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UNIMODALITY OF BOOLEAN AND MONOTONE STABLE DISTRIBUTIONS

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Abstract. We give a complete list of the Lebesgue–Jordan decomposition of Boolean and monotone stable distributions and a complete list of the mode of them. They are not always unimodal.

1. Introduction

Boolean and monotone stable distributions were defined in [SW97, H10a] in the context of non-commutative probability theory with Boolean and monotone independence, respectively [SW97, M01]. The aspect of domains of attraction for these distributions are studied in [W12, AW13] and [BP99], respectively.

They also play important roles in free probability theory: Positive monotone stable laws are the marginal laws of a free Lévy process of second kind [B98, Theorem 4.5, Corollary 4.5]; A compound free Poisson distribution having a monotone stable law as its free Lévy measure has explicit Cauchy and Voiculescu transforms [AH13]; A positive Boolean stable distribution is the law of the quotient X/Y of two i.i.d. classical stable random variables X, Y, and at the same time, it is the law of the "noncommutative quotient" $X^{-1/2}YX^{-1/2}$ of free random variables X, Y having the same free stable law [BP99, AH14].

In this paper, we first determine the absolutely continuous part and also the singular part of the monotone and Boolean stable laws. While part of

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this computation is known in the literature [AH13, AH14, H10a], there is no complete list of the formulas.

Second, we investigate the modality of them. A probability measure μ is unimodal with mode $a \in \mathbb{R}$ if μ is decomposed as $\mu(dx) = c\delta_a(dx) + f(x)dx$ where $c \in [0,1]$ and $f(x)$ is non-decreasing on $(-\infty, a)$ and is non-increasing on (a, ∞). In other words, μ is said to be unimodal with mode $a \in \mathbb{R}$ if $\mu((-\infty, x])$ is convex on $(-\infty, a)$ and concave on (a, ∞). We say that μ is unimodal if μ is unimodal with mode a for some $a \in \mathbb{R}$. A probability distribution μ is said to be bimodal with modes $a_1, a_2 \in \mathbb{R}$ if $\mu((-\infty, x])$ is convex on $(-\infty, a_1)$ and (b, a_2) and concave on (a_1, b) and (a_2, ∞) for some $a_1 < b < a_2$.

It is known that classical and free stable distributions are unimodal [Y78, BP99]. More generally, selfdecomposable and free selfdecomposable distributions, which respectively include all stable and free stable distributions, are unimodal [Y78, HT]. However, monotone stable laws and Boolean stable laws include the arcsine law and the Bernoulli law, respectively, and we cannot expect unimodality for all. We obtain the modes of all monotone and Boolean stable distributions. In some case, these distributions become bimodal.

Remark 1. The modes of a unimodal or bimodal distribution μ may not be unique. See page 394 in the book by Sato [Sa99] for details.

2. Unimodality of Boolean and monotone stable distributions

First, we gather analytic tools and their properties to compute Boolean and monotone stable distributions.

2.1. Analytic tools. Let $\mathcal{P}$ denote the set of all Borel probability measures on $\mathbb{R}$. In the following, we explain the main tool of free probability. Let $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\mathbb{C}^- := \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$. For $\mu \in \mathcal{P}$, the Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ is defined by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x} \mu(\mathrm{d}x), \quad z \in \mathbb{C}^+,$$

and the reciprocal Cauchy transform $F_\mu : \mathbb{C}^+ \to \mathbb{C}^+$of $\mu \in \mathcal{P}$ is defined by

$$F_\mu(z) = \frac{1}{G_\mu(z)}, \quad z \in \mathbb{C}^+.$$

In this paper, we apply the Stieltjes inversion formula [A65, T00] for Boolean and monotone stable distributions. For any Borel probability measure μ, we can recover the distribution from its Cauchy transform: if μ does not have

atoms at a, b, we have

$$\mu([a,b]) = -\frac{1}{\pi}\lim_{y\searrow 0}\operatorname{Im}\int_{[a,b]} G_\mu(x+iy)dx.$$

Especially, if $G_\mu(z)$ extends to a continuous function on $\mathbb{C}^+ \cup I$ for an open interval $I \subset \mathbb{R}$, then the distribution μ has continuous derivative $f_\mu = d\mu/dx$ with respect to the Lebesgue measure dx on I, and we obtain $f_\mu(x)$ by

$$f_\mu(x) = -\frac{1}{\pi}\lim_{y\searrow 0}\operatorname{Im} G_\mu(x+iy), \quad x \in I.$$

Atoms of μ may be computed by the formula

$$\mu(\{a\}) = \lim_{z\to a,\, z\in\mathbb{C}^+}(z-a)G_\mu(z) = \lim_{y\searrow 0} y\, G_\mu(a+iy), \quad a \in \mathbb{R}.$$

2.2. Boolean case. In this paper, the maps $z \mapsto z^p$ and $z \mapsto \log z$ always denote the principal values for $z \in \mathbb{C}\backslash(-\infty, 0]$. Correspondingly $\arg(z)$ is defined in $\mathbb{C}\backslash(-\infty, 0]$ so that it takes values in $(-\pi, \pi)$.

Definition 2. Let $\mathrm{b}_{\alpha,\rho}$ be a boolean stable law [SW97] characterized by the following.

(1) If $\alpha \in (0,1) \cup (1,2]$, then

$$F_{\mathrm{b}_{\alpha,\rho}}(z) = z + e^{i\rho\alpha\pi}z^{1-\alpha}, \qquad z \in \mathbb{C}^+, \qquad \rho \in [0,1] \cap [1-1/\alpha, 1/\alpha].$$

(2) If $\alpha = 1$, then

$$F_{\mathrm{b}_{\alpha,\rho}}(z) = z + 2\rho i - \frac{2(2\rho-1)}{\pi}\log z, \qquad z \in \mathbb{C}^+, \qquad \rho \in [0,1].$$

Remark 3. The case $\alpha = 1$ includes non strictly stable distributions which were considered in [AH14]. The parametrization for $\alpha > 1$ follows that in [HK], not the one in [BP99], in order to respect the correspondence to the parametrization of classical stable laws in the Zolotarev book [Z86].

