}(x) + \left(\frac{\alpha-\beta x}{n+\beta}\right)^2\right]\|g''\| + \omega_1\left(f, \frac{|\alpha-\beta x|}{n+\beta}\right).
\end{aligned}
$$

Hence, taking infimum on the right hand side over all $g \in W^2$, we get

$$
|B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)| \le K\left(f, \mu_{n,2,\gamma}^{\alpha,\beta}(x) + \left(\frac{\alpha-\beta x}{n+\beta}\right)^2\right) + \omega_1\left(f, \frac{|\alpha-\beta x|}{n+\beta}\right).
$$

In view of (3.2), we get

$$
|B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)| \le C\omega_2\left(f, \sqrt{\mu_{n,2,\gamma}^{\alpha,\beta}(x) + \left(\frac{\alpha-\beta x}{n+\beta}\right)^2}\right) + \omega_1\left(f, \frac{|\alpha-\beta x|}{n+\beta}\right).
$$

This completes the proof of the theorem. ∎

Our next result in this section is the Voronovskaja type asymptotic formula: Let $B_{x^2}[0,\infty) = \{f :\ \text{for every } x \in [0,\infty), |f(x)| \le M_f(1+x^2)\}$, $M_f$ being a constant depending on $f$. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous functions to $B_{x^2}[0,\infty)$. Also, $C^*_{x^2}[0,\infty)$ is a subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim\limits_{x\to\infty} \dfrac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $\|f\|_{x^2} = \sup\limits_{x\in[0,\infty)} \dfrac{|f(x)|}{1+x^2}$.

**Theorem 3.2.** *For any function $f \in C_{x^2}[0,\infty)$ such that $f', f'' \in C_{x^2}[0,\infty)$, we have*

$$
\lim_{n\to\infty} n[B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)] = (\alpha-\beta x)f'(x) + x(1+\gamma x)f''(x), \quad \textit{for every } x \ge 0.
$$

**Proof.** Let $f, f', f'' \in C^*_{x^2}[0,\infty)$ and $x \in [0,\infty)$ be fixed. By Taylor expansion, we can write

$$
f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + r(t,x)(t-x)^2,
$$

where $r(t,x)$ is the Peano form of the remainder, $r(t,x) \in C_B[0,\infty)$ and $\lim\limits_{t\to x} r(t,x) = 0$. Applying $B_{n,\gamma}^{\alpha,\beta}$, we get

$$
\begin{aligned}
&n[B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)] \\
&= f'(x)nB_{n,\gamma}^{\alpha,\beta}(t-x,x) + \frac{f''(x)}{2!}nB_{n,\gamma}^{\alpha,\beta}((t-x)^2,x) + nB_{n,\gamma}^{\alpha,\beta}(r(t,x)(t-x)^2,x),
\end{aligned}
$$

$$\begin{aligned}
&\lim_{n\to\infty} n[B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)]\\
&\quad= f'(x)\lim_{n\to\infty} nB_{n,\gamma}^{\alpha,\beta}((t-x),x)+\frac{f''(x)}{2}\lim_{n\to\infty} nB_{n,\gamma}^{\alpha,\beta}((t-x)^2,x)\\
&\qquad+\lim_{n\to\infty} nB_{n,\gamma}^{\alpha,\beta}(r(t,x)(t-x)^2,x)\\
&\quad=(\alpha-\beta x)f'(x)+x(1+\gamma x)f''(x)+\lim_{n\to\infty} nB_{n,\gamma}^{\alpha,\beta}(r(t,x)(t-x)^2,x).
\end{aligned}$$

By Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\lim_{n\to\infty} & nB_{n,\gamma}^{\alpha,\beta}(r(t,x)(t-x)^2,x)\\
&=\lim_{n\to\infty} nB_{n,\gamma}^{\alpha,\beta}(r^2(t,x),x)^{1/2}B_{n,\gamma}^{\alpha,\beta}((t-x)^4,x)^{1/2}.
\end{aligned}\tag{3.6}$$

