DE GRUYTER OPEN

DEMONSTRATIO MATHEMATICA
Vol. XLVIII No 3 2015

Rupali S. Jain, M. B. Dhakne

ON EXISTENCE OF SOLUTIONS OF IMPULSIVE NONLINEAR FUNCTIONAL NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITION

Communicated by E. Weber

Abstract. In the present paper, we investigate the existence, uniqueness and continuous dependence of mild solutions of an impulsive neutral integro-differential equations with nonlocal condition in Banach spaces. We use Banach contraction principle and the theory of fractional power of operators to obtain our results.

1. Introduction

Many evolution processes and phenomena experience abrupt changes of state through short term perturbations. Since the duration of the perturbations are negligible in comparison with the duration of each process, it is quite natural to assume that these perturbations act in terms of impulses. For more details see monographs [15], [18]. It is to be noted that the recent progress in the development of the qualitative theory of solutions of impulsive differential equations has been studied by many researchers, see [1]–[3], [11]–[16], [18]–[21]. Also neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last few decades, e.g. see [3], [11], [17], [20], [21].

On the other hand, the theory of functional differential equations with nonlocal conditions has been extensively studied in the literature, see [1], [4]–[10], [12]–[14], [19], as they have applications in physics and many other areas of applied mathematics. The nonlocal condition is more precise for describing natural phenomena than the classical condition because more

2010 *Mathematics Subject Classification*: Primary: 45J05; Secondary: 45N05, 47H10, 47B38.

Key words and phrases: impulsive, functional, neutral, integro-differential equation, fixed point, nonlocal condition.

DOI: 10.1515/dema-2015-0029

information is taken into account, thereby decreasing the negative effects incurred by a possibly single measurement taken at initial time.

In this paper, we study nonlinear impulsive neutral functional integro-differential equation with non local condition of the type :

$$\frac{d}{dt}[x(t) - u(t, x_t)] + Ax(t) = f(t, x_t, \int_0^t k(t,s)h(s,x_s)ds), \quad t \in (0,T], \quad t \neq \tau_k,\ k = 1,2,\ldots,m, \tag{1}$$

$$x(t) + (g(x_{t_1}, \ldots, x_{t_p}))(t) = \phi(t), \qquad -r \le t \le 0, \tag{2}$$

$$\Delta x(\tau_k + 0) = I_k x(\tau_k), \quad k = 1,2,\ldots,m, \tag{3}$$

where $0 < t_1 < t_2 < \cdots < t_p \le T$, $p \in \mathbb{N}$, A and $I_k (k = 1,2,\ldots,k)$ are the linear operators acting in a Banach space X. The functions f, h, g, k and ϕ are given functions satisfying some assumptions. $\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0)$ and the impulsive moments τ_k are such that $0 \le \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < T$, $k \in \mathbb{N}$. For any continuous function x and for any $t \in [0,T]$, we denote by x_t, the element of $C([-r,0],X)$ defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$.

Equations of the form (1)–(3) or their special forms arise in some physical applications as a natural generalization of the classical initial value problems. The results for semilinear functional differential evolution nonlocal problem are extended for the case of impulsive effect.

As usual, in the theory of impulsive differential equations at the points of discontinuity τ_i of the solution $t \to x(t)$, we assume that $x(\tau_i) \equiv x(\tau_i - 0)$. It is clear that, in general, the derivatives $x'(\tau_i)$ do not exist. On the other hand, according to (1), there exist the limits $x'(\tau_i \pm 0)$. According to the above convention, we assume that $x'(\tau_i) = x'(\tau_i - 0)$.

The aim of the present paper is to study the existence, uniqueness and continuous dependence of mild solution of nonlocal initial value problem for an impulsive neutral functional integro-differential equation. We are generalizing the results reported in [2], [11], [14]. The main tool used in our analysis is based on an application of the Banach contraction theorem and the theory of fractional power of operators.

This paper is organized as follows. Section 2 presents the preliminaries and hypotheses. In Section 3, we prove existence and uniqueness of mild solution. Finally in Section 4, we prove continuous dependence of solutions on initial data.