In the case $\alpha \in (0,1) \cup (1,2]$, for simplicity we also use a parameter θ, instead of ρ, defined by

$$\theta = \rho\alpha\pi. \tag{2.1}$$

The range of θ is $[0, \alpha\pi]$ for $\alpha \in (0,1)$ and $[(\alpha-1)\pi, \pi]$ for $\alpha \in (1,2]$. We will often use the following functions:

$$B_{\alpha,\rho}(x) = \frac{\sin\theta}{\pi}\frac{x^{\alpha-1}}{x^{2\alpha} + 2x^\alpha\cos\theta + 1}, \quad x > 0, \quad \alpha \in (0,1)\cup(1,2), \tag{2.2}$$

$$B_{1,\rho}(x) = \frac{2\rho}{\pi}\frac{1}{(x - \frac{2(2\rho-1)}{\pi}\log x)^2 + 4\rho^2}, \quad x > 0. \tag{2.3}$$

The probability measure $\mathrm{b}_{\alpha,\rho}$ is described as follows.

Proposition 4. *The Boolean stable distributions are as follows.*

(1) *If* $\alpha \in (0,1)$ *and* $\rho \in (0,1)$, *then*

$$\mathrm{b}_{\alpha,\rho}(dx) = B_{\alpha,\rho}(x)1_{(0,\infty)}(x)\,dx + B_{\alpha,1-\rho}(-x)1_{(-\infty,0)}(x)\,dx.$$

(2) *If* $\alpha \in (0,1)$ *and* $\rho \in \{0,1\}$, *then*

$$\mathrm{b}_{\alpha,0}(dx) = B_{\alpha,1}(-x)1_{(-\infty,0)}(x)\,dx,$$
$$\mathrm{b}_{\alpha,1}(dx) = B_{\alpha,1}(x)1_{(0,\infty)}(x)\,dx.$$

(3) *If* $\alpha = 1$ *and* $\rho \in (0,1)$, *then*

$$\mathrm{b}_{1,\rho}(dx) = B_{1,\rho}(x)1_{(0,\infty)}(x)\,dx + B_{1,1-\rho}(-x)1_{(-\infty,0)}(x)\,dx.$$

The case $\rho = \frac{1}{2}$ *is the Cauchy distribution*

$$\mathrm{b}_{1,1/2}(dx) = \frac{1}{\pi(x^2+1)}1_{\mathbb{R}}(x)\,dx.$$

(4) *If* $\alpha = 1$ *and* $\rho \in \{0,1\}$, *then*

$$\mathrm{b}_{1,0}(dx) = B_{1,1}(-x)1_{(-\infty,0)}(x)\,dx + \frac{u_+(0)}{u_+(0)+2/\pi}\delta_{u_+(0)},$$

$$\mathrm{b}_{1,1}(dx) = B_{1,1}(x)1_{(0,\infty)}(x)\,dx + \frac{u_+(0)}{u_+(0)+2/\pi}\delta_{-u_+(0)},$$

where $u_+(0) = 0.4745\ldots$ *is the unique solution* u *of the equation* $\pi u + 2\log u = 0$, $u \in (0,\infty)$.

(5) *If* $\alpha \in (1,2)$ *and* $\rho \in (1-1/\alpha, 1/\alpha)$, *then*

$$\mathrm{b}_{\alpha,\rho}(dx) = B_{\alpha,\rho}(x)1_{(0,\infty)}(x)\,dx + B_{\alpha,1-\rho}(-x)1_{(-\infty,0)}(x)\,dx.$$

(6) *If* $\alpha \in (1,2)$ *and* $\rho \in \{1-1/\alpha, 1/\alpha\}$, *then*

$$\mathrm{b}_{\alpha,1/\alpha}(dx) = B_{\alpha,1-1/\alpha}(-x)1_{(-\infty,0)}(x)\,dx + \frac{1}{\alpha}\delta_1,$$

$$\mathrm{b}_{\alpha,1-1/\alpha}(dx) = B_{\alpha,1-1/\alpha}(x)1_{(0,\infty)}(x)\,dx + \frac{1}{\alpha}\delta_{-1}.$$

(7) *If* $\alpha = 2$ *and* $\rho = 1/2$, *then*

$$\mathrm{b}_{2,1/2} = \frac{1}{2}(\delta_{-1} + \delta_1).$$

For each $\alpha \in (0,2)$, *the replacement* $\rho \mapsto 1-\rho$ *gives the reflection of the measure with respect to* 0. *Here the reflection* $\tilde{\mu}$ *of a measure* μ *with respect to* 0 *means* $\tilde{\mu}(B) = \mu(-B)$ *for any Borel set* B *on* $\mathbb{R}$, *where* $-B$ *is the set* $\{x \in \mathbb{R} : -x \in B\}$.

Proof. We can find the case $\alpha = 2$ on page 273 in [SW97].

(i) The case $\alpha \neq 1$. We write $G(z), F(z)$ instead of $G_{\mathrm{b}_{\alpha,\rho}}(z), F_{\mathrm{b}_{\alpha,\rho}}(z)$, respectively. If $x > 0$, then $\lim_{y \searrow 0}(x+iy)^{1-\alpha}$ is simply $x^{1-\alpha}$, while if $x < 0$, then the argument of $x + iy$ approaches to π, and so $\lim_{y\searrow 0}(x+iy)^{1-\alpha} = (-x)^{1-\alpha}e^{i\pi(1-\alpha)} = (-x)^{1-\alpha}e^{i\pi(1-\alpha)}$. So we have for $x > 0$ that

$$\lim_{y\searrow 0} G(x+iy) = \frac{1}{x + e^{i\theta}x^{1-\alpha}} = \frac{x^{\alpha-1}}{x^\alpha + e^{i\theta}} = \frac{x^{\alpha-1}(x^\alpha + \cos\theta - i\sin\theta)}{(x^\alpha + \cos\theta)^2 + \sin^2\theta}$$

and for $x < 0$

$$\begin{aligned}\lim_{y\searrow 0} G(x+iy) &= \frac{1}{x - e^{i(\theta-\alpha\pi)}(-x)^{1-\alpha}} = -\frac{(-x)^{\alpha-1}}{(-x)^\alpha + e^{i(\theta-\alpha\pi)}} \\ &= -\frac{(-x)^{\alpha-1}(x + \cos(\theta-\alpha\pi) - i\sin(\theta-\alpha\pi))}{((-x)^\alpha + \cos(\theta-\alpha\pi))^2 + \sin^2(\theta-\alpha\pi)}.\end{aligned}$$

Taking the imaginary part of these expressions, we obtain

$$\begin{aligned}(2.4)\qquad &-\frac{1}{\pi}\lim_{y\searrow 0}\operatorname{Im} G(x+iy) \\ &= \begin{cases} \dfrac{\sin\theta}{\pi}\dfrac{x^{\alpha-1}}{x^{2\alpha} + 2x^\alpha\cos\theta + 1}, & x > 0, \\ \dfrac{\sin(\alpha\pi-\theta)}{\pi}\dfrac{(-x)^{\alpha-1}}{(-x)^{2\alpha} + 2(-x)^\alpha\cos(\alpha\pi-\theta) + 1}, & x < 0. \end{cases}\end{aligned}$$

(i-1) The case $\alpha \in (0,1)$. If $\alpha \in (0,1)$ and $\rho \in [0,1]$, the function F extends continuously to $\mathbb{C}^+ \cup \mathbb{R}$ and does not have a zero except at $z = 0$ in $\mathbb{C}^+ \cup \mathbb{R}$, but since $\lim_{y\searrow 0} yG(iy) = 0$, there is no atom at 0. The above argument shows (1) and (2).

