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Adam Kubica

A REGULARITY CRITERION FOR POSITIVE PART OF RADIAL COMPONENT IN THE CASE OF AXIALLY SYMMETRIC NAVIER-STOKES EQUATIONS

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Abstract. We examine the conditional regularity of the solutions of Navier–Stokes equations in the entire three-dimensional space under the assumption that the data are axially symmetric. We show that if positive part of the radial component of velocity satisfies a weighted Serrin type condition and in addition angular component satisfies some condition, then the solution is regular.

1. Introduction

We will consider the Navier–Stokes equations in entire three-dimensional space

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{0} \ \text{ in } \ (0,T)\times\mathbb{R}^3, \\ \operatorname{div}\mathbf{u} = 0 \ \text{ in } \ (0,T)\times\mathbb{R}^3, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \ \text{ in } \ \mathbb{R}^3, \end{aligned} \tag{1}$$

where $\mathbf{u} : (0,T)\times\mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field, $p : (0,T)\times\mathbb{R}^3 \to \mathbb{R}$ is the pressure, $0 < T \le \infty$, ν is the viscosity coefficient, $\mathbf{u}_0$ is the initial velocity and the forcing term is, for the sake of simplicity, considered to be zero.

Up to now, it is not known whether equations (1) have global in time smooth solutions. In this paper, we analyze the special class of solutions which are axially symmetric, i.e. are in the form

$$u(t,x) = u_r(t,r,z)e_r + u_\theta(t,r,z)e_\theta + u_z(t,r,z)e_z,$$

where $r = \sqrt{x_1^2+x_2^2}$, $e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0)$, $e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0)$ and $e_z = (0,0,1)$, hence cylindrical coordinates u_r, u_θ, u_z do not depend on the angle θ. In this

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case, equations (1) have a simpler form. However, the issue of existence of smooth axially symmetric solutions is still open and there are only partial results. The first of them deal with flows without swirl (i.e. $u_\theta \equiv 0$) and there is proved that solutions remain smooth if the data were smooth (see [5], [11] and [6] for another proof). In the case of $u_\theta \not\equiv 0$, there are many conditional results. Let us mention some of them: there is no blow-up solutions if in addition one of the following conditions is satisfied

$$u_r \in L^q(0,T;L^s(\Omega)) \text{ with } \frac{2}{q}+\frac{3}{s}=1 \text{ for } s>3, \ ([8]), \tag{2}$$

$$u_\theta \in L^q(0,T;L^s(\Omega)) \text{ with } \frac{2}{q}+\frac{3}{s}<1 \text{ for } s>4, \ ([9]), \tag{3}$$

$$\frac{u_r}{r} \in L^q(0,T;L^s(\Omega)) \text{ with } \frac{2}{q}+\frac{3}{s}=2 \text{ for } s>\frac{3}{2}, \ ([1]), \tag{4}$$

$$r\mathbf{u} \in L^\infty(0,T;L^\infty(\Omega)), \ ([2]\ ), \tag{5}$$

$$r^d u_r^- \in L^q(0,T;L^s(\Omega)) \text{ with } \frac{2}{q}+\frac{3}{s}+d=1 \quad \text{for } s>\frac{3}{2}, q>1 \text{ and } d\in(-1,1) \ ([4]), \tag{6}$$

$$r^d u_\theta \in L^q(0,T;L^s(\Omega)) \text{ with } \frac{2}{q}+\frac{3}{s}+d<1 \quad \text{for } s>4, q>2 \text{ and } d\in\left[0,\frac{s-4}{2s}\right) \ ([4]), \tag{7}$$

$$r^d u_\theta \in L^\infty(0,T;L^\infty(\Omega)) \text{ with } d<\frac{5}{6} \ ([4]), \tag{8}$$

where $u_r^- = -\max\{-u_r,0\}$. We will denote $\Omega_{\delta_1} = \{x\in\mathbb{R}^3 : r<\delta_1\}$ and $u_r^+ = u+u_r^-$ is positive part of radial component. Our main result is following

Theorem 1. *Let $\mathbf{u}$ be a weak solution to problem (1) satisfying the energy inequality with $\mathbf{u}_0 \in W^{2,2}(\mathbb{R}^3)$, $ru_\theta(0)\in L^\infty(\mathbb{R}^3)$ and $(1+r)\nabla\mathbf{u}_0 \in L^1(\mathbb{R}^3)$. Let $\mathbf{u}_0$ be axisymmetric. If, in addition, u_r^+ a positive part of radial component of velocity satisfies*

$$r^d u_r^+ \in L^w(0,T;L^s(\Omega_{\delta_1})) \tag{9}$$

for some $s\in(\frac{3}{2},\infty)$, $w\in(1,\infty)$ and $d\in(-1,1)$ such that $\frac{2}{w}+\frac{3}{s}+d=1$ and for some positive δ_1 and

$$r^{1-\delta_0}u_\theta \in L^\infty((0,T)\times\mathbb{R}^3) \tag{10}$$

for some positive δ_0, then $(\mathbf{u},p)$, where p is the corresponding pressure, is axially symmetric strong solution to problem (1), which is unique in the class of all weak solutions satisfying the energy inequality.

