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KOROVKIN TYPE THEOREM FOR SEQUENCES OF OPERATORS DEPENDING ON A PARAMETER

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Abstract. We establish necessary and sufficient conditions for a parameter depending sequence $(L_{n,\lambda})_{n\geq 1}$ of positive linear operators such that $(L_{n,\lambda})_{n\geq 1}$ converges in the strong operator topology to its limit operator. Some applications of our theorem are also presented.

1. Introduction

The well-known Korovkin's theorem is applied to prove the convergence of sequences of positive linear operators to the identity in the strong operator topology. Let us denote by $C[0,1]$, the Banach space of all continuous functions on $[0,1]$ equipped with the norm $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$ and by e_s, the power function $e_s(x) = x^s$, $x \in [0,1]$, $s \geq 0$. Then Korovkin's theorem is the following (see [1, p. 8]): *let* $(L_n)_{n\geq 1}$ *be a sequence of positive linear operators such that* $L_n : C[0,1] \to C[0,1]$. *Then* $\|L_n(f) - f\| \to 0$ *as* $n \to \infty$ *for all* $f \in C[0,1]$ *if and only if* $\|L_n(e_i) - e_i\| \to 0$ *as* $n \to \infty$ *for* $i \in \{0,1,2\}$. Specifically we recover the Weierstrass' approximation theorem if we choose, for the positive linear operators L_n, the Bernstein operators $B_n : C[0,1] \to C[0,1]$ defined by

$$(1.1)\quad (B_n(f))(x) = \sum_{k=0}^{n} p_{n,k}(x)\, f\left(\frac{k}{n}\right) \equiv \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

The development of the q-calculus has led to the discovery of new Bernstein type operators involving q-integers. The so-called q-Bernstein operators were introduced by Phillips [9] in 1997 and they are generalization of (1.1) based on q-integers. To present these operators we recall some notions of the q-calculus

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(see e.g. [5]). Let $q > 0$. For each non-negative integer n, the q-integers $[n] \equiv [n]_q$ and the q-factorials $[n]!$ are defined by

$$[n] = \begin{cases} 1 + q + \ldots + q^{n-1}, & \text{if } n \geq 1, \\ 0, & \text{if } n = 0, \end{cases}$$

and

$$[n]! = \begin{cases} [1][2]\ldots[n], & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

Then the q-Bernstein operators $B_{n,q} : C[0,1] \to C[0,1]$ are introduced as follows:

$$\begin{aligned} (1.2) \quad (B_{n,q}(f))(x) &= \sum_{k=0}^{n} p_{n,k}(q;x)\, f\left(\frac{[k]}{[n]}\right) \\ &\equiv \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)(1-qx)\ldots(1-q^{n-k-1}x) f\left(\frac{[k]}{[n]}\right). \end{aligned}$$

For $q = 1$, we recover the operators (1.1). If $0 < q < 1$, then $B_{n,q}$ are positive linear operators. Taking into account [9, pp. 513–514], we have $(B_{n,q}(e_0))(x) = e_0(x) = 1$, $(B_{n,q}(e_1))(x) = e_1(x) = x$ and

$$(B_{n,q}(e_2))(x) = e_2(x) + \frac{1}{[n]}(e_1 - e_2)(x) = x^2 + \frac{1}{[n]}x(1-x).$$

Hence $\|B_{n,q}(e_0) - e_0\| \to 0$, $\|B_{n,q}(e_1) - e_1\| \to 0$ as $n \to \infty$, but $\|B_{n,q}(e_2) - e_2\| = \frac{1-q}{4(1-q^n)} \to \frac{1-q}{4} \neq 0$ as $n \to \infty$ for $q \in (0,1)$ fixed. Thus Korovkin's theorem cannot be applied for $(B_{n,q})_{n \geq 1}$.

Now we consider a sequence of operators $(L_{n,\lambda})_{n\geq 1}$ such that $L_{n,\lambda} : C[0,1] \to C[0,1]$ and λ is a parameter belonging to a set Λ. The goal of the paper is to establish necessary and sufficient conditions which insure the convergence of $(L_{n,\lambda})_{n\geq 1}$ in the strong operator topology to a limit operator $L_{\infty,\lambda} : C[0,1] \to C[0,1]$, i.e. $\|L_{n,\lambda}(f) - L_{\infty,\lambda}(f)\| \to 0$ as $n \to \infty$ for all $f \in C[0,1]$. In this way, we obtain a new Korovkin type theorem. This will be the subject of Section 2. Finally, in Section 3 we will apply our result for some parameter depending sequences of operators.