2. Preliminaries and hypotheses

Throughout this paper, X will be a Banach space with norm $\|\cdot\|$. Let $C([a,b],X)$ denote the set $\{x : [a,b] \to X | x(t)$ is continuous at $t \neq \tau_k$, left continuous at $t = \tau_k$, and the right limit $x(\tau_k + 0)$ exists for $k = 1, 2, \ldots, m\}$. Clearly, $C([a,b], X)$ is a Banach space with the supremum norm $\|x\|_{C([a,b],X)} = \sup\{\|x(t)\| : t \in [a,b] \backslash \{\tau_1, \tau_2, \ldots, \tau_m\}\}$. Let $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup of linear operators $\{T(t)\}_{t \geq 0}$ on X. It is well known that there exists a constant K such that $\|T(t)\| \leq K$, $t \geq 0$. Also assume that $k : [0,T] \times [0,T] \to \mathbb{R}$ is a continuous function and as the set $[0,T] \times [0,T]$ is compact, there exists a constant $L > 0$ such that $|k(t,s)| \leq L$, for $0 \leq s \leq t \leq T$. If T is uniformly bounded analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X and the expression $\|x\|_\alpha = \|(-A)^\alpha x\|$ defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|.\|$, then the following properties are well known.

Lemma 2.1. ([17], p. 74) *Let $0 < \alpha \leq \beta \leq 1$. Then the following properties hold.*

(i) *X_β is a Banach space and $X_\beta \hookrightarrow X_\alpha$ is continuous.*
(ii) *The function $S \mapsto (A)^\alpha T(S)$ is continuous in the uniform operator topology on $(0, \infty)$ and there exists a positive constant C_α such that $\|(-A)^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}$, for every $t > 0$.*

The following inequality will be useful while proving our results.

Lemma 2.2. ([18], p. 12) *Let a nonnegative piecewise continuous function $u(t)$ satisfy, for $t \geq t_0$, the inequality*

$$u(t) \leq C + \int_{t_0}^{t} v(s)u(s)ds + \sum_{t_0 < \tau_i < t} \beta_i u(\tau_i),$$

where $C \geq 0$, $\beta_i \geq 0$, $v(t) > 0$, τ_i are the first kind discontinuity points of the function $u(t)$. Then the following estimate holds for the function $u(t)$

$$u(t) \leq C \prod_{t_0 < \tau_i < t} (1 + \beta_i) \exp\biggl(\int_{t_0}^{t} v(s)ds\biggr).$$

Definition 2.1. A function $x \in C([-r,T], X)$, $T > 0$ is called the mild solution of the problem (1)–(3) if $x(t) + (g(x_{t_1}, \ldots, x_{t_p}))(t) = \phi(t)$, $-r \leq t \leq 0$, the restriction of $x(.)$ to the interval $[0,T)$ is continuous and for each $0 \leq t < T$, the function $AT(t-s)u(s, x_s)$, $s \in [0,t)$ is integrable and the following integral equation

$$\begin{aligned}x(t) = T(t)[\phi(0) - g(x_{t_1}, \ldots, x_{t_p})(0) - u(0, \phi(0) - g(x_{t_1}, \ldots, x_{t_p})(0))]\\ + u(t, x_t) + \int_0^t T(t-s) f\left(s, x_s, \int_0^s k(s,\tau) h(\tau, x_\tau) d\tau\right) ds\\ + \int_0^t AT(t-s) u(s, x_s) ds\\ + \sum_{0<\tau_k<t} T(t-\tau_k) I_k x(\tau_k), \quad t \in (0, T]\end{aligned}$$

is satisfied.

Now we introduce the following hypotheses.