Remark 1. If $ru_\theta(0) \in L^\infty(\mathbb{R}^3)$, then weak solutions of (1) satisfies $ru_\theta \in L^\infty(0,T;\mathbb{R}^3)$. Hence, the assumption (10) is arbitrary close to this properties of weak solutions.

Remark 2. It is worth to mention that the norm

$$\|u\|_{L^{w,s}_d} \equiv \|r^d u\|_{L^w(\mathbb{R}_+;L^s(\mathbb{R}^3))}$$

is *scaling invariant* if and only if $\frac{2}{w} + \frac{3}{s} + d = 1$. Indeed, for such exponents we get $\|u\|_{L^{w,s}_d} = \|u_\lambda\|_{L^{w,s}_d}$, where $u_\lambda(t,x) = \lambda u(\lambda t, \lambda^2 x)$. Therefore, conditions (2), (4), (5), (6) and (9) involve scaling invariant norms.

In order to prove Theorem 1, we write the equations (1) in cylindrical coordinates

$$u_{r,t} + u_r u_{r,r} + u_z u_{r,z} - \frac{1}{r}u_\theta^2 + p_{,r} - \nu\Big[\frac{1}{r}\big(ru_{r,r}\big)_{,r} + u_{r,zz} - \frac{u_r}{r^2}\Big] = 0, \tag{11}$$

$$u_{\theta,t} + u_r u_{\theta,r} + u_z u_{\theta,z} + \frac{1}{r}u_\theta u_r - \nu\Big[\frac{1}{r}\big(ru_{\theta,r}\big)_{,r} + u_{\theta,zz} - \frac{u_\theta}{r^2}\Big] = 0, \tag{12}$$

$$u_{z,t} + u_r u_{z,r} + u_z u_{z,z} + p_{,z} - \nu\Big[\frac{1}{r}\big(ru_{z,r}\big)_{,r} + u_{z,zz}\Big] = 0, \tag{13}$$

$$u_{r,r} + \frac{u_r}{r} + u_{z,z} = 0, \tag{14}$$

where the last one is a continuity equation. If we denote $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$, then in cylindrical coordinates, we have

$$\omega_r = -u_{\theta,z}, \quad \omega_\theta = u_{r,z} - u_{z,r}, \quad \omega_z = u_{\theta,r} + \frac{u_\theta}{r}.$$

Therefore, the equations for $\boldsymbol{\omega}$ in cylindrical coordinates are following

$$\omega_{r,t} + u_r\omega_{r,r} + u_z\omega_{r,z} - u_{r,r}\omega_r - u_{r,z}\omega_z - \nu\Big[\frac{1}{r}\big(r\omega_{r,r}\big)_{,r} + \omega_{r,zz} - \frac{\omega_r}{r^2}\Big] = 0, \tag{15}$$

$$\omega_{\theta,t} + u_r\omega_{\theta,r} + u_z\omega_{\theta,z} - \frac{u_r}{r}\omega_\theta + 2\frac{u_\theta}{r}\omega_r - \nu\Big[\frac{1}{r}\big(r\omega_{\theta,r}\big)_{,r} + \omega_{\theta,zz} - \frac{\omega_\theta}{r^2}\Big] = 0, \tag{16}$$

$$\omega_{z,t} + u_r\omega_{z,r} + u_z\omega_{z,z} - u_{z,r}\omega_r - u_{z,z}\omega_z - \nu\Big[\frac{1}{r}\big(r\omega_{z,r}\big)_{,r} + \omega_{z,zz}\Big] = 0. \tag{17}$$

We will prove Theorem 1 by contradiction. Therefore, suppose that $0 < t^* < T$ is the time of the first blow-up of solution, i.e. the smaller positive number such that $\sup_{t\in(0,t^*)} \|\nabla\mathbf{u}(t,\cdot)\|_{L^2(\mathbb{R}^3)} = \infty$. Then, for $0 < \bar{t} < t^*$, the equations (11)–(17) are satisfied in $((0,\bar{t}) \times \mathbb{R}^3)$ in strong sense. We will show that if u_r^+ and u_θ satisfy assumptions of Theorem 1, then $\|\nabla\mathbf{u}(t,\cdot)\|_{L^2(\mathbb{R}^3)}$ remains uniformly bounded for $t \in (0,t^*)$ and we will get contradiction.