2. Main results

Our Korovkin type theorem is the following:

Theorem 2.1. *Let Λ be a set of parameters. For $\lambda \in \Lambda$ let $(L_{n,\lambda})_{n\geq 1}$ be a sequence of positive linear operators on $C[0,1]$ satisfying the following conditions:*

- (i) *the sequence $(\|L_{n,\lambda}(e_0)\|)_{n\geq 1}$ is bounded,*
- (ii) *$(L_{n,\lambda}(g))_{n\geq 1}$ is a Cauchy sequence for all $g \in X$, where X is a dense set in $C[0,1]$.*

Then there exists a positive linear operator $L_{\infty,\lambda} : C[0,1] \to C[0,1]$ such that $\|L_{n,\lambda}(f) - L_{\infty,\lambda}(f)\| \to 0$ as $n \to \infty$ for all $f \in C[0,1]$. Moreover, if there exists $\lambda_0 \in \Lambda$ such that $\|L_{n,\lambda_0}(e_i) - e_i\| \to 0$ as $n \to \infty$ for $i \in \{0,1,2\}$, then $L_{\infty,\lambda_0}(f) = f$ for all $f \in C[0,1]$.

Conversely: if $L_{n,\lambda}$, $L_{\infty,\lambda}$ are positive linear operators on $C[0,1]$ for $n \geq 1$ and $\lambda \in \Lambda$ such that $\|L_{n,\lambda}(f) - L_{\infty,\lambda}(f)\| \to 0$ as $n \to \infty$ for all $f \in C[0,1]$, then we obtain the statements (i) and (ii). Moreover, if the condition $L_{\infty,\lambda_0}(f) = f$ for all $f \in C[0,1]$ and for some $\lambda_0 \in \Lambda$ is also satisfied, then $\|L_{n,\lambda_0}(e_i) - e_i\| \to 0$ as $n \to \infty$ for $i \in \{0,1,2\}$.

Proof. By (i), there exists $M > 0$ such that $\|L_{n,\lambda}(e_0)\| \leq M$ for all $n \geq 1$. The positivity of $L_{n,\lambda}$ implies that

$$\begin{aligned} |(L_{n,\lambda}(f))(x)| &\leq (L_{n,\lambda}(|f|))(x) \leq (L_{n,\lambda}(\|f\|e_0))(x) \\ &= \|f\|(L_{n,\lambda}(e_0))(x) \leq \|f\|\|L_{n,\lambda}(e_0)\|, \end{aligned} \tag{2.1}$$

for $f \in C[0,1]$. Hence $\|L_{n,\lambda}(f)\| \leq M\|f\|$, where $f \in C[0,1]$ and $n \geq 1$. Thus $\|L_{n,\lambda}\| = \sup\{\|L_{n,\lambda}(f)\| : \|f\| \leq 1\} \leq M$ for every $n \geq 1$. Further, in view of (ii), $(L_{n,\lambda}(g))_{n\geq 1}$ is a Cauchy sequence in $C[0,1]$, therefore $(L_{n,\lambda}(g))_{n\geq 1}$ converges in $C[0,1]$ for all $g \in X$. Then the well-known Banach–Steinhaus theorem [1, p. 29] implies that there exists a positive linear operator $L_{\infty,\lambda} : C[0,1] \to C[0,1]$ such that $\|L_{n,\lambda}(f) - L_{\infty,\lambda}(f)\| \to 0$ as $n \to \infty$.

If there exists $\lambda_0 \in \Lambda$ such that $\|L_{n,\lambda_0}(e_i) - e_i\| \to 0$ as $n \to \infty$ for $i \in \{0,1,2\}$, then, by Korovkin's theorem, $\|L_{n,\lambda_0}(f) - f\| \to 0$ as $n \to \infty$. But $\|L_{n,\lambda_0}(f) - L_{\infty,\lambda_0}(f)\| \to 0$ as $n \to \infty$ (see the proof above), therefore $L_{\infty,\lambda_0}(f) = f$ for all $f \in C[0,1]$.