(i-2) The case $\alpha \in (1,2)$. First note that the measure does not have an atom at 0 since $\lim_{y\searrow 0} G(iy) = 0$. If $\theta \in ((\alpha-1)\pi, \pi)$, then the same computation (2.4) is valid and so we obtain (5). If $\theta = (\alpha-1)\pi$ (or equivalently $\rho = 1 - 1/\alpha$), then (2.4) is valid for $x > 0$. Now note that $z \mapsto F(z+i0) = z + e^{i(\alpha-1)\pi}z^{1-\alpha} = z + (-z)^{1-\alpha}$ has a zero $z = -1$ in $(-\infty, 0)$. Hence G extends to a continuous function on $\mathbb{C}^+ \cup \mathbb{R}\backslash\{-1\}$, and we have $\operatorname{Im} G(x+i0) = 0$ for $x < 0$, $x \neq -1$. There is an atom at -1 with weight $1/\alpha$ since

$$\lim_{z\to -1, z\in\mathbb{C}^+}(z+1)G(z) = \lim_{z\to -1, z\in\mathbb{C}^+}\frac{1}{z}\cdot\frac{1-(-z)}{1-(-z)^{-\alpha}} = \frac{1}{\alpha}.$$

This implies the second part of (6). The first part is similar.

(ii) The case $\alpha = 1$. We can easily see that $\lim_{y\searrow 0} yG(iy) = 0$ and so there is no atom at 0. Assume first that $\rho \in (0,1)$. For $x > 0$, we have that

$$\begin{aligned}(2.5)\qquad \lim_{y\searrow 0}\operatorname{Im} G(x+iy) &= \lim_{y\searrow 0}\operatorname{Im}\frac{1}{(x+iy)+2\rho i-\frac{2(2\rho-1)}{\pi}\log(x+iy)}\\ &= \operatorname{Im}\frac{1}{x-\frac{2(2\rho-1)}{\pi}\log x+2\rho i}\\ &= \operatorname{Im}\frac{x-\frac{2(2\rho-1)}{\pi}\log x-2\rho i}{(x-\frac{2(2\rho-1)}{\pi}\log x)^2+4\rho^2}\\ &= \frac{-2\rho}{(x-\frac{2(2\rho-1)}{\pi}\log x)^2+4\rho^2}\end{aligned}$$

and so we get $-\frac{1}{\pi}\lim_{y\searrow 0}\operatorname{Im} G(x+iy) = B_{1,\rho}(x)$. For $x<0$, note that $\log(x+i0) = \log(-x)+i\pi$ and then, similarly to (2.5), we get $-\frac{1}{\pi}\lim_{y\searrow 0}\operatorname{Im} G(x+iy) = B_{1,1-\rho}(-x)$. Since G extends to a continuous function on $\mathbb{C}^+\cup\mathbb{R}\backslash\{0\}$, we get the formula (3) by the Stieltjes inversion.

If $\rho=1$, then the computation for $x>0$ is the same as (2.5). For $x<0$, note that

$$F(x+i0) = x+2i-\frac{2}{\pi}(\log(-x)+i\pi) = -((-x)+\frac{2}{\pi}\log(-x)),$$

which has the unique zero at $x=-u_+(0)$. So G extends to a continuous function on $\mathbb{C}^+\cup\mathbb{R}\backslash\{0,-u_+(0)\}$ and $\operatorname{Im} G(x+i0)=0$ for $x<0, x\neq -u_+(0)$. We have the series expansion

$$x-\frac{2}{\pi}\log(-x) = a(x+u_+(0))+b(x+u_+(0))^2+\cdots.$$

Then $a = \frac{d}{dx}\Big|_{x=-u_+(0)}(x-\frac{2}{\pi}\log(-x)) = \frac{u_+(0)+2/\pi}{u_+(0)}$. The weight of the atom at $-u_+(0)$ is equal to $1/a$, and so we have the second part of (4). The first part is similar.

The reflection property is clear from the formulas. ∎

Theorem 5. *Let $\alpha_0 = 0.7364\ldots$ be the unique solution of the equation $\sin(\pi\alpha)=\alpha$, $\alpha\in(0,1)$. Let*

$$(2.6)\qquad x_+ := \left(\frac{-\cos\rho\alpha\pi+\sqrt{\alpha^2-\sin^2\rho\alpha\pi}}{1+\alpha}\right)^{1/\alpha},$$

$$(2.7)\qquad x_- := -\left(\frac{-\cos((1-\rho)\alpha\pi)+\sqrt{\alpha^2-\sin^2((1-\rho)\alpha\pi)}}{1+\alpha}\right)^{1/\alpha}.$$

Let $u_+ = u_+(\rho)$ be the unique solution x of the equation $\pi x+2(1-2\rho)\log x = 0$, $x\in(0,\infty)$, $\rho\in[0,\frac{1}{2})$, and let $u_- = u_-(\rho) := -u_+(1-\rho)$, $\rho\in(\frac{1}{2},1]$.