We shall obtain a uniform estimate for $t \in (0,t^*)$. For this purpose, we multiply equation (12) by $\left|\frac{u_\theta}{r^{m_u}}\right|^{p-2}\frac{u_\theta}{r^{2\mu}}$ and integrate over $\mathbb{R}^3$. Then integrating by parts and continuity equation (14) yield

$$
\begin{aligned}
(18)\qquad & \frac{1}{p}\frac{d}{dt}\left\|\frac{u_\theta}{r^\mu}\right\|_p^p + \frac{4(p-1)\nu}{p^2}\int\left|\nabla\left|\frac{u_\theta}{r^\mu}\right|^{\frac{p}{2}}\right|^2 \\
&\quad + \nu(1-\mu^2)\int\left|\frac{u_\theta}{r^\mu}\right|^p\frac{1}{r^2} + (1+\mu)\int\frac{u_r^+}{r}\left|\frac{u_\theta}{r^\mu}\right|^p \\
&\qquad = (1+\mu)\int\frac{u_r^-}{r}\left|\frac{u_\theta}{r^\mu}\right|^p.
\end{aligned}
$$

Next, we multiply equation (16) by $\left|\frac{\omega_\theta}{r^\alpha}\right|^{q-2}\frac{\omega_\theta}{r^{2\alpha}}$ and then in a similar way, we get

$$
\begin{aligned}
(19)\qquad & \frac{1}{q}\frac{d}{dt}\left\|\frac{\omega_\theta}{r^\alpha}\right\|_q^q + \frac{4\nu(q-1)}{q^2}\int\left|\nabla\left|\frac{\omega_\theta}{r^\alpha}\right|^{\frac{q}{2}}\right|^2 \\
&\quad + \nu(1-\alpha^2)\int\left|\frac{\omega_\theta}{r^\alpha}\right|^q\frac{1}{r^2} + (1-\alpha)\int\frac{u_r^-}{r}\left|\frac{\omega_\theta}{r^\alpha}\right|^q \\
&\qquad = (1-\alpha)\int\frac{u_r^+}{r}\left|\frac{\omega_\theta}{r^\alpha}\right|^q + 2\int\frac{u_\theta}{r}u_{\theta,z}\left|\frac{\omega_\theta}{r^\alpha}\right|^{q-2}\frac{\omega_\theta}{r^{2\alpha}}.
\end{aligned}
$$

Summing up above equalities, we obtain

$$
\begin{aligned}
(20)\qquad & \frac{1}{p}\frac{d}{dt}\left\|\frac{u_\theta}{r^\mu}\right\|_p^p + \frac{1}{q}\frac{d}{dt}\left\|\frac{\omega_\theta}{r^\alpha}\right\|_q^q + \frac{4(p-1)\nu}{p^2}\int\left|\nabla\left|\frac{u_\theta}{r^\mu}\right|^{\frac{p}{2}}\right|^2 \\
&+ \frac{4\nu(q-1)}{q^2}\int\left|\nabla\left|\frac{\omega_\theta}{r^\alpha}\right|^{\frac{q}{2}}\right|^2 + \nu(1-\mu^2)\int\left|\frac{u_\theta}{r^\mu}\right|^p\frac{1}{r^2} + \nu(1-\alpha^2)\int\left|\frac{\omega_\theta}{r^\alpha}\right|^q\frac{1}{r^2} \\
&+ (1+\mu)\int\frac{u_r^+}{r}\left|\frac{u_\theta}{r^\mu}\right|^p + (1-\alpha)\int\frac{u_r^-}{r}\left|\frac{\omega_\theta}{r^\alpha}\right|^q \\
&= 2\int\frac{u_\theta}{r}u_{\theta,z}\left|\frac{\omega_\theta}{r^\alpha}\right|^{q-2}\frac{\omega_\theta}{r^{2\alpha}} + (1+\mu)\int\frac{u_r^-}{r}\left|\frac{u_\theta}{r^\mu}\right|^p \\
&+ (1-\alpha)\int\frac{u_r^+}{r}\left|\frac{\omega_\theta}{r^\alpha}\right|^q \equiv 2I_1 + (1+\mu)I_2 + (1-\alpha)I_3.
\end{aligned}
$$

We will show that for some exponents p, q, α and μ, the right hand side can be estimated.

2. Estimate of I_1

Proposition 1. *Assume that $\gamma \in (0,3)$, $q \in (\frac{2}{4-\gamma}, 2)$, $p = \frac{(4-\gamma)q}{2}$, $\mu \in (-1,1)$ and $a \in (0,1)$. Then for $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, the following estimate*

$$|I_1| \le \varepsilon_1 \int \left|\frac{u_\theta}{r^\mu}\right|^{p-2} \left|\frac{u_{\theta,z}}{r^\mu}\right|^2 + \varepsilon_2 \int \left|\frac{u_\theta}{r^\mu}\right|^p \frac{1}{r^2} + \varepsilon_3 \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \frac{1}{r^2} + C \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q, \tag{21}$$

holds, where

$$\alpha = 2\mu - \frac{\gamma}{2}(1+\mu) - \frac{2(q-1)}{q}(1-a), \tag{22}$$

and $C = C(\gamma, q, a, \varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Proof. Using Young inequality, we get