Conversely: if $\|L_{n,\lambda}(f) - L_{\infty,\lambda}(f)\| \to 0$ as $n \to \infty$ for all $f \in C[0,1]$, then $|\|L_{n,\lambda}(e_0)\| - \|L_{\infty,\lambda}(e_0)\|| \leq \|L_{n,\lambda}(e_0) - L_{\infty,\lambda}(e_0)\| \to 0$ as $n \to \infty$, which means that $(\|L_{n,\lambda}(e_0)\|)_{n\geq 1}$ is a convergent sequence. Therefore $(\|L_{n,\lambda}(e_0)\|)_{n\geq 1}$ is a bounded sequence, thus we obtain the statement (i). Further, because $\|L_{n,\lambda}(g) - L_{\infty,\lambda}(g)\| \to 0$ as $n \to \infty$ for all $g \in X$, and $\|L_{n,\lambda}(g) - L_{n+p,\lambda}(g)\| \leq \|L_{n,\lambda}(g) - L_{\infty,\lambda}(g)\| + \|L_{\infty,\lambda}(g) - L_{n+p,\lambda}(g)\|$ for every $n, p \geq 1$, we obtain that $(\|L_{n,\lambda}(g)\|)_{n\geq 1}$ is a Cauchy sequence, thus we find the statement (ii).

If $L_{\infty,\lambda_0}(f) = f$, $f \in C[0,1]$, then, by $\|L_{n,\lambda_0}(f) - L_{\infty,\lambda_0}(f)\| \to 0$ as $n \to \infty$, we get $\|L_{n,\lambda_0}(f) - f\| \to 0$ as $n \to \infty$ for all $f \in C[0,1]$. Using Korovkin's theorem, we obtain that $\|L_{n,\lambda_0}(e_i) - e_i\| \to 0$ as $n \to \infty$, $i \in \{0,1,2\}$. This completes the proof of the theorem. ∎

The next result is formulated with the aid of the first order modulus of smoothness and the second order modulus of smoothness of $f \in C[0,1]$, defined as follows:

$$\omega(f,\delta) \equiv \omega_1(f,\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,1], |x-y| \le \delta\},$$
$$\omega_2(f,\delta) = \sup_{0<t\le\delta} \sup_{x\in[0,1-2t]} |f(x+2t) - 2f(x+t) + f(x)|, \quad \delta > 0.$$

Corollary 2.1. *For $\lambda \in \Lambda$ let $(L_{n,\lambda})_{n\ge1}$ be a sequence of positive linear operators on $C[0,1]$. If there exist the positive sequences $(\alpha_n)_{n\ge1}$ and $(\beta_n)_{n\ge1}$ such that*

(a) *$\alpha_n \to 0$ as $n \to \infty$,*
(b) *there exists $C_1 > 0$ with $\beta_n + \beta_{n+1} + \ldots + \beta_{n+p-1} \le C_1\alpha_n$ for all $n, p \ge 1$,*
(c) *there exists $C_2 > 0$ with $\|L_{n,\lambda}(g) - L_{n+1,\lambda}(g)\| \le C_2\beta_n\|g^{(j)}\|$ for all $n \ge 1$ and $g \in C^j[0,1]$, where $j \in \{1,2\}$ is given,*

then there exists $C_3 = C_3(\|L_{1,\lambda}(e_0)\|) > 0$ and a positive linear operator $L_{\infty,\lambda} : C[0,1] \to C[0,1]$ such that

$$\|L_{n,\lambda}(f) - L_{\infty,\lambda}(f)\| \le C_3\omega_j(f, \alpha_n^{1/j}), \tag{2.2}$$

for all $f \in C[0,1]$ and $n \ge 1$.