(H_1) There exists $\beta \in (0,1)$. For the functions

$$f : [0,T] \times C([-r,0], X_\beta) \times X_\beta \to X_\beta,$$
$$u, h : [0,T] \times C([-r,0], X_\beta) \to X_\beta, \quad I_k : X_\beta \to X_\beta,$$

there exist positive constants F, U, H, L_k such that

$$\begin{aligned}\|(-A)^\beta u(t, x_t) - (-A)^\beta u(t, y_t)\| &\le U \|x - y\|_{C([-r,t], X_\beta)},\\ \|f(t, x_t, \phi) - f(t, y_t, \psi)\| &\le F(\|x - y\|_{C([-r,t], X_\beta)} + \|\phi - \psi\|),\\ \|h(t, x_t) - h(t, y_t)\| &\le H \|x - y\|_{C([-r,t], X_\beta)},\\ \|I_k(v)\|_{X_\beta} &\le L_k \|v\|_{X_\beta}, \ \phi, \psi, v \in X_\beta, \ k = 1, 2, \ldots, m.\end{aligned}$$

(H_2) For the function $g : C([-r,0], X_\beta)^P \to C([-r,0], X_\beta)$, there exists a constant $G > 0$ such that

$$\|g(x_{t_1}, \ldots, x_{t_p})(t) - g(y_{t_1}, \ldots, y_{t_p})(t)\| \le G \|x - y\|_{C([-r,T], X_\beta)}.$$

(H_3) Assume that $\phi \in C([-r,0], X_\beta)$.

3. Existence and uniqueness

Theorem 3.1. *Suppose that hypotheses (H_1)–(H_3) are satisfied and $\Gamma < 1$, where*

$$\begin{aligned}\Gamma = KG + K\|(-A)^{-\beta}\| UG + \|(-A)^{-\beta}\| U + C_{1-\beta} T^\beta U\\ + KF[1 + LHT]T + K \sum_{0<\tau_k<t} L_k,\end{aligned}$$

then the nonlocal impulsive Cauchy problem (1)–(3) has a unique mild solution x on $[-r, T]$.

Proof. We introduce an operator $\mathcal{F}$ on a Banach space $C([-r,T], X_\beta)$ as follows:

$$(4)\quad (\mathcal{F}x)(t) = \begin{cases} \phi(t) - (g(x_{t_1}, \ldots, x_{t_p}))(t), & \text{if } -r \le t \le 0, \\ T(t)[\phi(0) - g(x_{t_1}, \ldots, x_{t_p})(0) - u(0, \phi(0) - g(x_{t_1}, \ldots, x_{t_p})(0)] & \\ \quad + u(t, x_t) + \int_0^t AT(t-s)u(s, x_s)ds & \\ \quad + \int_0^t T(t-s) f\left(s, x_s, \int_0^s k(s,\tau)h(\tau, s_\tau)d\tau\right) ds & \\ \quad + \sum_{0<\tau_k<t} T(t-\tau_k)I_k x(\tau_k), & \text{if } t \in (0, T]. \end{cases}$$

It is easy to see that $\mathcal{F} : C([-r,T], X_\beta) \to C([-r,T], X_\beta)$.

Now, we will show that $\mathcal{F}$ is a contraction on $C([-r,T], X_\beta)$. Let $x, y \in C([-r,T], X_\beta)$. Then for $t \in [-r, 0]$,

$$(5)\quad \begin{aligned} \|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| &= \|g(x_{t_1}, \ldots, x_{t_p})(t) - g(y_{t_1}, \ldots, y_{t_p})(t)\| \\ &\le G\|x-y\|_{C([-r,T],X_\beta)} \end{aligned}$$