$$\begin{aligned}
|I_1| &\le \int \left|\frac{u_\theta}{r}\right| |u_{\theta,z}| \left|\frac{\omega_\theta}{r^\alpha}\right|^{q-2} \frac{|\omega_\theta|}{r^{2\alpha}} = \int \left|\frac{u_\theta}{r^\mu}\right|^{\frac{p}{2}-1} \left|\frac{u_{\theta,z}}{r^\mu}\right| \cdot \frac{|u_\theta|^{2-\frac{p}{2}}}{r^{1+\alpha-\frac{\mu p}{2}}} \left|\frac{\omega_\theta}{r^\alpha}\right|^{q-1} \\
&\le \varepsilon_1 \int \left|\frac{u_\theta}{r^\mu}\right|^{p-2} \left|\frac{u_{\theta,z}}{r^\mu}\right|^2 + C(1/\varepsilon_1) \int \frac{|u_\theta|^{4-p}}{r^{2[1+\alpha-\frac{\mu p}{2}]}} \left|\frac{\omega_\theta}{r^\alpha}\right|^{2(q-1)} \\
&= \varepsilon_1 \int \left|\frac{u_\theta}{r^\mu}\right|^{p-2} \left|\frac{u_{\theta,z}}{r^\mu}\right|^2 + C(1/\varepsilon_1) \int |ru_\theta|^\gamma \\
&\quad \times \frac{|u_\theta|^{4-p-\gamma}}{r^{2+2\alpha-\mu p+\gamma-\frac{4(q-1)}{q}a}} \cdot \frac{|\omega_\theta|^{2(q-1)a}}{r^{2\alpha(q-1)a+\frac{4(q-1)}{q}a}} \cdot \frac{|\omega_\theta|^{2(q-1)(1-a)}}{r^{2\alpha(q-1)(1-a)}}.
\end{aligned}$$

It is well known that (18) leads to the estimate $\|ru_\theta\|_{L^\infty} \le \|r(\mathbf{u}_0)_\theta\|_{L^\infty}$, thus applying this estimate and next Young inequality with exponents $(\frac{q}{2-q}, \frac{q}{2(q-1)a}, \frac{q}{2(q-1)(1-a)})$, we obtain

$$\begin{aligned}
|I_1| &\le \varepsilon_1 \int \left|\frac{u_\theta}{r^\mu}\right|^{p-2} \left|\frac{u_{\theta,z}}{r^\mu}\right|^2 + \varepsilon_2 \int \frac{|u_\theta|^{[4-p-\gamma]\frac{q}{2-q}}}{r^{\frac{q}{2-q}[2+2\alpha-\mu p+\gamma-\frac{4(q-1)}{q}a]}} \\
&\quad + \varepsilon_3 \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \frac{1}{r^2} + C \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q,
\end{aligned}$$

where C depends on a, ε_1, ε_2, ε_3 and $\|r(\mathbf{u}_0)_\theta\|_{L^\infty}$. By simple calculations, we get $[4-p-\gamma]\frac{q}{2-q} = p$ and $\frac{q}{2-q}[2+2\alpha-\mu p+\gamma-\frac{4(q-1)}{q}a] = p\mu + 2$. ∎

3. Estimate of I_2

We begin with the following

Remark 3. Assume that $q \in (1, \infty)$, α and ε_0 satisfy $-2 + \varepsilon_0 < \alpha < \varepsilon_0$. Then there exists a constant $C = C(q, \alpha, \varepsilon_0)$ such that

$$\int \left|\frac{u_r}{r^{1+\alpha}}\right|^q \cdot \frac{1}{r^{2-\varepsilon_0 q}} \le C \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \cdot \frac{1}{r^{2-\varepsilon_0 q}}. \tag{23}$$

Proof. From Lemma 2.2 [3], we have $\|\frac{u_r}{r}\|_q \le c(q)\|\omega_\theta\|_q$ for all $q \in (1, \infty)$. Thus, we only have to verify that $r^{-q(\alpha+\frac{2}{q}-\varepsilon_0)}$ is A_q weight (see comment before Lemma 2.6 [3]). In view of Example 1.2.5 [10], it holds if

$$-2 < -q(\alpha + \frac{2}{q} - \varepsilon_0) < 2(q-1),$$

i.e. $-2 + \varepsilon_0 < \alpha < \varepsilon_0$. ∎

Proposition 2. *Assume that* $\varpi \equiv \|r^{1-\delta_0} u_\theta\|_{L^\infty} < \infty$ *for some* $\delta_0 \in (0, \frac{1}{6})$. *Then for all* $\gamma \in (0,3)$, $q \in (\frac{2}{4-\gamma}, 2)$, $a \in (1 - q\frac{(4-\gamma)\delta_0}{4(q-1)}, \frac{4}{4-\gamma} - \delta_0) \cap (0,1)$,

$$\mu \in (\delta_0 - 1, \delta_0 + \frac{\gamma}{4-\gamma}) \cap (-1,1), \tag{24}$$

and for ε_4, $\varepsilon_5 \in (0,1)$, *the following estimate holds*

$$|I_2| = \int \frac{u_r^-}{r}\left|\frac{u_\theta}{r^\mu}\right|^p \leq \varepsilon_4 \int \left|\frac{u_\theta}{r^\mu}\right|^p \frac{1}{r^2} + \varepsilon_5 \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \frac{1}{r^2} + C\int \left|\frac{\omega_\theta}{r^\alpha}\right|^q, \tag{25}$$