Proof. Applying (c) for $g = e_0$, we find

$$L_{n,\lambda}(e_0) = L_{n+1,\lambda}(e_0), \tag{2.3}$$

for all $n \ge 1$. Hence $\|L_{n,\lambda}(e_0)\| = \|L_{1,\lambda}(e_0)\| < +\infty$ for all $n \ge 1$. Therefore, the sequence $(\|L_{n,\lambda}(e_0)\|)_{n\ge1}$ is bounded, thus we obtain the condition (i) of Theorem 2.1

On the other hand, by (b) and (c), we have for all $n, p \ge 1$ and $g \in C^j[0,1]$ that

$$\begin{aligned} &\|L_{n,\lambda}(g) - L_{n+p-1,\lambda}(g)\| \\ &\quad \le \|L_{n,\lambda}(g) - L_{n+1,\lambda}(g)\| + \ldots + \|L_{n+p-2,\lambda}(g) - L_{n+p-1,\lambda}(g)\| \\ &\quad \le C_2(\beta_n + \beta_{n+1} + \ldots + \beta_{n+p-1})\|g^{(j)}\| \\ &\quad \le C_1C_2\alpha_n\|g^{(j)}\|. \end{aligned} \tag{2.4}$$

Taking into account (a), we find that $(\|L_{n,\lambda}(g)\|)_{n\geq 1}$ is a Cauchy sequence for all $g \in C^j[0,1]$, where $C^j[0,1]$ is dense in $C[0,1]$. Thus we obtain the condition (ii) of Theorem 2.1.

In conclusion, by Theorem 2.1, we have the existence of a positive linear operator $L_{\infty,\lambda} : C[0,1] \to C[0,1]$ such that $\|L_{n,\lambda}(f) - L_{\infty,\lambda}(f)\| \to 0$ as $n \to \infty$ for all $f \in C[0,1]$.

Now using (2.1) and (2.3), we have $\|L_{n,\lambda}(f)\| \leq \|L_{1,\lambda}(e_0)\|\|f\|$, $f \in C[0,1]$. Hence

$$\|L_{\infty,\lambda}(f)\| \leq \|L_{1,\lambda}(e_0)\|\|f\|, \quad f \in C[0,1]. \tag{2.5}$$

Let $p \to \infty$ in (2.4), then we obtain

$$\|L_{n,\lambda}(g) - L_{\infty,\lambda}(g)\| \leq C_1 C_2 \alpha_n \|g^{(j)}\|. \tag{2.6}$$

Taking into account (2.5)–(2.6), and using the equivalence between the K-functionals and the modulus of smoothness (see [1, p. 217, Theorem 5.2]), we get (2.2). This completes the proof of the corollary. ∎

Remark 2.1. In [11, p. 259, Theorem 2] is established a Korovkin type theorem with the following conditions:

1) the sequence $(\|L_n(e_2)\|)_{n\geq 1}$ converges to a function $L_\infty(e_2)$ in $C[0,1]$,
2) the sequence $((L_n(f))(x))_{n\geq 1}$ is non-increasing for any convex function f and any $x \in [0,1]$.

We prove that 1) and 2) are only sufficient conditions for $\|L_n(f) - L_\infty(f)\| \to 0$ as $n \to \infty$, $f \in C[0,1]$.

Indeed, let us consider the operators $L_{n,q} : C[0,1] \to C[0,1]$,

$$(L_{n,q}(f))(x) = \sum_{k=0}^{n} p_{n,k}(q;x)\, f\left(\tau\left(\frac{[k]}{[n]}\right)\right),$$

where $0 < q < 1$, $p_{n,k}(q;x)$ is defined by (1.2) and $\tau : C[0,1] \to C[0,1]$ is a given continuously differentiable function such that $e_2 \circ \tau$ is concave on $[0,1]$ (for example $\tau(x) = \sqrt{\ln(x+e-1)}$, $x \in [0,1]$ satisfies the enumerated conditions). Using the procedure of [8, p. 412, (3.2)–(3.3)], we find

$$\begin{aligned}(L_{n,q}(f))(x) &- (L_{n+1,q}(f))(x) \\ &= \sum_{k=1}^{n} p_{n+1,k}(q;x)\left\{\frac{[n+1-k]}{[n+1]} f\left(\tau\left(\frac{[k]}{[n]}\right)\right) + q^{n+1-k}\right. \\ &\quad \left.\times \frac{[k]}{[n+1]} f\left(\tau\left(\frac{[k-1]}{[n]}\right)\right) - f\left(\tau\left(\frac{[k]}{[n+1]}\right)\right)\right\}.\end{aligned} \tag{2.7}$$