and for $t \in (0, T]$,

$$(6)\quad \begin{aligned} &\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \\ &= \Big\| T(t)\Big[g(x_{t_1}, \ldots, x_{t_p})(0) - g(y_{t_1}, \ldots, y_{t_p})(0) \\ &\quad + u(0, \phi(0) - g(x_{t_1}, \ldots, x_{t_p})(0) \\ &\quad - u(0, \phi(0) - g(y_{t_1}, \ldots, y_{t_p})(0))\Big] + u(t, x_t) - u(t, y_t) \\ &\quad + \int_0^t AT(t-s)[u(s, x_s) - u(s, y_s)]ds \\ &\quad + \int_0^t T(t-s)[f(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau) \\ &\quad - f(s, y_s, \int_0^s k(s,\tau)h(\tau, y_\tau)d\tau)]ds + \sum_{0<\tau_k<t} T(t-\tau_k)[I_k x(\tau_k) - I_k y(\tau_k)] \Big\| \\ &\le \|T(t)\|\|g(x_{t_1}, \ldots, x_{t_p})(0) - g(y_{t_1}, \ldots, y_{t_p})(0)\| \\ &\quad + \|T(t)\|\|u(0, \phi(0) - g(x_{t_1}, \ldots, x_{t_p})(0)) - u(0, \phi(0) - g(y_{t_1}, \ldots, y_{t_p})(0))\| \\ &\quad + \|u(t, x_t) - u(t, y_t)\| + \int_0^t \|AT(t-s)\|\|u(s, x_s) - u(s, y_s)\|ds \\ &\quad + \int_0^t \|T(t-s)\| \Big\| f\Big(s, x_s, \int_0^s k(s,\tau)h(\tau, x_\tau)d\tau\Big) \\ &\quad - f\Big(s, y_s, \int_0^s k(s,\tau)h(\tau, y_\tau)d\tau\Big)\Big\| ds + \sum_{0<\tau_k<t} \|T(t-\tau_k)\|\|[I_k x(\tau_k) - I_k y(\tau_k)]\| \\ &\le KG\|x-y\|_{C([-r,T],X_\beta)} + KJ_1 + J_2 + J_3 + J_4 + J_5, \end{aligned}$$

where

$$J_1 = \|u(0,\phi(0)-g(x_{t_1},\dots,x_{t_p})(0))-u(0,\phi(0)-g(y_{t_1},\dots,y_{t_p})(0))\| \tag{7}$$
$$= \|(-A)^{-\beta}\|\|(-A)^{\beta}u(0,\phi(0)-g(x_{t_1},\dots,x_{t_p})(0))$$
$$-(-A)^{\beta}u(0,\phi(0)-g(y_{t_1},\dots,y_{t_p})(0))\|$$
$$\leq \|(-A)^{-\beta}\|U\|\phi(0)-g(x_{t_1},\dots,x_{t_p})(0)-\phi(0)+g(y_{t_1},\dots,y_{t_p})(0)\|$$
$$\leq \|(-A)^{-\beta}\|UG\|x-y\|_{C([-r,T],X_\beta)},$$

$$J_2 = \|u(t,x_t)-u(t,y_t)\| \leq \|(-A)^{-\beta}\|U\|x-y\|_{C([-r,T],X_\beta)}, \tag{8}$$

$$J_3 = \int_0^t \|AT(t-s)\|\|u(s,x_s)-u(s,y_s)\|ds \tag{9}$$
$$= \int_0^t \|-AT(t-s)\|\|(-A)^{-\beta}\|\|(-A)^{\beta}[u(s,x_s)-u(s,y_s)]\|ds$$
$$\leq \int_0^t \|(-A)^{1-\beta}T(t-s)\|U\|x-y\|_{C([-r,s],X_\beta)}ds$$
$$\leq \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}U\|x-y\|_{C([-r,s],X_\beta)}ds$$
$$= \int_0^t \frac{C_{1-\beta}}{T^{1-\beta}}U\|x-y\|_{C([-r,s],X_\beta)}ds$$
$$\leq \frac{C_{1-\beta}}{T^{1-\beta}}U\|x-y\|_{C([-r,T],X_\beta)}T$$
$$= C_{1-\beta}T^{\beta}U\|x-y\|_{C([-r,T],X_\beta)},$$