where

$$p = \frac{(4-\gamma)q}{2}, \qquad \alpha = 2\mu - \frac{\gamma}{2}(1+\mu) - \frac{2(q-1)}{q}(1-a)$$

and

$$C = C(\varepsilon_4, \varepsilon_5, a, q, \delta_0, \gamma, \varpi).$$

Proof. We denote $\kappa = -\frac{2(q-1)}{q}(1-a)$. Then the assumption concerning u_θ and Young inequality with exponents $(\frac{q}{q-1}, q)$ yield

$$\begin{aligned}|I_2| &= \int \frac{u_r^-}{r}\left|\frac{u_\theta}{r^\mu}\right|^p = \int \left|\frac{u_\theta}{r^{\mu+\frac{2}{p}}}\right|^{\frac{p(q-1)}{q}} \cdot |r^{1-\delta_0}u_\theta|^{\frac{p}{q}} \cdot \frac{u_r^-}{r^{1+\alpha+\frac{2}{q}-\kappa-\frac{p}{q}\delta_0}} \\ &\leq \varepsilon_4 \int \left|\frac{u_\theta}{r^\mu}\right|^p \frac{1}{r^2} + C(q,\gamma,\varpi,\varepsilon_4)\int \left|\frac{u_r}{r^{1+\alpha}}\right|^q \frac{1}{r^{2-\kappa q-\delta_0 p}}.\end{aligned}$$

We define ε_0 by equality $-\kappa q - \delta_0 p = -q[\frac{(4-\gamma)}{2}\delta_0 - \frac{2(q-1)}{q}(1-a)] \equiv -q\varepsilon_0$. Then the assumption (24) leads to $-2+\varepsilon_0 < \alpha < \varepsilon_0$, hence we can apply Remark 3 and we get

$$|I_2| \leq \varepsilon_4 \int \left|\frac{u_\theta}{r^\mu}\right|^p \frac{1}{r^2} + C(q,\gamma,\varpi,\varepsilon_4,a,\mu)\int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \frac{1}{r^{2-\varepsilon_0 q}}.$$

Using the assumption on a, we deduce that $b = 1 - \frac{q\varepsilon_0}{2}$ satisfies $b \in (0,1)$, hence we can apply Young inequality with exponents $(\frac{1}{b}, \frac{1}{1-b})$ and then we get

$$\int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \frac{1}{r^{2-\varepsilon_0 q}} = \int \left|\frac{\omega_\theta}{r^{\alpha+\frac{2}{q}}}\right|^{bq} \cdot \left|\frac{\omega_\theta}{r^\alpha}\right|^{(1-b)q} \leq \varepsilon_5 \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \frac{1}{r^2} + C(\varepsilon_5, b)\int \left|\frac{\omega_\theta}{r^\alpha}\right|^q. \ ∎$$

Propositions 1 and 2 immediately give

Corollary 1. *Assume that* $\varpi \equiv \|r^{1-\delta_0} u_\theta\|_{L^\infty} < \infty$ *for some* $\delta_0 \in (0, \frac{1}{6})$. *Then for all* $\gamma \in (0,3)$, $q \in (\frac{2}{4-\gamma}, 2)$, $a \in (1 - \frac{(4-\gamma)\delta_0}{4(q-1)}, \frac{4}{4-\gamma} - \delta_0) \cap (0,1)$,

$$\mu \in (\delta_0 - 1, \delta_0 + \frac{\gamma}{4-\gamma}) \cap (-1, 1), \tag{26}$$

and for ε_1, ε_2, $\varepsilon_3 \in (0,1)$, *the following estimate holds*

$$|I_1| + |I_2| \le \varepsilon_1 \int \left|\frac{u_\theta}{r^\mu}\right|^p \frac{1}{r^2} + \varepsilon_2 \int \left|\frac{\omega_\theta}{r^\alpha}\right|^p \frac{1}{r^2} + \varepsilon_3 \int \left|\nabla \left|\frac{u_\theta}{r^\mu}\right|^{\frac{p}{2}}\right|^2 + C \int \left|\frac{\omega_\theta}{r^\alpha}\right|^p, \tag{27}$$

where

$$p = \frac{(4-\gamma)q}{2}, \qquad \alpha = 2\mu - \frac{\gamma}{2}(1+\mu) - \frac{2(q-1)}{q}(1-a),$$

and

$$C = C(\varepsilon_1, \varepsilon_2, \varepsilon_3, a, q, \delta_0, \gamma, \varpi).$$

The above estimate involve many exponents and it is not clear at once, whether we can get the estimate with useful range of exponents. Therefore, we formulate