Hence, by $[n+1-k]+q^{n+1-k}[k]=[n+1]$ and Taylor's formula, we have for $g\in C^1[0,1]$ that

$$
\begin{aligned}
&|(L_{n,q}(g))(x)-(L_{n+1,q}(g))(x)|\\
&\le \sum_{k=1}^{n} p_{n+1,k}(q;x)\left\{\frac{[n+1-k]}{[n+1]}\left|g\left(\tau\left(\frac{[k]}{[n]}\right)\right)-g\left(\tau\left(\frac{[k]}{[n+1]}\right)\right)\right|\right.\\
&\quad\left.+\ q^{n+1-k}\frac{[k]}{[n+1]}\left|g\left(\tau\left(\frac{[k-1]}{[n]}\right)\right)-g\left(\tau\left(\frac{[k]}{[n+1]}\right)\right)\right|\right\}\\
&\le \sum_{k=1}^{n} p_{n+1,k}(q;x)\left\{\frac{[n+1-k]}{[n+1]}\left|\int_{\tau([k]/[n+1])}^{\tau([k]/[n])}|g'(t)|\,dt\right|\right.\\
&\quad\left.+\ q^{n+1-k}\frac{[k]}{[n+1]}\left|\int_{\tau([k]/[n+1])}^{\tau([k-1]/[n])}|g'(t)|\,dt\right|\right\}\\
&\le \|g'\|\sum_{k=1}^{n} p_{n+1,k}(q;x)\left\{\frac{[n+1-k]}{[n+1]}\left|\tau\left(\frac{[k]}{[n]}\right)-\tau\left(\frac{[k]}{[n+1]}\right)\right|\right.\\
&\quad\left.+\,q^{n+1-k}\frac{[k]}{[n+1]}\left|\tau\left(\frac{[k-1]}{[n]}\right)-\tau\left(\frac{[k]}{[n+1]}\right)\right|\right\}\\
&\le \|\tau'\|\|g'\|\sum_{k=1}^{n} p_{n+1,k}(q;x)\left\{\frac{[n+1-k]}{[n+1]}\left|\frac{[k]}{[n]}-\frac{[k]}{[n+1]}\right|\right.\\
&\quad\left.+\,q^{n+1-k}\frac{[k]}{[n+1]}\left|\frac{[k-1]}{[n]}-\frac{[k]}{[n+1]}\right|\right\}\\
&= \|\tau'\|\|g'\|\sum_{k=1}^{n} p_{n+1,k}(q;x)\left\{\frac{[n+1-k]}{[n+1]}\frac{q^n[k]}{[n][n+1]}\right.\\
&\quad\left.+\,q^{n+1-k}\frac{[k]}{[n+1]}\frac{q^{k-1}[n+1-k]}{[n][n+1]}\right\}\\
&\le 2\|\tau'\|\frac{q^n}{[n]}\|g'\|.
\end{aligned}
$$

If $\beta_n=\frac{q^n}{[n]}$, $n\ge 1$, then $\beta_n+\beta_{n+1}+\ldots+\beta_{n+p-1}\le\frac{q^n}{[n]}(1+q+\ldots+q^{p-1})\le\frac{q^n}{1-q^n}$, where $n,p\ge 1$. For $\alpha_n=\frac{q^n}{1-q^n}$, $n\ge 1$, we have $\alpha_n\to 0$ as $n\to\infty$. Thus, by Corollary 2.1, there exists a positive linear operator $L_{\infty,q}:C[0,1]\to C[0,1]$ such that $\|L_{n,q}(f)-L_{\infty,q}(f)\|\le\max\{2,2\|\tau'\|\}\omega(f,\sqrt{q^n/(1-q^n)})$, i.e. $\|L_{n,q}(f)-L_{\infty,q}(f)\|\to 0$ as $n\to\infty$ for $f\in C[0,1]$.