(1) *If* $\alpha \in (0, \alpha_0]$, *then* $\mathrm{b}_{\alpha,\rho}$ *is unimodal with mode 0.*
(2) *If* $\alpha \in (\alpha_0, 1)$, *then there are sub cases.*
  (a) *If* $\rho \in [0, \arcsin(\alpha)/(\alpha\pi) + 1 - 1/\alpha)$, *then* $\mathrm{b}_{\alpha,\rho}$ *is bimodal with modes 0 and* x_+.
  (b) *If* $\rho \in [\arcsin(\alpha)/(\alpha\pi) + 1 - 1/\alpha, 1/\alpha - \arcsin(\alpha)/(\alpha\pi)]$, *then* $\mathrm{b}_{\alpha,\rho}$ *is unimodal with mode 0.*
  (c) *If* $\rho \in (1/\alpha - \arcsin(\alpha)/(\alpha\pi), 1]$, *then* $\mathrm{b}_{\alpha,\rho}$ *is bimodal with modes* x_- *and* 0.
(3) *If* $\alpha = 1$, *then there are sub cases.*
  (a) *If* $\rho \in [0, \frac{1}{2})$, *then* $\mathrm{b}_{\alpha,\rho}$ *is bimodal with modes* $-\frac{2(1-2\rho)}{\pi}$ *and* u_+. *If* $\rho = 0$, *then the mode at* u_+ *is an atom.*
  (b) *If* $\rho = \frac{1}{2}$, *then* $\mathrm{b}_{\alpha,\rho}$ *is unimodal with mode 0.*
  (c) *If* $\rho \in (\frac{1}{2}, 1]$, *then* $\mathrm{b}_{\alpha,\rho}$ *is bimodal with modes* u_- *and* $\frac{2(2\rho-1)}{\pi}$. *If* $\rho = 1$, *then the mode at* u_- *is an atom.*
(4) *If* $\alpha \in (1, 2]$, *then* $\mathrm{b}_{\alpha,\rho}$ *is bimodal with modes* x_- *and* x_+. *If* $\rho = 1 - 1/\alpha$, *then* $\mathrm{b}_{\alpha,\rho}$ *has an atom at* $x_- = -1$. *If* $\rho = 1/\alpha$, *then* $\mathrm{b}_{\alpha,\rho}$ *has an atom at* $x_+ = 1$.

Proof. (i) The case $\alpha \in (0, \alpha_0]$. Note that 0 is a mode since the density diverges to ∞ at $x = 0$. We can easily compute

$$\begin{aligned} (2.8) \qquad & \frac{\partial}{\partial x} B_{\alpha,\rho}(x) \\ & = -\frac{x^{\alpha-2}\sin\theta}{\pi} \cdot \frac{(1+\alpha)(x^\alpha + \frac{1}{1+\alpha}\cos\theta)^2 + \frac{1}{1+\alpha}(\sin^2\theta - \alpha^2)}{(x^{2\alpha} + 2x^\alpha\cos\theta + 1)^2}, \quad x > 0. \end{aligned}$$

Let

$$f(x) := (1+\alpha)\left(x^\alpha + \frac{1}{1+\alpha}\cos\theta\right)^2 + \frac{1}{1+\alpha}(\sin^2\theta - \alpha^2).$$

Note that $f(0) = 1 - \alpha > 0$. Since $f(x)$ is a polynomial on x^α of degree 2, it is easy to see that if $\theta \in [0, \frac{\pi}{2}]$, then $\cos\theta \geq 0$ and $f(x)$ does not have a zero in $(0, \infty)$. If $\theta \in (\frac{\pi}{2}, \alpha\pi)$, then $f(x)$ attains a local minimal value at $x = -\frac{1}{1+\alpha}\cos\theta$, but now $\sin^2\theta - \alpha^2 \geq \sin^2(\alpha\pi) - \alpha^2 \geq 0$ for $\alpha \leq \alpha_0$. Hence $f(x) \geq 0$ for $x > 0$ and the map $x \mapsto B_{\alpha,\rho}(x)$ is strictly decreasing on $(0, \infty)$ for any $\theta \in (0, \alpha\pi]$. By the reflection property (see the last statement of Proposition 4) the density is strictly increasing on $(-\infty, 0)$ for any $\theta \in [0, \alpha\pi)$ and hence $\mathrm{b}_{\alpha,\rho}$ is unimodal, the conclusion (1).

(ii) The case $\alpha \in (\alpha_0, 1)$. In this case 0 is still a mode of $\mathrm{b}_{\alpha,\rho}$. From (2.8), we have that

(2.9) $x \mapsto B_{\alpha,\rho}(x)$ takes a local maximum in $(0,\infty)$
$$\Leftrightarrow \theta \in \left(\frac{\pi}{2}, \alpha\pi\right] \text{ and } \sin\theta < \alpha$$
$$\Leftrightarrow \theta \in (\pi - \arcsin(\alpha), \alpha\pi],$$

and if this condition is satisfied, then the local maximum is attained at $x = x_+$. By reflection, it holds that

(2.10) $x \mapsto B_{\alpha,1-\rho}(-x)$ takes a local maximum in $(-\infty, 0)$
$$\Leftrightarrow \theta \in [0, \arcsin(\alpha) - (1-\alpha)\pi),$$

and if this condition is satisfied, the local maximum is attained at $x = x_-$. The two conditions (2.9) and (2.10) cannot be satisfied for the same θ. Hence we have the conclusion (2).

(iii) The case $\alpha \in (1,2)$. Note that 0 is not a mode of $\mathrm{b}_{\alpha,\rho}$ since the density function takes 0 at $x = 0$. Since now $f(0) = 1 - \alpha < 0$, we conclude that the map $x \mapsto B_{\alpha,\rho}(x)$ in $(0,\infty)$ takes a unique local maximum at $x = x_+$ for any $\theta \in [(\alpha-1)\pi, \pi)$. By reflection and by Proposition 4(5)–(7), the conclusion follows.

(iv) The case $\alpha = 1$, $\rho \neq \frac{1}{2}$. Note that the density takes 0 at $x = 0$. We have

$$\frac{\partial}{\partial x} B_{1,\rho}(x) = -\frac{4\rho\left(x - \frac{2(2\rho-1)}{\pi}\right)\left(x - \frac{2(2\rho-1)}{\pi}\log x\right)}{\pi x\left((x - \frac{2(2\rho-1)}{\pi}\log x)^2 + 4\rho^2\right)^2}, \qquad x > 0, \tag{2.11}$$

and then we obtain zeros of $\frac{\partial}{\partial x} B_{1,\rho}(x)$. The conclusion follows from increase and decrease of $B_{1,\rho}(x)$. ∎

Remark 6. If $\alpha \in (0,1)$, the density $\mathrm{b}_{\alpha,\rho}$ diverges at 0. So $x = 0$ always becomes a mode. In addition, at $x = 0$ the density cannot be differentiable. On the other hand, if $\alpha \in (\alpha_0, 1)$ and $\theta \in [0, \arcsin(\alpha) - (1-\alpha)\pi]$, at x_+ the density is differentiable.