Corollary 2. *Assume that* $\varpi \equiv \|r^{1-\delta_0} u_\theta\|_{L^\infty} < \infty$ *for some* $\delta_0 \in (0, \frac{1}{6})$. *Then for* $\varepsilon \in (0, \frac{1}{7})$ *such that*

$$4(1-2\varepsilon)(1-\varepsilon)\varepsilon \le \delta_0, \tag{28}$$

and for all $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in (0,1)$, *the following estimate holds*

$$|I_1| + |I_2| \le \varepsilon_1 \int \left|\frac{u_\theta}{r^\mu}\right|^p \frac{1}{r^2} + \varepsilon_2 \int \left|\frac{\omega_\theta}{r^\alpha}\right|^p \frac{1}{r^2} + \varepsilon_3 \int \left|\nabla \left|\frac{u_\theta}{r^\mu}\right|^{\frac{p}{2}}\right|^2 + C \int \left|\frac{\omega_\theta}{r^\alpha}\right|^p, \tag{29}$$

where $p = 2(1-\varepsilon^2)$, $q = 2(1-\varepsilon)$, $\mu = \frac{1-\varepsilon}{1+\varepsilon}$ *and* $\alpha = -2(1-2\varepsilon)(1+\varepsilon)\varepsilon$ *and* $C = C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_0, \varpi, \varepsilon)$. *In particular, for such exponents we have*

$$\int \left|\frac{u_r}{r^{1+\alpha}}\right|^q \le c(q, \alpha) \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q, \tag{30}$$

and

$$\frac{2}{\infty} + \frac{3}{q} - (1+\alpha) \le \frac{1}{2}(1+7\varepsilon) < 1. \tag{31}$$

Proof. We have to verify the assumptions of Corollary 1. Therefore we put $\gamma = 2(1-\varepsilon)$. Then $\gamma \in (0,3)$ and $q \in (\frac{2}{4-\gamma}, 2)$ and we set $a = 1 - 2(1-\varepsilon^2)\varepsilon$. Then condition (28) implies $a \in (1 - \frac{(4-\gamma)\delta_0}{4(q-1)}, \frac{4}{4-\gamma} - \delta_0)$. Finally, δ_0 is positive, hence $\mu = \frac{1-\varepsilon}{1+\varepsilon}$ satisfies (26), thus assumptions of Corollary 1 are satisfied and we get (27) with $p = \frac{(4-\gamma)q}{2} = 2(1-\varepsilon^2)$ and $\alpha = 2\mu - \frac{\gamma}{2}(1+\mu) - \frac{2(q-1)}{q}(1-a) = -2(1-2\varepsilon)(1+\varepsilon)\varepsilon$.

In order to get (30), we have to verify that $r^{-q\alpha}$ is $\mathcal{A}_q$ weight. It is equivalent to $\frac{2}{q} - 2 < \alpha < \frac{2}{q}$ and holds true, because ε is small enough. Finally, by direct calculations we obtain inequality (31). ■

4. Estimate of I_3

Proposition 3. *Assume that $s \in (\frac{3}{2}, \infty)$, $w \in (1, \infty)$ and $d \in (-1, 1)$ are such that $\frac{2}{w} + \frac{3}{s} + d = 1$. If $q \in (1, \infty)$, $\alpha \in (-1, 1)$ and $\delta_1 > 0$, then for all $\varepsilon_4, \varepsilon_5 \in (0, 1)$ the following estimate*

$$|I_3| \leq \varepsilon_4 \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \frac{1}{r^2} + \varepsilon_5 \int \left|\nabla \left|\frac{\omega_\theta}{r^\alpha}\right|^{\frac{q}{2}}\right|^2 + C[f(t) + g(t)] \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q, \tag{32}$$

holds, where

$$g(t) = \int |u_r^+|^{\frac{10}{3}}, \qquad f(t) \equiv \left[\int_{\Omega_1} |r^d u_r^+|^s dx\right]^{\frac{w}{s}}$$

and

$$C = C(\varepsilon_4, \varepsilon_5, \delta_1, s, w, q).$$

Remark 4. Recall that Ω_{δ_1} denotes $\{x \in \mathbb{R}^3 : r < \delta_1\}$. It is known that if $\mathbf{u}$ is weak solutions of (1), then with the help of Sobolev embedding theorem, we deduce that $\mathbf{u} \in L^{\frac{10}{3}}((0, T) \times \Omega)$, hence function $g(t)$ is integrable on $(0, T)$. However, up to now, there is no proof of integrability of $f(t)$ on $(0, T)$ and its integrability is our main assumption in proving smoothness of axially symmetric solutions. For weak solutions, we have $u_r \in L^\infty(0, T; L^2(\Omega_1))$ and $\frac{u_r}{r} \in L^2(0, T; L^2(\Omega_1))$, i.e. the exponents satisfy too weak conditions: $\frac{2}{\infty} + \frac{3}{2} = \frac{3}{2}$ and $\frac{2}{2} + \frac{3}{2} - 1 = \frac{3}{2}$.