$$J_4 = \int_0^t \|T(t-s)\|\|f(s,x_s,\int_0^s k(s,\tau)h(\tau,x_\tau)d\tau) \tag{10}$$
$$-f(s,y_s,\int_0^s k(s,\tau)h(\tau,y_\tau)d\tau)\|ds$$
$$\leq K\int_0^t F\left[\|x-y\|_{C([-r,s],X_\beta)}+\int_0^s |k(s,\tau)|\|h(\tau,x_\tau)-h(\tau,y_\tau)\|d\tau\right]ds$$
$$\leq KF\int_0^t \left[\|x-y\|_{C([-r,s],X_\beta)}+L\int_0^s H\|x-y\|_{C([-r,\tau],X_\beta)}d\tau\right]ds$$
$$\leq KF\int_0^t \left[\|x-y\|_{C([-r,s],X_\beta)}+LHT\|x-y\|_{C([-r,s],X_\beta)}\right]ds$$
$$\leq KF\int_0^t [1+LHT]\|x-y\|_{C([-r,s],X_\beta)}ds$$
$$\leq KF[1+LHT]T\|x-y\|_{C([-r,T],X_\beta)},$$

$$(11)\quad \begin{aligned} J_5 &= \sum_{0<\tau_k<t} \|T(t-\tau_k)\| \|I_k x(\tau_k) - I_k y(\tau_k)\|_{X_\beta} \\ &\le \sum_{0<\tau_k<t} K \|I_k x(\tau_k) - I_k y(\tau_k)\|_{X_\beta} \\ &\le K \sum_{0<\tau_k<t} L_k \|x(\tau_k) - y(\tau_k)\|_{X_\beta} \\ &\le K \sum_{0<\tau_k<t} L_k \|x-y\|_{C([-r,T],X_\beta)}. \end{aligned}$$

Using (7)–(11), inequality (6) becomes

$$(12)\quad \begin{aligned} &\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \\ &\le KG\|x-y\|_{C([-r,T],X_\beta)} \\ &\quad + \big(K\|(-A)^{-\beta}\|UG + \|(-A)^{-\beta}\|U + C_{1-\beta}T^\beta U \\ &\quad + KF[1+LHT]T + K\sum_{0<\tau_k<t} L_k\big)\|x-y\|_{C([-r,T],X_\beta)}, \quad t\in[0,T]. \end{aligned}$$

In view of inequality (5) and (12), we can say that inequality (12) holds good for $t \in [-r, T]$. Therefore, for $t \in [-r, T]$,

$$\begin{aligned} &\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \\ &\le KG\|x-y\|_{C([-r,T],X_\beta)} + \Big(K\|(-A)^{-\beta}\|UG + \|(-A)^{-\beta}\|U \\ &\quad + C_{1-\beta}T^\beta U + KF[1+LHT]T + K\sum_{0<\tau_k<t} L_k\Big)\|x-y\|_{C([-r,T],X_\beta)} \\ &\le \Big(KG + K\|(-A)^{-\beta}\|UG + \|(-A)^{-\beta}\|U + C_{1-\beta}T^\beta u_\beta \\ &\quad + KF[1+LHT]T + K\sum_{0<\tau_k<t} L_k\Big)\|x-y\|_{C([-r,T],X_\beta)}, \end{aligned}$$

which implies

$$\|\mathcal{F}x - \mathcal{F}y\|_{C([-r,T],X_\beta)} \le \Gamma\|x-y\|_{C([-r,T],X_\beta)},$$

where

$$\begin{aligned} \Gamma = {} & KG + K\|(-A)^{-\beta}\|UG + \|(-A)^{-\beta}\|U + C_{1-\beta}T^\beta U \\ & + KF[1+LHT]T + K\sum_{0<\tau_k<t} L_k. \end{aligned}$$

Since $\Gamma < 1$, the operator $\mathcal{F}$ satisfies all assumptions of the Banach contraction theorem and therefore, $\mathcal{F}$ has a unique fixed point in the space $C([-r,T], X_\beta)$, which is the mild solution of nonlocal neutral initial value problem (1)–(3) with impulse effect. This completes the proof of the theorem. ∎