Observe that $r^2(x,x)=0$ and $r^2(\cdot,x)\in C^*_{x^2}[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty} nB_{n,\gamma}^{\alpha,\beta}(r^2(t,x),x)=r^2(x,x)=0,\tag{3.7}$$

uniformly with respect to $x\in[0,A]$. Now from (3.6) and (3.7), we obtain

$$\lim_{n\to\infty} nB_{n,\gamma}^{\alpha,\beta}(r(t,x)(t-x)^2,x)=0.$$

Thus, we have

$$\lim_{n\to\infty} n[B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)]=(\alpha-\beta x)f'(x)+x(1+\gamma x)f''(x),$$

which completes the proof. ∎

## 4. Rate of convergence

For any positive $a$, by

$$\omega_a(f,\delta)=\sup_{|t-x|\le\delta}\ \sup_{x,t\in[0,a]}|f(t)-f(x)|,$$

we denote the usual modulus of continuity of $f$ on the closed interval $[0,a]$. We know that for a function $f\in C_{x^2}[0,\infty)$, the modulus of the continuity $\omega_a(f,\delta)$ tends to zero.

Now, we give a rate of convergence theorem for the operators $B_{n,\gamma}^{\alpha,\beta}$.

**Theorem 4.1.** *Let $f\in C_{x^2}[0,\infty)$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1]\subset[0,\infty)$, where $a>0$. Then for every $n>3$,*

$$\|B_{n,\gamma}^{\alpha,\beta}(f)-f\|_{C[0,a]}\le 6M_f(1+a^2)\mu_{n,2,\gamma}^{\alpha,\beta}(x)+\omega_{a+1}\big(f,\sqrt{\mu_{n,2,\gamma}^{\alpha,\beta}(x)}\big).$$

**Proof.** For $x\in[0,a]$ and $t>a+1$, since $t-x>1$, we have

$$\begin{aligned}
|f(t)-f(x)|&\le M_f(2+x^2+t^2)\\
&\le M_f\left(2+3x^2+2(t-x)^2\right)\\
&\le 6M_f(1+a^2)(t-x)^2.
\end{aligned}\tag{4.1}$$

For $x \in [0,a]$ and $t \leq a+1$, we have

$$|f(x)-f(t)| \leq \omega_{a+1}(f,|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta) \tag{4.2}$$

with $\delta > 0$.
From (4.1) and (4.2), we can write

$$|f(t)-f(x)| \leq 6M_f(1+a^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \tag{4.3}$$

for $x \in [0,a]$ and $t \geq 0$. Thus

$$\begin{aligned}|B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)| &\leq B_{n,\gamma}^{\alpha,\beta}\left(|f(t)-f(x)|,x\right)\\ &\leq 6M_f(1+a^2)B_{n,\gamma}^{\alpha,\beta}((t-x)^2,x)\\ &\quad +\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}B_{n,\gamma}^{\alpha,\beta}((t-x)^2,x)^{1/2}\right).\end{aligned}$$

Hence, by Schwarz's inequality and Lemma 2.3, for $x \in [0,a]$

$$\begin{aligned}&|B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)|\\ &\qquad \leq 6M_f(1+a^2)\mu_{n,2,\gamma}^{\alpha,\beta}(x)+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\mu_{n,2,\gamma}^{\alpha,\beta}(x)}\right).\end{aligned}$$

Taking $\delta = \sqrt{\mu_{n,2,\gamma}^{\alpha,\beta}(x)}$, we get

$$\|B_{n,\gamma}^{\alpha,\beta}(f)-f\|_{C[0,a]} \leq 6M_f(1+a^2)\mu_{n,2,\gamma}^{\alpha,\beta}(x)+\omega_{a+1}\left(f,\sqrt{\mu_{n,2,\gamma}^{\alpha,\beta}(x)}\right).$$

This completes the proof of theorem. ∎

**Corollary 4.2.** *If $f \in Lip_{M_1}\rho$ on $[0,a+1]$, then for $n > \gamma$*

$$\|B_{n,\gamma}^{\alpha,\beta}(f)-f\|_{C[0,a]} \leq (1+2M_1)\sqrt{\mu_{n,2,\gamma}^{\alpha,\beta}(x)}.$$

**Proof.** For sufficiently large $n$,

$$\mu_{n,2,\gamma}^{\alpha,\beta}(x) \leq \sqrt{\mu_{n,2,\gamma}^{\alpha,\beta}(x)}.$$

Hence, by $f \in Lip_{M_1}\rho$, we obtain the assertion of the corollary. ∎

## 5. Weighted approximation

Now we shall discuss the weighted approximation theorem, where the approximation formula holds true on the interval $[0,\infty)$.