On the other hand, by (2.7),

$$
\begin{aligned}
&(L_{n,q}(e_2))(x) - (L_{n+1,q}(e_2))(x) \\
&\quad = \sum_{k=1}^{n} p_{n+1,k}(q;x) \left\{ \frac{[n+1-k]}{[n+1]} (e_2 \circ \tau) \left(\tau \left(\frac{[k]}{[n]} \right) \right) + q^{n+1-k} \frac{[k]}{[n+1]} \right. \\
&\qquad \left. \times (e_2 \circ \tau) \left(\tau \left(\frac{[k-1]}{[n]} \right) \right) - (e_2 \circ \tau) \left(\tau \left(\frac{[k]}{[n+1]} \right) \right) \right\} \leq 0,
\end{aligned}
$$

because $e_2 \circ \tau$ is a concave function and $\frac{[n+1-k]}{[n+1]} \frac{[k]}{[n]} + q^{n+1-k} \frac{[k]}{[n+1]} \frac{[k-1]}{[n]} = \frac{[k]}{[n+1]}$. Thus 2) is not satisfied for $f = e_2$.

Remark 2.2. In [2, p. 752, Theorem 2.1], we established a new Korovkin type theorem using the first order Ditzian–Totik modulus of smoothness, while in [2, p. 753, Theorem 2.2], we obtained its converse theorem. Our main result (Theorem 2.1) is different from the above mentioned theorems of [2].

3. Applications

In this section, we apply our results for some parameter depending sequences of positive linear operators.

1° The following q-Kantorovich type operators were introduced in [7]: $B^*_{n,q} : C[0,1] \to C[0,1]$,

$$
(B^*_{n,q}(f))(x) = \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 f\left(\frac{[k] + q^k t}{[n+1]} \right) d_q t,
$$

where $0 < q < 1$, $f \in C[0,1]$, $x \in [0,1]$, $p_{n,k}(q;x)$ is defined by (1.2) and the integral with the aid of Jackson integral (see [5, p. 69, Definition]). Taking into account [7, p. 722, Lemma 2.1], we have for all $m = 0, 1, 2, \ldots$ and $x \in [0,1]$ that

$$
\begin{aligned}
(3.1) \qquad (B^*_{n,q}(e_m))(x) &= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \\
&\quad \times \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n - 1)^i (B_{n,q}(e_{j+i}))(x).
\end{aligned}
$$

We set

$$
\begin{aligned}
(3.2) \qquad (B^*_{\infty,q}(e_m))(x) &= \sum_{j=0}^{m} \binom{m}{j} \frac{(1-q)^{m-j}}{[m-j+1]} \\
&\quad \times \sum_{i=0}^{m-j} \binom{m-j}{i} (-1)^i (B_{\infty,q}(e_{j+i}))(x),
\end{aligned}
$$

where $B_{\infty,q}: C[0,1] \to C[0,1]$ is the limit q-Bernstein operator for $q \in (0,1)$ fixed. It is known that $\|B_{n,q}(f) - B_{\infty,q}(f)\| \to 0$ as $n \to \infty$ for each $f \in C[0,1]$ (for details see [4]). Then, by (3.1) and (3.2), we have

$$\begin{aligned}
(3.3)\qquad & |(B^*_{n,q}(e_m))(x) - (B^*_{\infty,q}(e_m))(x)| \\
&\le \sum_{j=0}^{m} \binom{m}{j} \left| \frac{[n]^j}{[n+1]^m [m-j+1]} - \frac{(1-q)^{m-j}}{[m-j+1]} \right| \\
&\quad \times \sum_{i=0}^{m-j} \binom{m-j}{i} |(q^n - 1)^i| |(B_{n,q}(e_{j+i}))(x)| \\
&\quad + \sum_{j=0}^{m} \binom{m}{j} \frac{(1-q)^{m-j}}{[m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} |(q^n-1)^i - (-1)^i| |(B_{n,q}(e_{j+i}))(x)| \\
&\quad + \sum_{j=0}^{m} \binom{m}{j} \frac{(1-q)^{m-j}}{[m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} |(B_{n,q}(e_{j+i}))(x) - (B_{\infty,q}(e_{j+i}))(x)|.
\end{aligned}$$