4. Continuous dependence of a mild solution

Theorem 4.1. *Suppose that hypotheses* (H_1)–(H_3) *are satisfied and* $\Gamma < 1$. *Then for each* $\phi_1, \phi_2 \in C([-r,T], X_\beta)$ *and for the corresponding mild solutions* x_1, x_2 *of the problems*

$$\frac{d}{dt}[x(t)+u(t,x_t)] = Ax(t) + f\left(t, x_t, \int_0^t k(t,s)h(s,x_s)ds\right), \ t \in (0,T], \tag{13}$$

$$x(\tau_k + 0) = Q_k x(\tau_k) = x(\tau_k) + I_k x(\tau_k), \quad k = 1,2,\ldots,m, \tag{14}$$

$$x(t) + g(x_{t_1}, \ldots, x_{t_p})(t) = \phi_i(t), \quad i = 1,2, \quad t \in [-r,0], \tag{15}$$

the following inequality holds

$$\begin{aligned} &\|x_1 - x_2\|_{C([-r,T],X_\beta)} \\ &\le \frac{\prod_{0<\tau_k<t}(1+KL_k)\exp(KFT)[K + KU\|(-A)^{-\beta}]}{[1 - \Lambda_1 \prod\limits_{0<\tau_k<t}(1+KL_k)\exp(KFT)]} \times \|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)}, \end{aligned} \tag{16}$$

where

$$\Lambda_1 = GK + K\|(-A)^{-\beta}\|UG + \|(-A)^{-\beta}\|U + C_{1-\beta}UT^\beta + KFLHT^2.$$

Moreover, if $U = 0$ *and* $G = 0$, *the above inequality reduces to classical inequality*

$$\begin{aligned} &\|x_1 - x_2\|_{C([-r,T],X_\beta)} \\ &\le \frac{K\prod_{0<\tau_k<t}(1+KL_k)\exp(KFT)}{[1 - KFLHT^2\prod_{0<\tau_k<t}(1+KL_k)\exp(KFT)]} \times \|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)}. \end{aligned} \tag{17}$$

Proof. Let $\phi_1, \phi_2 \in C([-r,T], X_\beta)$ be arbitrary functions and let x_1, x_2 be the mild solutions of the problem(13)–(15). Then we have

$$\begin{aligned} &x_1(t) - x_2(t) \\ &= T(t)[\phi_1(0) - \phi_2(0)] - T(t)[g(x_{1_{t_1}}, \ldots, x_{1_{t_p}})(0) - g(x_{2_{t_1}}, \ldots, x_{2_{t_p}})(0)] \\ &\quad - T(t)[u(0, \phi_1(0) - g(x_{1_{t_1}}, \ldots, x_{1_{t_p}})(0)) - u(0, \phi_2(0) - g(x_{2_{t_1}}, \ldots, x_{2_{t_p}})(0))] \\ &\quad + u(t, x_{1_t}) - u(t, x_{2_t}) \\ &\quad + \int_0^t (-A)^{1-\beta} T(t-s)\big[(-A)^{-\beta}u(s, x_{1_s}) - (-A)^{-\beta}u(s, x_{2_s})\big]ds \\ &\quad + \int_0^t T(t-s)\Big[f(s, x_{1_s}, \int_0^s k(s,\tau)h(s, x_{1_s})d\tau) \\ &\quad - f(s, x_{2_s}, \int_0^s k(s,\tau)h(s, x_{2_s})d\tau)\Big]ds \\ &\quad + \sum_{0<\tau_k<t} T(t-\tau_k)(I_k x_1(\tau_k) - I_k x_2(\tau_k)), \quad t \in (0,T], \end{aligned} \tag{18}$$

and for $t \in [-r, 0]$,

$$(19) \qquad x_1(t) - x_2(t) = \phi_1(t) - \phi_2(t) - [g(x_{1_{t_1}}, \ldots, x_{1_{t_p}})(t) - g(x_{2_{t_1}}, \ldots, x_{2_{t_p}})(t)].$$