Proof. Let $\eta = \eta(r)$ be smooth cut off function such that $\eta(r) = 1$ for $r < \delta_1/2$ and $\eta(r) = 0$ for $r > \delta_1$. Then we can write

$$I_3 = \int \frac{\eta u_r^+}{r} \left|\frac{\omega_\theta}{r^\alpha}\right|^q + \int \frac{(1-\eta) u_r^+}{r} \left|\frac{\omega_\theta}{r^\alpha}\right|^q \equiv I_{3,1} + I_{3,2}.$$

We first estimate $I_{3,1}$. We set $a = \frac{2}{2 - (\frac{2}{w} + \frac{3}{s})}$, $b = \frac{2s}{w} + 3$. Then $a > 1$ and $b > 3$ and applying Young inequality with exponents $(a, \frac{a}{a-1})$, we obtain

$$\begin{aligned} I_{3,1} &\equiv \int \frac{\eta u_r^+}{r} \left|\frac{\omega_\theta}{r^\alpha}\right|^q = \int \left|\frac{\omega_\theta}{r^\alpha}\right|^{\frac{q}{a}} \frac{1}{r^{\frac{2}{a}}} \cdot \eta u_r^+ r^{\frac{2-a}{a}} \left|\frac{\omega_\theta}{r^\alpha}\right|^{q\frac{a-1}{a}} \\ &\leq \varepsilon_1 \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \frac{1}{r^2} + c(\varepsilon_1, a) \int |\eta u_r^+|^{\frac{a}{a-1}} r^{\frac{2-a}{a-1}} \cdot \left|\frac{\omega_\theta}{r^\alpha}\right|^q. \end{aligned}$$

To estimate the last integral on the right hand side, we use twice Hölder inequality with exponents $(\frac{b}{2}, \frac{b}{b-2})$ and $(\frac{b-2}{b-3}, b-2)$

$$\int |\eta u_r^+|^{\frac{a}{a-1}} r^{\frac{2-a}{a-1}} \cdot \left|\frac{\omega_\theta}{r^\alpha}\right|^q \le \left[\int |\eta u_r^+|^{\frac{ab}{2(a-1)}} r^{\frac{b(2-a)}{2(a-1)}}\right]^{\frac{2}{b}} \cdot \left[\int \left|\frac{\omega_\theta}{r^\alpha}\right|^{\frac{qb}{b-2}}\right]^{\frac{b-2}{b}}$$
$$= \left[\int |\eta u_r^+|^{\frac{ab}{2(a-1)}} r^{\frac{b(2-a)}{2(a-1)}}\right]^{\frac{2}{b}} \cdot \left[\int \left|\frac{\omega_\theta}{r^\alpha}\right|^{q\frac{b-3}{b-2}} \cdot \left|\frac{\omega_\theta}{r^\alpha}\right|^{\frac{3q}{b-2}}\right]^{\frac{b-2}{b}}$$
$$\le \left[\int |\eta u_r^+|^{\frac{ab}{2(a-1)}} r^{\frac{b(2-a)}{2(a-1)}}\right]^{\frac{2}{b}} \cdot \left[\int \left|\frac{\omega_\theta}{r^\alpha}\right|^{q}\right]^{\frac{b-3}{b}} \cdot \left[\int \left|\frac{\omega_\theta}{r^\alpha}\right|^{3q}\right]^{\frac{1}{b}}$$
$$\le \varepsilon_2\left[\int \left|\frac{\omega_\theta}{r^\alpha}\right|^{3q}\right]^{\frac{1}{3}} + c(\varepsilon_2, b)\left[\int |\eta u_r^+|^{\frac{ab}{2(a-1)}} r^{\frac{b(2-a)}{2(a-1)}}\right]^{\frac{2}{b-3}} \cdot \left[\int \left|\frac{\omega_\theta}{r^\alpha}\right|^{q}\right],$$

where we also applied Young inequality with exponents $(\frac{b}{3}, \frac{b}{b-3})$. By definition we have $\frac{ab}{2(a-1)} = s$, $\frac{b(2-a)}{2(a-1)} = ds$ and $\frac{2}{b-3} = \frac{w}{s}$, thus

$$I_{3,1} \le \varepsilon_1 \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \frac{1}{r^2} + \varepsilon_2\left[\int \left|\frac{\omega_\theta}{r^\alpha}\right|^{3q}\right]^{\frac{1}{3}} + c(\varepsilon_1, \varepsilon_2, w, s) f(t) \cdot \left[\int \left|\frac{\omega_\theta}{r^\alpha}\right|^{q}\right]. \tag{33}$$

In order to estimate $I_{3,2}$, we put $a = 4$ and $b = 5$ and proceeding analogously we get

$$I_{3,2} \le \varepsilon_1 \int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \frac{1}{r^2} + \varepsilon_2\left[\int \left|\frac{\omega_\theta}{r^\alpha}\right|^{3q}\right]^{\frac{1}{3}} + c(\varepsilon_1, \varepsilon_2)\left[\int |(1-\eta)u_r^+|^{\frac{10}{3}} r^{-\frac{5}{3}}\right] \cdot \left[\int \left|\frac{\omega_\theta}{r^\alpha}\right|^{q}\right]. \tag{34}$$