But

$$\begin{aligned}
\left| \frac{[n]^j}{[n+1]^m} - (1-q)^{m-j} \right| &= \left| \frac{(1-q)^{m-j}}{(1-q^{n+1})^m} \left\{ (1-q^n)^j - (1-q^{n+1})^m \right\} \right| \\
&\le (1-q)^{-j} |(1-q^n)^j - (1-q^{n+1})^j + (1-q^{n+1})^j - (1-q^{n+1})^m| \\
&\le q^n (1-q)^{-j+1} \left\{ (1-q^n)^{j-1} + \ldots + (1-q^{n+1})^{j-1} \right\} \\
&\quad + q^{n+1}(1-q)^{-j}(1-q^{n+1})^j \left\{ 1 + (1-q^{n+1}) + \ldots \right. \\
&\quad \left. + (1-q^{n+1})^{m-j-1} \right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
(3.4)\qquad \left| \frac{[n]^j}{[n+1]^m} - (1-q)^{m-j} \right| &\le j(1-q)^{-j+1} q^n + q^{n+1}(1-q)^{-j}(m-j) \\
&\le q^n (1-q)^{-j} m.
\end{aligned}$$

Further

$$\begin{aligned}
(3.5)\qquad |(q^n-1)^i - (-1)^i| &= \left| q^{ni} - \binom{i}{1} q^{n(i-1)} + \ldots + (-1)^{i-1} \binom{i}{i-1} q^n \right| \\
&\le q^n 2^i
\end{aligned}$$

and, by [12, p. 153, Theorem 1] and $\omega(f,\delta) \le \delta \|f'\|$, we have

$$(3.6)\qquad \|B_{n,q}(e_{j+i}) - B_{\infty,q}(e_{j+i})\| \le C_q \omega(e_{j+i}, q^n) \le C_q (j+i) q^n,$$

where $C_q = 2 + \frac{4 \ln \frac{1}{1-q}}{q(1-q)}$. Using (1.2), we find that

$$(3.7)\qquad |(B_{n,q}(e_s))(x)| \le 1,$$

for all $x \in [0,1]$ and $s \geq 0$.

Now (3.3)–(3.7) imply that

$$
\begin{aligned}
&|(B^*_{n,q}(e_m))(x) - (B^*_{\infty,q}(e_m))(x)| \\
&\qquad \leq \sum_{j=0}^{m} \binom{m}{j} q^n \frac{(1-q)^{-j} m}{[m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} \\
&\qquad\quad + \sum_{j=0}^{m} \binom{m}{j} \frac{(1-q)^{m-j}}{[m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} q^n 2^i \\
&\qquad\quad + \sum_{j=0}^{m} \binom{m}{j} \frac{(1-q)^{m-j}}{[m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} C_q(j+i) q^n \\
&\qquad \leq q^n \left\{ 3^m m (1-q)^{-m} + 4^m + 3^m m C_q \right\} \to 0,
\end{aligned}
$$

as $n \to \infty$, for every $x \in [0,1]$ and $m = 0,1,2,\ldots$ This means that the conditions (i) and (ii) of Theorem 2.1 are satisfied, the second one for $X = \{e_m |\, m = 0,1,2,\ldots\}$ dense in $C[0,1]$. In conclusion: *there exists the positive linear operator $B^*_{\infty,q} : C[0,1] \to C[0,1]$ such that $\|B^*_{n,q}(f) - B^*_{\infty,q}(f)\| \to 0$ as $n \to \infty$, where $f \in C[0,1]$ is arbitrary and $q \in (0,1)$ is fixed.*

2° The following q-Durrmeyer operators $D_{n,q} : C[0,1] \to C[0,1]$ were introduced in [3] and are defined with the aid of Jackson integral:

$$
(D_{n,q}(f))(x) = [n+1] \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_0^1 f(t) p_{n,k}(q;qt)\, d_q t,
$$

where $0 < q < 1$, $f \in C[0,1]$, $x \in [0,1]$ and $p_{n,k}(q;x)$ is given by (1.2). Because of [8, p. 412, (3.2)–(3.3)], we may write

$$
\begin{aligned}
(3.8)\qquad &(D_{n,q}(f))(x) - (D_{n+1,q}(f))(x) \\
&= \sum_{k=0}^{n+1} p_{n+1,k}(q;x) \left\{ \frac{[n+1-k]}{[n+1]} [n+1] q^{-k} \int_0^1 f(t) p_{n,k}(q;qt)\, d_q t \right. \\
&\quad + q^{n+1-k} \frac{[k]}{[n+1]} [n+1] q^{-k+1} \int_0^1 f(t) p_{n,k-1}(q;qt)\, d_q t \\
&\quad \left. - [n+2] q^{-k} \int_0^1 f(t) p_{n+1,k}(q;qt)\, d_q t \right\}.
\end{aligned}
$$