**Theorem 5.1.** *For each $f \in C^*_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty} \|B^{\alpha,\beta}_{n,\gamma}(f) - f\|_{x^2} = 0.$$

**Proof.** Using the theorem in [5], we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \|B^{\alpha,\beta}_{n,\gamma}(t^r,x) - x^r\|_{x^2} = 0, \quad r = 0,1,2. \tag{5.1}$$

Since $B^{\alpha,\beta}_{n,\gamma}(1,x) = 1$, the first condition of (5.1) is fulfilled for $r = 0$.

By Lemma 2.2, we have

$$\begin{aligned}
\|B^{\alpha,\beta}_{n,\gamma}(t,x) - x\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{|B^{\alpha,\beta}_{n,\gamma}(t,x) - x|}{1+x^2} \\
&\le \left|\frac{nx}{n+\beta} - x\right| \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{\alpha}{n+\beta} \le \frac{\beta x+\alpha}{n+\beta}
\end{aligned}$$

and the second condition of (5.1) holds for $r = 1$ as $n \to \infty$.

Similarly, we can write for $n > \gamma$

$$\begin{aligned}
\|B^{\alpha,\beta}_{n,\gamma}(t^2,x) - x^2\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{|B^{\alpha,\beta}_{n,\gamma}(t^2,x) - x^2|}{1+x^2} \\
&\le \left|\frac{n^2(n+\gamma)}{(n+\beta)^2(n-\gamma)} - 1\right| \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\
&\quad + \left|\frac{2n\,(n+\alpha(n-\gamma))}{(n+\beta)^2(n-\gamma)}\right| \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{\alpha^2}{(n+\beta)^2} \\
&\le \left|\frac{-2n^2\beta - n\beta^2 + 2n^2\gamma + 2n\beta\gamma + \beta^2\gamma}{(n+\beta)^2(n-\gamma)}\right| \\
&\quad + \left|\frac{2n\,(n+\alpha(n-\gamma))}{(n+\beta)^2(n-\gamma)}\right| + \frac{\alpha^2}{(n+\beta)^2},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|B^{\alpha,\beta}_{n,\gamma}(t^2,x) - x^2\|_{x^2} = 0.$$

Thus the proof is completed. ∎

We give the following to approximate all functions in $C_{x^2}[0,\infty)$. This type of results are given in [6] for locally integrable functions.

**Theorem 5.2.** *For each $f \in C_{x^2}[0,\infty)$ and $\nu > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|B^{\alpha,\beta}_{n,\gamma}(f,x) - f(x)|}{(1+x^2)^{\nu+1}} = 0.$$

**Proof.** For any fixed $x_0 > 0$,

$$\sup_{x\in[0,\infty)} \frac{|B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)|}{(1+x^2)^{\nu+1}} \leq \sup_{x\leq x_0} \frac{|B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)|}{(1+x^2)^{\nu+1}} + \sup_{x\geq x_0} \frac{|B_{n,\gamma}^{\alpha,\beta}(f,x)-f(x)|}{(1+x^2)^{\nu+1}}$$

$$\leq \|B_{n,\gamma}^{\alpha,\beta}(f)-f\|_{C[0,x_0]} + \|f\|_{x^2} \sup_{x\geq x_0} \frac{|B_{n,\gamma}^{\alpha,\beta}(1+t^2,x)|}{(1+x^2)^{\nu+1}} + \sup_{x\geq x_0} \frac{|f(x)|}{(1+x^2)^{1+\nu}}.$$

The first term of the above inequality tends to zero from Theorem 4.1. By Lemma 2.2, for any fixed $x_0 > 0$, it is easily seen that

$$\sup_{x\geq x_0} \frac{|B_{n,\gamma}^{\alpha,\beta}(1+t^2,x)|}{(1+x^2)^{\nu+1}}$$

tends to zero $n \to \infty$. We can choose $x_0 > 0$ so large that the last part of the above inequality can be made small enough.