2.3. Monotone case. As before, the map $z \mapsto z^p$ denotes the principal value (defined in $\mathbb{C}\backslash(-\infty, 0]$). We also use a different branch

$$z \mapsto z^p_{(0,2\pi)} = e^{p\log|z| + ip\arg_{(0,2\pi)} z}, \qquad z \in \mathbb{C}\backslash[0,\infty),$$

where $\arg_{(0,2\pi)} z$ is defined continuously so that $\arg_{(0,2\pi)} z \in (0, 2\pi)$.

Let $\mathrm{m}_{\alpha,\rho}$ be a monotone strictly stable law [H10a] characterized by

Definition 7. (1) If $\alpha \in (0,1) \cup (1,2]$, then

$$F_{\mathrm{m}_{\alpha,\rho}}(z) = (z^\alpha + e^{i\rho\alpha\pi})^{1/\alpha}_{(0,2\pi)}, \qquad z \in \mathbb{C}^+, \qquad \rho \in [0,1] \cap [1 - 1/\alpha, 1/\alpha].$$

(2) If $\alpha = 1$, then we only consider $\rho = \frac{1}{2}$:

$$F_{\mathrm{m}_{1,1/2}}(z) = z + i, \qquad z \in \mathbb{C}^+.$$

Remark 8. (a) If $\alpha \in (0,1)$, then $z^\alpha + e^{i\rho\alpha\pi}$ stays in $\mathbb{C}^+$ for $z \in \mathbb{C}^+$, and hence we may define $\mathrm{m}_{\alpha,\rho}$ by $F_{\mathrm{m}_{\alpha,\rho}}(z) = (z^\alpha + e^{i\rho\alpha\pi})^{1/\alpha}$. However for $\alpha \in (1,2]$, $z^\alpha + e^{i\rho\alpha\pi}$ may be in $\mathbb{C}^-$, and so we need to use the branch $(\cdot)^{1/\alpha}_{(0,2\pi)}$ to define $F_{\mathrm{m}_{\alpha,\rho}}$ analytically (or continuously) in $\mathbb{C}^+$.

(b) The above definition does not respect the monotone-classical Bercovici–Pata bijection. If we hope to let $\mathrm{m}_{\alpha,\rho}$ correspond to $\mathrm{b}_{\alpha,\rho}$ regarding the monotone-classical Bercovici–Pata bijection, then we have to consider $\mathbf{D}_{\alpha^{1/\alpha}}(\mathrm{m}_{\alpha,\rho})$, which is the induced measure of $\mathrm{m}_{\alpha,\rho}$ by the map $x \mapsto \alpha^{1/\alpha}x$. About the Bercovici–Pata bijection between monotone and classical infinitely divisible distributions, see [H10b].

(c) All the above distributions are strictly stable. Non strictly stable distributions are not defined in the literature, and so we do not consider the non-symmetric case in $\alpha = 1$.

We will describe the probability measure $\mathrm{m}_{\alpha,\rho}$. Let $\theta = \theta(\alpha,\rho)$ be (2.1) as before. For $\alpha \in (0,2], \theta \in (0,\pi)$, let

$$M_{\alpha,\rho}(x) = \frac{\sin[\frac{1}{\alpha}\varphi(x^\alpha,\theta)]}{\pi(x^{2\alpha} + 2x^\alpha\cos\theta + 1)^{1/(2\alpha)}}, \qquad x > 0,$$

where

$$\varphi(x,\theta) = \begin{cases} \arctan(\frac{\sin\theta}{x+\cos\theta}), & x > -\cos\theta, \\ \frac{\pi}{2}, & x = -\cos\theta, \\ \arctan(\frac{\sin\theta}{x+\cos\theta}) + \pi, & 0 < x < -\cos\theta. \end{cases}$$

The second and the third cases do not appear if $\cos\theta \geq 0$. We can also write $\varphi(x,\theta) = \arg(x + e^{i\theta})$.

Proposition 9. *The strictly monotone stable distributions are as follows.*

(1) *If $\alpha \in (0,1)$ and $\rho \in (0,1)$, then*

$$\mathrm{m}_{\alpha,\rho}(dx) = M_{\alpha,\rho}(x)1_{(0,\infty)}(x)\,dx + M_{\alpha,1-\rho}(-x)1_{(-\infty,0)}(x)\,dx.$$

(2) *If $\alpha \in (0,1)$ and $\rho \in \{0,1\}$, then*

$$\mathrm{m}_{\alpha,0}(dx) = M_{\alpha,1}(-x)1_{(-\infty,0)}(x)\,dx,$$
$$\mathrm{m}_{\alpha,1}(dx) = M_{\alpha,1}(x)1_{(0,\infty)}(x)\,dx.$$

(3) *If $\alpha = 1$ and $\rho = \frac{1}{2}$, then*

$$\mathrm{m}_{1,1/2}(dx) = \frac{1}{\pi(x^2+1)}1_{\mathbb{R}}(x)\,dx.$$

(4) *If* $\alpha \in (1,2)$ *and* $\rho \in (1-1/\alpha, 1/\alpha)$, *then*

$$\mathrm{m}_{\alpha,\rho}(dx) = M_{\alpha,\rho}(x)1_{(0,\infty)}(x)\,dx + M_{\alpha,1-\rho}(-x)1_{(-\infty,0)}(x)\,dx.$$

(5) *If* $\alpha \in (1,2)$ *and* $\rho \in \{1-1/\alpha, 1/\alpha\}$, *then*

$$\mathrm{m}_{\alpha,1/\alpha}(dx) = M_{\alpha,1-1/\alpha}(-x)1_{(-\infty,0)}(x)\,dx + \frac{\sin(\pi/\alpha)}{\pi(1-x^\alpha)^{1/\alpha}}1_{[0,1)}(x)\,dx,$$

$$\mathrm{m}_{\alpha,1-1/\alpha}(dx) = M_{\alpha,1-1/\alpha}(x)1_{(0,\infty)}(x)\,dx + \frac{\sin(\pi/\alpha)}{\pi(1-(-x)^\alpha)^{1/\alpha}}1_{(-1,0]}(x)\,dx.$$

(6) *If* $\alpha = 2$ *and* $\rho = 1/2$, *then*

$$\mathrm{m}_{2,1/2}(dx) = \frac{1}{\pi\sqrt{1-x^2}}1_{(-1,1)}(x)\,dx.$$

They are all absolutely continuous with respect to the Lebesgue measure. In the case $\alpha \in (1,2], \rho \in \{1-1/\alpha, 1/\alpha\}$, *the density function diverges to infinity at the edge of the support, but in the other cases the density function is either continuous on* $\mathbb{R}$, *or extends to a continuous function on* $\mathbb{R}$ *(if the support is not* $\mathbb{R}$*). The density function is real analytic except at the edge of the support and at 0. The replacement* $\rho \mapsto 1-\rho$ *gives the reflection of the measure with respect to* $x = 0$.