From (18) and using hypothesis (H_1)–(H_3), we get

$$\begin{aligned}
(20) \qquad & \|x_1(t) - x_2(t)\| \\
&\leq K\|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} + GK\|x_1 - x_2\|_{C([-r,T],X_\beta)} \\
&\quad + KU\|(-A)^{-\beta}\|\|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} \\
&\quad + K\|(-A)^{-\beta}\|UG\|x_1 - x_2\|_{C([-r,T],X_\beta)} + \|(-A)^{-\beta}\|U\|x_1 - x_2\|_{C([-r,T],X_\beta)} \\
&\quad + \int_0^t C_{1-\beta}UT^{\beta-1}\|x_1 - x_2\|_{C([-r,s],X_\beta)}ds + K\int_0^t F\|x_1 - x_2\|_{C([-r,s],X_\beta)}ds \\
&\quad + K\int_0^t FLHT\|x_1 - x_2\|_{C([-r,T],X_\beta)}ds \\
&\quad + K\sum_{0<\tau_k<t} L_k\|x_1(\tau_k) - x_2(\tau_k)\|, \quad 0 \leq t \leq T \\
&\leq [K + KU\|(-A)^{-\beta}]\|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} + GK\|x_1 - x_2\|_{C([-r,T],X_\beta)} \\
&\quad + \|(-A)^{-\beta}\|UG\|x_1 - x_2\|_{C([-r,T],X_\beta)} + \|(-A)^{-\beta}\|U\|x_1 - x_2\|_{C([-r,T],X_\beta)} \\
&\quad + C_{1-\beta}UT^\beta\|x_1 - x_2\|_{C([-r,T],X_\beta)} + \int_0^t KF\|x_1 - x_2\|_{C([-r,s],X_\beta)}ds \\
&\quad + KFLHT^2\|x_1 - x_2\|_{C([-r,T],X_\beta)} \\
&\quad + K\sum_{0<\tau_k<t} L_k\|x_1(\tau_k) - x_2(\tau_k)\|, \quad 0 \leq t \leq T \\
&\leq [K + KU\|(-A)^{-\beta}]\|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} + \Lambda_1\|x_1 - x_2\|_{C([-r,T],X_\beta)} \\
&\quad + \int_0^t KF\|x_1 - x_2\|_{C([-r,s],X_\beta)}ds \\
&\quad + K\sum_{0<\tau_k<t} L_k\|x_1(\tau_k) - x_2(\tau_k)\|, \quad 0 < \tau_k \leq t.
\end{aligned}$$

Simultaneously, by (19) and hypothesis (H_3), we get

$$\begin{aligned}
(21) \qquad & \|x_1(t) - x_2(t)\| \\
&\leq \|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} + G\|x_1 - x_2\|_{C([-r,T],X_\beta)}, \quad t \in [-r, 0].
\end{aligned}$$

Since $K \geq 1$, the inequalities (20) and (21) imply

$$
\begin{aligned}
(22)\quad & \|x_1 - x_2\|_{C([-r,t],X_\beta)} \\
&\le \big[[K + KU\|(-A)^{-\beta}]\|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} + \Lambda_1\|x_1 - x_2\|_{C([-r,T],X_\beta)}\big] \\
&\quad + \int_0^t KF\|x_1 - x_2\|_{C([-r,s],X_\beta)}ds \\
&\quad + K\sum_{0<\tau_k<t} L_k\|x_1(\tau_k) - x_2(\tau_k)\|, \quad 0<\tau_k\le t, \quad t\in[0,T].
\end{aligned}
$$

Now applying Lemma 2.2 to the inequality (22), we get

$$
\begin{aligned}
\|x_1 - x_2\|_{C([-r,t],X_\beta)} \le & \prod_{0<\tau_k<t}(1+KL_k)\exp(KFT) \\
& \times \big([K + KU\|(-A)^{-\beta}]\|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} \\
& + \Lambda_1\|x_1 - x_2\|_{C([-r,T],X_\beta)}\big),
\end{aligned}
$$

hence, we get

$$
\begin{aligned}
\|x_1 - x_2\|_{C([-r,T],X_\beta)} \le & \prod_{0<\tau_k<t}(1+KL_k)\exp(KFT) \\
& \times \big([K + KU\|(-A)^{-\beta}]\|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} \\
& + \Lambda_1\|x_1 - x_2\|_{C([-r,T],X_\beta)}\big).
\end{aligned}
$$

The inequalities given by (16) and (17) are easy consequences of the above inequality. This completes the proof. ■

References

[1] H. Akca, A. Boucherif, V. Covachev, *Impulsive functional differential equations with nonlocal conditions*, Int. J. Math. Math. Sci. 29(5) (2002), 251–256.