Thus the proof is completed. ∎

Remark 5.3. The following theorem is inverse theorem in simultaneous approximation for the operators $B_{n,\gamma}^{\alpha,\beta}$.

Theorem. *Let* $0 < \zeta < 2$, $0 < a_1 < a_2 < b_2 < b_1 < \infty$ *and suppose* $f \in C_\gamma[0,\infty)$. *Then in the following statements* (i)⇒(ii)

(i) $\|(B_{n,\gamma}^{\alpha,\beta})^{(r)}(f,\cdot) - f^{(r)}\|_{C[a_1,b_1]} = O\left(n^{-\zeta/2}\right)$,
(ii) $f^{(r)} \in Lip^*(\alpha, a_2, b_2)$,

*where* $Lip^*(\alpha, a_2, b_2)$ *denotes the Zygmund class satisfying* $\omega_2(f,\delta,a_2,b_2) \leq M_2\delta^\zeta$.

The proof of the above Theorem follows along line of [12, Theorem 4.1]; thus, we omit the details.

## 6. Better approximation

Many well-known operators preserve the linear as well as constant function for example Bernstein, Baskakov, Szász–Mirakyan and Szász-Beta operators possess these properties i.e. $L_n(e_i,x) = e_i(x)$ where $e_i(x) = x^i$ $(i = 0,1)$. To make the convergence faster, King [14] proposed an approach to modify the classical Bernstein polynomial, so that the sequence preserve test function $e_0$ and $e_2$. Later this approach was applied to some well-known operators. For detail see [3, 17].

As the operators $B_{n,\gamma}^{\alpha,\beta}$ preserve only the constant function, further modification of these operators is proposed such that the modified operators preserves the constant as well as linear function, for this purpose the modification of $B_{n,\gamma}^{\alpha,\beta}$ is as follows:

$$B_{n,\gamma}^{*(\alpha,\beta)}(f,x) = \sum_{k=1}^{\infty} p_{n,k,\gamma}(r_n(x)) \int_0^{\infty} b_{n,k,\gamma}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt + p_{n,0,\gamma}(r_n(x)) f\left(\frac{\alpha}{n+\beta}\right), \tag{6.1}$$

where $r_n(x) = \dfrac{(n+\beta)x-\alpha}{n}$ and $x \in I_n = \left[\dfrac{\alpha}{n+\beta}, \infty\right)$.

**Lemma 6.1.** *For each $x \in I_n$, we have*

$$B_{n,\gamma}^{*(\alpha,\beta)}(1,x) = 1, \quad B_{n,\gamma}^{*(\alpha,\beta)}(t,x) = x,$$

$$B_{n,\gamma}^{*(\alpha,\beta)}(t^2,x) = \frac{x^2(n+\gamma)}{n-\gamma} + \frac{2x(n-2\alpha\gamma)}{(n+\beta)(n-\gamma)} + \frac{2\left(-n\alpha+\alpha^2\gamma\right)}{(n+\beta)^2(n-\gamma)}.$$

**Lemma 6.2.** *For each $x \in I_n$ and $n > \gamma$, we have*

$$B_{n,\gamma}^{*(\alpha,\beta)}(t-x,x) = 0,$$

$$\begin{aligned}\mu_{n,2,\gamma}^{*(\alpha,\beta)}(x) &= B_{n,\gamma}^{*(\alpha,\beta)}((t-x)^2,x)\\ &= \frac{2\left(-n\alpha+\alpha^2\gamma\right)}{(n+\beta)^2(n-\gamma)} + \frac{2x\left(n-2\alpha\gamma\right)}{(n+\beta)^2(n-\gamma)} + \frac{2\gamma x^2}{(n-\gamma)}.\end{aligned}$$

**Theorem 6.3.** *Let $f \in C_B(I_n)$ and $x \in I_n$. Then for $n > \gamma$, there exists a positive constant such that*

$$|B_{n,\gamma}^{*(\alpha,\beta)}(f,x) - f(x)| \leq M\omega_2\left(f, \sqrt{\mu_{n,2,\gamma}^{*(\alpha,\beta)}(x)}\right).$$