Proof. Let $G(z), F(z)$ denote $G_{\mathrm{m}_{\alpha,\rho}}(z), F_{\mathrm{m}_{\alpha,\rho}}(z)$, respectively. First, note that $\mathrm{m}_{\alpha,\rho}$ does not have an atom at $x = 0$ since $\lim_{y\searrow 0} F(iy) \neq 0$.

In the cases $\alpha = 1$ and $\alpha = 2$, see [M00, Example 4.8].

Proof of the case (1). The case $\alpha \in (0,1)$, $\rho \in (0,1)$. For $x > 0$, we have that

$$\lim_{y\searrow 0}(x+iy)^\alpha + e^{i\theta} = x^\alpha + \cos\theta + i\sin\theta \in \mathbb{C}^+, \tag{2.12}$$

which equals $r(x^\alpha,\theta)e^{i\varphi(x^\alpha,\theta)}$, where

$$r(x,\theta) = \sqrt{x^2 + 2x\cos\theta + 1}.$$

So we get

$$\begin{aligned} -\frac{1}{\pi}\lim_{y\searrow 0}\operatorname{Im} G(x+iy) \\ &= -\frac{1}{\pi}\operatorname{Im}\frac{1}{(x^\alpha + e^{i\theta})^{1/\alpha}_{(0,2\pi)}} = -\frac{1}{\pi}\operatorname{Im} r(x^\alpha,\theta)^{-1/\alpha}e^{-i\varphi(x^\alpha,\theta)/\alpha} \\ &= \frac{1}{\pi}r(x^\alpha,\theta)^{-1/\alpha}\sin\left(\frac{1}{\alpha}\varphi(x^\alpha,\theta)\right), \quad x > 0, \end{aligned} \tag{2.13}$$

which is strictly positive since now $\theta \in (0,\pi)$. This implies that $\mathrm{m}_{\alpha,\rho}$ is absolutely continuous on $(0,\infty)$ with respect to the Lebesgue measure, and the density function is given by $M_{\alpha,\rho}(x)$.

Note that, for $x<0$, it holds that

$$\lim_{y\searrow 0}(x+iy)^\alpha + e^{i\theta} = (-x)^\alpha e^{i\alpha\pi} + e^{i\theta} = e^{i\alpha\pi}((-x)^\alpha + e^{i(\theta-\alpha\pi)}). \tag{2.14}$$

Now $\theta-\alpha\pi\in(-\pi,0)$. Since $\theta-\alpha\pi<\arg((-x)^\alpha+e^{i(\theta-\alpha\pi)})<0$, the following can be justified:

$$\begin{aligned}\left(e^{i\alpha\pi}((-x)^\alpha+e^{i(\theta-\alpha\pi)})\right)^{1/\alpha}_{(0,2\pi)} &= \left(e^{i\alpha\pi}\right)^{1/\alpha}\left((-x)^\alpha+e^{i(\theta-\alpha\pi)}\right)^{1/\alpha}\\ &= -\left((-x)^\alpha+e^{i(\theta-\alpha\pi)}\right)^{1/\alpha}, \qquad x<0.\end{aligned} \tag{2.15}$$

We can show that

$$(-x)^\alpha + e^{i(\theta-\alpha\pi)} = r((-x)^\alpha, \alpha\pi-\theta)e^{-i\varphi((-x)^\alpha,\alpha\pi-\theta)}, \tag{2.16}$$

and so we get (1) from a computation similar to (2.13).

Proof of the case (2). The case $\alpha\in(0,1)$, $\rho=1$. The formula (2.13) still holds for $x>0$. For $x<0$, we have

$$\begin{aligned}F(x+i0) &= ((x+i0)^\alpha + e^{i\alpha\pi})^{1/\alpha}_{(0,2\pi)} = ((-x)^\alpha e^{i\alpha\pi} + e^{i\alpha\pi})^{1/\alpha}\\ &= -((-x)^\alpha+1)^{1/\alpha} < 0,\end{aligned}$$

and hence $\mathrm{m}_{\alpha,1}$ does not have support on $(-\infty,0)$. We can prove the case $\rho=0$ similarly. So we get (2).

Proof of the case (4). The case $\alpha\in(1,2)$, $\rho\in(1-1/\alpha,1/\alpha)$. The formula (2.13) still holds for $x>0$. The computation of the density function for $x<0$ is now delicate. For $x<0$, the formula (2.14) still holds true. Note that now again $\theta-\alpha\pi\in(-\pi,0)$. By $-\pi<\theta-\alpha\pi<\arg((-x)^\alpha+e^{i(\theta-\alpha\pi)})<0$ and $\arg_{(0,2\pi)}e^{i\alpha\pi}\in(\pi,2\pi)$, the formula

$$\begin{aligned}\left(e^{i\alpha\pi}((-x)^\alpha+e^{i(\theta-\alpha\pi)})\right)^{1/\alpha}_{(0,2\pi)} &= \left(e^{i\alpha\pi}\right)^{1/\alpha}_{(0,2\pi)}\left((-x)^\alpha+e^{i(\theta-\alpha\pi)}\right)^{1/\alpha}\\ &= -\left((-x)^\alpha+e^{i(\theta-\alpha\pi)}\right)^{1/\alpha}, \qquad x<0,\end{aligned} \tag{2.17}$$

is valid. A delicate point is that we should use the principal value in the last expression, not the branch $(\cdot)^{1/\alpha}_{(0,2\pi)}$. Then the formula (2.16) still holds and then the Stieltjes inversion formula implies (4).

Proof of the case (5). The case $\alpha\in(1,2)$, $\rho=1-1/\alpha$. For $x>0$, the computation (2.13) holds without any change. For $x<-1$, we have

$$F(x+i0) = ((x+i0)^\alpha + e^{i(\alpha-1)\pi})^{1/\alpha}_{(0,2\pi)} = ((-x)^\alpha e^{i\alpha\pi} - e^{i\alpha\pi})^{1/\alpha}_{(0,2\pi)} \tag{2.18}$$
$$= -((-x)^\alpha - 1)^{1/\alpha} < 0,$$

and so $\mathrm{m}_{\alpha,1}$ does not have support on $(-\infty,-1)$. We can also show that $\lim_{z\to-1,z\in\mathbb{C}^+}(z+1)G(z+1)=0$, which implies that there is no atom at -1. For $x\in(-1,0)$, by using part of (2.18), we have

$$F(x+i0) = \left(e^{i(\alpha-1)\pi}(1-(-x)^\alpha)\right)^{1/\alpha} = e^{i\frac{\alpha-1}{\alpha}\pi}(1-(-x)^\alpha)^{1/\alpha},$$

and then by the Stieltjes inversion formula, we get its probability density function. We can prove the case $\rho = 1/\alpha$ similarly. So we obtain (5).