[2] A. Anguraj, K. Karthikeyan, *Existence of solutions for impulsive functional differential equations with nonlocal conditions*, Nonlinear Anal. 70 (2009), 2717–2721.

[3] M. Benchora, J. Henderson, *Existence results for impulsive multivalued semilinear neutral functional differential inclusions in Banach spaces*, J. Math. Anal. Appl. 263 (2001), 763–780.

[4] K. Balachandran, J. Y. Park, *Nonlocal Cauchy problem for Sobolev type functional integro-differential equations*, Bull. Korean Math. Soc. 39(1) (2002), 561–569.

[5] K. Balachandran, J. Y. Park, *Existence of a mild solution of a functinal integro-differential equations with nonlocal condition*, Bull. Korean Math. Soc. 38(1) (2001), 175–182.

[6] L. Byszewski, *On mild solution of a semilinear functional differential evolution nonlocal problem*, J. Appl. Math. Stochastic Anal. 10(3) (1997), 265–271.

[7] L. Byszewski, H. Akca, *Existence of solutions of a semilinear functional evolution nonlocal problem*, Nonlinear Anal. 34 (1998), 65–72.

[8] L. Byszewski, V. Lakshamikantham, *Theorems about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal. 40 (1990), 11–19.

[9] J. P. Daur, K. Bhalchandran, *Existance of solutions of nonlinear neutral integrodifferential equations in Banach spaces*, J. Math. Anal. Appl. 251 (2000), 93–105.

[10] M. B. Dhakne, K. D. Kucche, *Existence of mild solution of mixed Volterra-Fredholm functional integrodifferential equation with nonlocal condition*, Appl. Math. Sci. 5(8) (2011), 359–366.

[11] E. Hernandez, M. Hernan, R. Henriquez, *Impulsive partial neutral differential equations*, Appl. Math. Lett. 19 (2006), 215–222.

[12] R. S. Jain, M. B. Dhakne, *On global existence of solutions for abstract nonlinear functional integro-differential equations with nonlocal condition*, Contemp. Math. Stat. 1 (2013), 44–53.

[13] R. S. Jain, M. B. Dhakne, *On mild solutions of nonlocal semilinear functional integro-differential equations*, Malaya J. Math. 3(1) (2013), 27–33.

[14] J. Liang, Z. Fan, *Nonlocal impulsive Cauchy problems for evolution equations*, Advances in Difference Equations, Vol. 2011, Article ID 784161, 17 pages.

[15] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[16] B. Liu, *Controllability of impulsive neutral functional differential inclusions with infinte delay*, Nonlinear Anal. 60 (2005), 1533–1552.

[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer Verlag, New York, 1983.

[18] A. M. Samoilenko, N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[19] S. C. Ji, S. Wen, *Nonlocal Cauchy problem for impulsive differential equations in Banach spaces*, Int. J. Nonlinear Sci. 10(1) (2010), 88–95.

[20] S. Sivasankaran, V. Vijaykumar, M. M. Arjunan, *Existence of global solutions for impulsive abstract partial neutral functional differential equations*, Int. J. Nonlinear Sci. 11(4) (2011), 412–426.

[21] V. Vijaykumar, S. Sivasankaran, M. M. Arjunan, *Global existence for Volterra-Fredholm type neutral impulsive functional integro-differential equations*, Surveys Math. Appl. 7 (2012), 49–68.

R. S. Jain
SCHOOL OF MATHEMATICAL SCIENCES
SWAMI RAMANAND TEERTH MARATHWADA UNIVERSITY
NANDED-431606, INDIA
E-mail: rupalisjain@gmail.com

M. B. Dhakne
DEPARTMENT OF MATHEMATICS
DR. BABASAHEB AMBEDKAR MARATHWADA UNIVERSITY
AURANGABAD-431004, INDIA
E-mail: mbdhakne@yahoo.com

Received May 20, 2013; revised version October 21, 2013.