**Proof.** Let $g \in C_B(I_n)$ and $x, t \in I_n$. By Taylor's expansion, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Applying $B_{n,\gamma}^{*(\alpha,\beta)}$, we get

$$\begin{aligned}&B_{n,\gamma}^{*(\alpha,\beta)}(g,x) - g(x)\\ &\qquad = g'(x)B_{n,\gamma}^{*(\alpha,\beta)}((t-x),x) + B_{n,\gamma}^{*(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du, x\right).\end{aligned}$$

Obviously, we have $\left|\displaystyle\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2\|g''\|$,

$$|B_{n,\gamma}^{*(\alpha,\beta)}(g,x) - g(x)| \leq B_{n,\gamma}^{*(\alpha,\beta)}\left((t-x)^2,x\right)\|g''\| = \mu_{n,2,\gamma}^{*(\alpha,\beta)}(x)\|g''\|.$$

Since $|B_{n,\gamma}^{*(\alpha,\beta)}(f,x)| \leq \|f\|$

$$\begin{aligned} |B_{n,\gamma}^{*(\alpha,\beta)}(f,x) - f(x)| &\leq |B_{n,\gamma}^{*(\alpha,\beta)}(f-g,x) - (f-g)(x)| \\ &\quad + |B_{n,\gamma}^{*(\alpha,\beta)}(g,x) - g(x)| \\ &\leq 2\|f-g\| + \mu_{n,2,\gamma}^{*(\alpha,\beta)}(x)\|g''\|. \end{aligned}$$

Taking infimum over all $g \in C_{x^2}(I_n)$, we obtain

$$|B_{n,\gamma}^{*(\alpha,\beta)}(f,x) - f(x)| \leq K_2\left(f, \sqrt{\mu_{n,2,\gamma}^{*(\alpha,\beta)}(x)}\right).$$

In view of (3.2), we have

$$|B_{n,\gamma}^{*(\alpha,\beta)}(f,x) - f(x)| \leq M\omega_2\left(f, \sqrt{\mu_{n,2,\gamma}^{*(\alpha,\beta)}(x)}\right),$$

which proves the theorem. ∎

**Theorem 6.4.** *For $f \in C_{x^2}(I_n)$ such that $f', f'' \in C_{x^2}(I_n)$, we have*

$$\lim_{n\to\infty} n\left[B_{n,\gamma}^{*(\alpha,\beta)}(f,x) - f(x)\right] = (x+\gamma x^2)f''(x).$$

The proof follows along the lines of Theorem 3.2.

**Remark 6.5.** One can discuss rate of approximation in weighted space for the operators $B_{n,\gamma}^{*(\alpha,\beta)}$. We omit the details as it is similar to Theorem 5.1 and 5.2.

**Remark 6.6.** Very recently Gupta and Aral [10] introduced the $q$-analogue of original Baskakov-Beta operators [8]. We can now propose the $q$-analogue of operators (1.2) as

$$B_{n,\gamma}^q(f,x) = \sum_{k=1}^{\infty} p_{n,k,\gamma}^q(x) \int_0^{\infty/A} q^{-k} b_{n,k,\gamma}^q(t) f(t)dt + p_{n,0,\gamma}^q(x)f(0), \tag{6.2}$$

where

$$p_{n,k,\gamma}^q(x) = q^{\frac{k^2}{2}} \frac{\Gamma_q(n/\gamma+k)}{\Gamma_q(k+1)\Gamma_q(n/\gamma)} \cdot \frac{(q\gamma x)^k}{(1+q\gamma x)_q^{(n/\gamma)+k}},$$

$$b_{n,k,\gamma}^q(x) = \gamma q^{\frac{k^2}{2}} \frac{\Gamma_q(n/\gamma+k+1)}{\Gamma_q(k)\Gamma_q(n/\gamma+1)} \cdot \frac{(\gamma t)^{k-1}}{(1+\gamma t)_q^{(n/\gamma)+k+1}}$$

and the $q$ improper integral is defined as

$$\int_0^{\infty/A} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \quad A > 0.$$

For further details, we refer the reader to [10, 1]. For the operators (6.2), one can study their local approximation properties and Voronovskaja type asymptotic formula results based on $q$-integer.

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