The remaining statements can be proved as follows.

The reflection property of $\rho\mapsto 1-\rho$. This is clear.

The continuity of the density function at $x=0$. For $\alpha\in(0,1)\cup(1,2)$ and $\theta\in(0,\pi)$, we can show that $\lim_{x\searrow 0}\varphi(x,\theta)=\theta$ and hence

$$\lim_{x\searrow 0} M_{\alpha,\rho}(x) = \frac{\sin(\rho\pi)}{\pi} = \frac{\sin((1-\rho)\pi)}{\pi} = \lim_{x\searrow 0} M_{\alpha,1-\rho}(x).$$

This shows the continuity.

The remaining statements are easy consequences of (1)–(6). ∎

Theorem 10. *Let*

$$v_+ := \left(\frac{\sin(\alpha\pi(\rho-\frac{1}{1+\alpha}))}{\sin(\frac{\alpha\pi}{1+\alpha})}\right)^{1/\alpha}, \quad v_- := -\left(\frac{\sin(\alpha\pi(\frac{\alpha}{1+\alpha}-\rho))}{\sin(\frac{\alpha\pi}{1+\alpha})}\right)^{1/\alpha}. \tag{2.19}$$

(1) *If* $\alpha\in(0,1)$, *then* $\mathrm{m}_{\alpha,\rho}$ *is unimodal. The mode is described as follows.*
  (a) *If* $\rho\in[0,\frac{\alpha}{1+\alpha}]$, *then the mode is* v_-.
  (b) *If* $\rho\in[\frac{\alpha}{1+\alpha},\frac{1}{1+\alpha}]$, *then the mode is* 0.
  (c) *If* $\rho\in[\frac{1}{1+\alpha},1]$, *then the mode is* v_+.

(2) *If* $\alpha=1$ *and* $\rho=\frac{1}{2}$, *then* $\mathrm{m}_{1,1/2}$ *is unimodal with mode 0.*

(3) *If* $\alpha\in(1,\frac{1+\sqrt{5}}{2}]$, *then there are sub cases.*
  (a) *If* $\rho\in[1-\frac{1}{\alpha},\frac{1}{1+\alpha}]$, *then* $\mathrm{m}_{\alpha,\rho}$ *is unimodal with mode* v_-.
  (b) *If* $\rho\in(\frac{1}{1+\alpha},\frac{\alpha}{1+\alpha})$, *then* $\mathrm{m}_{\alpha,\rho}$ *is bimodal with modes* v_- *and* v_+.
  (c) *If* $\rho\in[\frac{\alpha}{1+\alpha},\frac{1}{\alpha}]$, *then* $\mathrm{m}_{\alpha,\rho}$ *is unimodal with mode* v_+.

(4) *If* $\alpha\in(\frac{1+\sqrt{5}}{2},2]$, *then* $\mathrm{m}_{\alpha,\rho}$ *is bimodal with modes* v_- *and* v_+.

Note that $\frac{1+\sqrt{5}}{2} = 1.6180\ldots$.

Proof. We assume that $\alpha\neq 1,2$.

(0) (Arguments valid for $\alpha\in(0,1)\cup(1,2), \rho\in(0,1)\cap(1-1/\alpha,1/\alpha)$) Let $p(x)$ be the density function of $\mathrm{m}_{\alpha,\rho}$ and let $q(x,\theta) := x^2+2(\cos\theta)x+1$.

The assumption $\rho \in (0,1) \cap (1-1/\alpha, 1/\alpha)$ implies $\theta \in (0, \alpha\pi)$. Then $\mathrm{m}_{\alpha,\rho}$ has the support $\mathbb{R}$ and $p(x) > 0$ in $\mathbb{R}$. We can prove that

$$p'(x) = \frac{\partial}{\partial x} M_{\alpha,\rho}(x) = -\frac{x^{\alpha-1}}{\pi q(x^\alpha,\theta)^{\frac{\alpha+1}{2\alpha}}} \sin\left(\frac{\alpha+1}{\alpha}\varphi(x^\alpha,\theta)\right), \quad x > 0. \tag{2.20}$$

Since $x \mapsto \varphi(x,\theta)$ is strictly decreasing, mapping $(0,\infty)$ onto $(0,\theta)$, the following equivalence holds true:

$$p'(x) \text{ changes the sign at a point } x \in (0,\infty) \Leftrightarrow \theta > \frac{\alpha\pi}{1+\alpha}. \tag{2.21}$$

Moreover, $\frac{\alpha+1}{\alpha}\theta \le (\alpha+1)\pi < 2\pi$ for $\alpha \in (0,1)$, and also $\frac{\alpha+1}{\alpha}\theta \le (1/\alpha+1)\pi < 2\pi$ for $\alpha \in (1,2)$, and so the sign of $p'(x)$ changes at most once in $(0,\infty)$. If the sign changes, the critical point is given by $x = v_+$.

For the density function on the negative line, it suffices to study $\frac{\partial}{\partial x} M_{\alpha,1-\rho}(x)$, $x > 0$, and it follows from (2.21) that

$$p'(x) \text{ changes the sign at a point } x \in (-\infty,0) \Leftrightarrow \theta < \frac{\alpha^2\pi}{1+\alpha}. \tag{2.22}$$

The sign changes at most once, and if it changes, the critical point is $x = -v_-$.

(i) The case $\alpha \in (0,1)$. The conditions (2.21) and (2.22) cannot be satisfied at the same time. Note that p may have a mode at 0. On the other hands, the sign of p' cannot be changed even number of times because p is a probability density function. Combining these arguments, the claim (1) follows for $\rho \in (0,1)$. The formula (2.20) holds also for $\theta = \alpha\pi$ and so the case $\rho = 1$ is covered. The case $\rho = 0$ is the reflection of $\rho = 1$.