Clearly $\int |(1-\eta)u_r^+|^{\frac{10}{3}} r^{-\frac{5}{3}} \le (2/\delta_1)^{5/3} \int |u_r^+|^{\frac{10}{3}}$. Finally, applying Sobolev embedding theorem in estimates (33) and (34) we get (32). ∎

Corollary 3. *Assume that $s \in (\frac{3}{2}, \infty)$, $w \in (1, \infty)$ and $d \in (-1, 1)$ are such that $\frac{2}{w} + \frac{3}{s} + d = 1$ and $\varpi \equiv \|r^{1-\delta_0} u_\theta\|_{L^\infty} < \infty$ for some $\delta_0 \in (0, \frac{1}{6})$ and δ_1 is positive. Then for all $\varepsilon \in (0, \frac{1}{14})$ such that*

$$4(1-2\varepsilon)(1-\varepsilon)\varepsilon \le \delta_0, \tag{35}$$

the following estimate

$$\frac{d}{dt}\left\|\frac{u_\theta}{r^\mu}\right\|_p^p + \frac{d}{dt}\left\|\frac{\omega_\theta}{r^\alpha}\right\|_q^q + \frac{4(p-1)\nu}{p}\int \left|\nabla\left|\frac{u_\theta}{r^\mu}\right|^{\frac{p}{2}}\right|^2 + \frac{4\nu(q-1)}{q}\int \left|\nabla\left|\frac{\omega_\theta}{r^\alpha}\right|^{\frac{q}{2}}\right|^2 + \nu(1-\mu^2)\int \left|\frac{u_\theta}{r^\mu}\right|^p \frac{1}{r^2} + \nu(1-\alpha^2)\int \left|\frac{\omega_\theta}{r^\alpha}\right|^q \frac{1}{r^2} \le C[1 + f(t) + g(t)]\left\|\frac{\omega_\theta}{r^\alpha}\right\|_q^q \tag{36}$$

holds, where $p = 2(1-\varepsilon^2)$, $q = 2(1-\varepsilon)$, $\mu = \frac{1-\varepsilon}{1+\varepsilon}$ *and* $\alpha = -2(1-2\varepsilon)(1+\varepsilon)\varepsilon$, $g(t) = \int |u_r^+|^{\frac{10}{3}}$, $f(t) \equiv [\int_{\Omega_1} |r^d u_r^+|^s dx]^{\frac{w}{s}}$ *and* $C = C(\nu, \varepsilon, \delta_0, \delta_1, \varpi, s, w)$. *In particular, if* $f(t)$ *is integrable on* $(0, t^*)$, *then*

$$\sup_{t\in(0,t^*)} \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q \le C' \left[\left\| \frac{u_\theta}{r^\mu}(0) \right\|_p + \left\| \frac{\omega_\theta}{r^\alpha}(0) \right\|_q \right], \tag{37}$$

where $C' = C'(C, \|f\|_{L^1(0,T)}, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)})$.

Proof. The assumptions of Corollary 2 and Proposition 3 are satisfied, hence we get (36). If $f(t)$ is integrable on $(0, t^*)$, then by Gronwall lemma, we obtain (37). ■

Proof of Theorem 1. Under the assumptions of Theorem 1, we get estimate (37), where the right hand side is finite by the assumption concerning $\mathbf{u}_0$. For such q and α inequality (30) holds, thus we deduce that $\sup_{t\in(0,t^*)} \left\| \frac{u_r}{r^{1+\alpha}} \right\|_q$ is finite, where exponents q and α satisfy (31). Therefore, we can apply Theorem 1(i) ([4]) and we deduce that $\mathbf{u}$ is regular on $(0, t^*)$, i.e. there is no blow-up at time $t = t^*$, which gives the contradiction. ■

Remark 5. The condition (9) can be weakened a bit. Namely, it is enough to assume that

$$t \mapsto \widetilde{f}(t) \equiv \frac{\left[\int_{\Omega_1} |r^d u_r^+|^s dx\right]^{\frac{w}{s}}}{1 + \ln^+\left(\left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q \right)} \quad \text{is integrable on} \quad (0, T), \tag{38}$$

where d, w, s, p, q are as in Theorem 1. Indeed, from (36) we have

$$\frac{d}{dt}\left(\left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q \right) \le C[1 + f(t) + g(t)] \left(\left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q \right),$$

hence arguing similarly as in [7], we can write

$$\frac{d}{dt} \ln\left[1 + \ln^+\left(\left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q \right) \right] \le C \frac{[1 + f(t) + g(t)]}{1 + \ln^+\left(\left\| \frac{u_\theta}{r^\mu} \right\|_p^p + \left\| \frac{\omega_\theta}{r^\alpha} \right\|_q^q \right)}.$$

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A. Kubica
FACULTY OF MATHEMATICS AND INFORMATION SCIENCE
WARSAW UNIVERSITY OF TECHNOLOGY
Pl. Politechniki 1
00-661 WARSAW
E-mail: a.kubica@mini.pw.edu.pl

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