(ii) The case $\alpha \in (1,2)$. If $\rho \in (1-1/\alpha, 1/\alpha)$, then by looking at the formula (2.20) and the fact $\lim_{x\searrow 0}\varphi(x,\theta) = \theta$, we have that $p'(+0) = 0$, and from the replacement $\theta \mapsto \alpha\pi - \theta$ we have $p'(-0) = 0$, and hence $p'(0) = 0$. It is also true that $p'(0) = 0$ for $\rho \in \{1-1/\alpha, 1/\alpha\}$.

(ii-1) The case $\alpha \in (1, \frac{1+\sqrt{5}}{2}]$. Note that $(\frac{\alpha\pi}{1+\alpha}, \frac{\alpha^2\pi}{1+\alpha}) \subset [(\alpha-1)\pi, \pi]$. If $\theta \in (\frac{\alpha\pi}{1+\alpha}, \frac{\alpha^2\pi}{1+\alpha})$, then p has two modes at $x = v_-, v_+$ from (2.21), (2.22). The density function p takes a local minimum at $x = 0$, and hence we get (3b). If $\theta \in ((\alpha-1)\pi, \frac{\alpha\pi}{1+\alpha}]$, then $p'(x) = 0$ only for $x = 0, v_-$. If we assume that the sign of p' changes at $x = 0$, p cannot be a probability density function. Thus p is unimodal with mode $x = v_-$. For $\theta = (\alpha-1)\pi$ (or equivalently $\rho = 1-1/\alpha$), by using the formula (2.20) for $x > 0$ and Proposition 9(5) for $x \in (-1,0)$, we can show that p is strictly decreasing in $(-1,\infty)$. Hence $\mathrm{m}_{\alpha,1}$ is unimodal with mode $-1 = v_-$. Thus we have (3a). (3c) is obtained by reflection.

(ii-2) The case $\alpha \in (\frac{1+\sqrt{5}}{2}, 2)$. Note that $[(\alpha-1)\pi, \pi] \subset (\frac{\alpha\pi}{1+\alpha}, \frac{\alpha^2\pi}{1+\alpha})$. So, if $\rho \in (1-1/\alpha, 1/\alpha)$, p' changes its sign at $x = v_+, v_-$, and also at $x = 0$. If

$\rho = 1 - 1/\alpha$, then p' changes its sign at $x = v_+$ and p is strictly decreasing in $(-1, 0)$. If we assume that the sign of p' does not change at $x = 0$, p cannot be a probability density function. Therefore, we find that p takes a local minimum at 0. The case $\rho = 1/\alpha$ follows by refection. Hence we proved (4). ∎

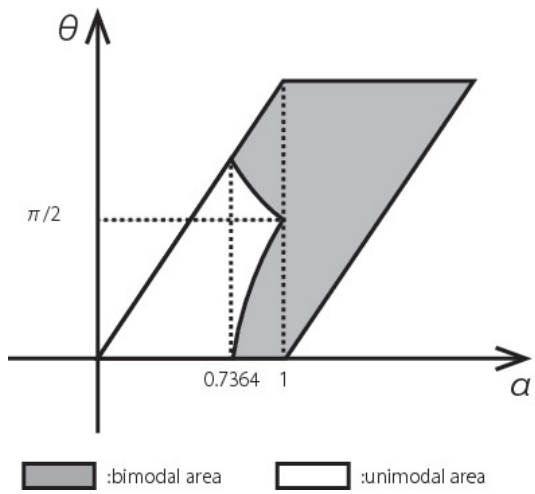

Fig. 1. Unimodality of boolean stable distributions

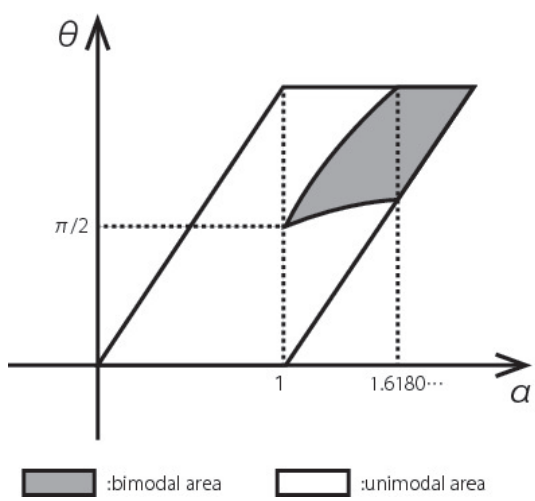

Fig. 2. Unimodality of monotone stable distributions

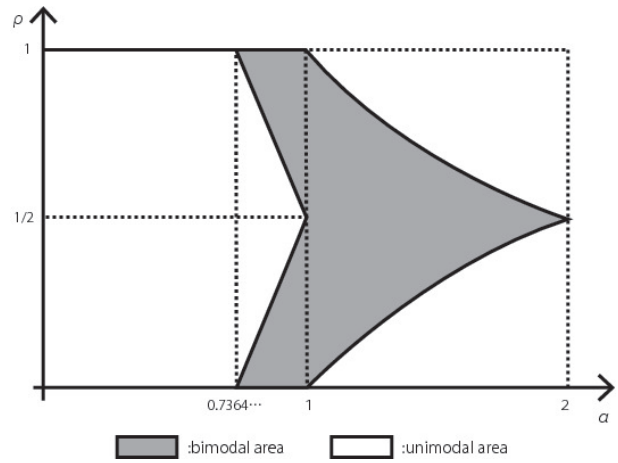

Fig. 3. Unimodality of boolean stable distributions

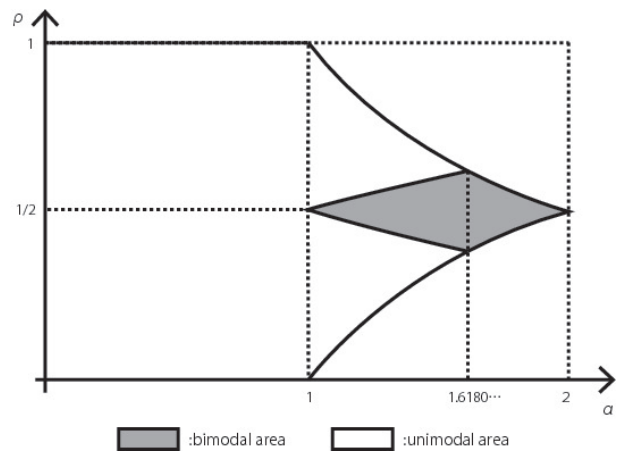

Fig. 4. Unimodality of monotone stable distributions

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