In view of $[n+1-k] + q^{n+1-k}[k] = [n+1]$, $[n+1]q^{-k} \int_0^1 p_{n,k}(q;qt)\, d_q t = 1$ (see [3, p. 173, (3)]) and Taylor's formula: $g(t) = g(x_0) + \int_{x_0}^t g'(u)\, du$, where $g \in C^1[0,1]$ and $x_0 \in [0,1]$, we have

$$\begin{aligned}
(3.9)\quad & \Bigg|\frac{[n+1-k]}{[n+1]}[n+1]q^{-k}\int_0^1 g(t)p_{n,k}(q;qt)\,d_qt + q^{n+1-k}\frac{[k]}{[n+1]}[n+1] \\
& \times q^{-k+1}\int_0^1 g(t)p_{n,k-1}(q;qt)\,d_qt - [n+2]q^{-k}\int_0^1 g(t)p_{n+1,k}(q;qt)\,d_qt\Bigg| \\
& \le \frac{[n+1-k]}{[n+1]}\Bigg|[n+1]q^{-k}\int_0^1 p_{n,k}(q;qt)\left\{\int_{x_0}^t g'(u)\,du\right\}d_qt \\
& - [n+2]q^{-k}\int_0^1 p_{n+1,k}(q;qt)\left\{\int_{x_0}^t g'(u)\,du\right\}d_qt\Bigg| + q^{n+1-k} \\
& \frac{[k]}{[n+1]}\Bigg|[n+1]q^{-k+1}\int_0^1 p_{n,k-1}(q;qt)\left\{\int_{x_0}^t g'(u)\,du\right\}d_qt \\
& - [n+2]q^{-k}\int_0^1 p_{n+1,k}(q;qt)\left\{\int_{x_0}^t g'(u)\,du\right\}d_qt\Bigg| \\
& \le \frac{[n+1-k]}{[n+1]}\|g'\|\int_0^1 [n+2]q^{-k}p_{n+1,k}(q;qt)\Bigg|\frac{[n+1]p_{n,k}(q;qt)}{[n+2]p_{n+1,k}(q;qt)} \\
& - 1\Bigg||t-x_0|\,d_qt + q^{n+1-k}\frac{[k]}{[n+1]}\|g'\|\int_0^1 [n+2]q^{-k}p_{n+1,k}(q;qt) \\
& \times\left|\frac{[n+1]qp_{n,k-1}(q;qt)}{[n+2]p_{n+1,k}(q;qt)} - 1\right||t-x_0|\,d_qt.
\end{aligned}$$

Further

$$\begin{aligned}
(3.10)\quad & \left|\frac{[n+1]p_{n,k}(q;qt)}{[n+2]p_{n+1,k}(q;qt)} - 1\right| \\
& = \left|\frac{[n+1-k]}{[n+2]}\frac{q^{n+1-k}t}{1-q^{n+1-k}t} - q^{n+1-k}\frac{[k+1]}{[n+2]}\right| \\
& \le q^{n+1-k}\left(\frac{[n+1-k]}{[n+2]}\frac{1}{1-q} + \frac{[k+1]}{[n+2]}\right) \\
& \le q^{n+1-k}\frac{2-q}{1-q},
\end{aligned}$$

for $k = 0, 1, \ldots, n$ and

$$(3.11)\qquad \left|\frac{[n+1]qp_{n,k-1}(q;qt)}{[n+2]p_{n+1,k}(q;qt)} - 1\right| = \left|\frac{[k]}{[n+1]}\frac{1}{t} - 1\right| \le \frac{2}{t},$$

for $k = 1, 2, \ldots, n+1$.

Now combining (3.8)–(3.11), and applying the identity $p_{n+1,k}(q;x) = x[n+1]p_{n,k-1}(q;x)$ and Hölder's inequality, respectively, we find

$$
\begin{aligned}
&|(D_{n,q}(g))(x)-(D_{n+1,q}(g))(x)| \\
&\le \frac{2-q}{1-q}\|g'\| \sum_{k=0}^{n} p_{n+1,k}(q;x)\frac{[n+1-k]}{[n+1]} \\
&\quad \times \int_0^1 [n+2]q^{-k}p_{n+1,k}(q;qt)q^{n+1-k}|t-x_0|d_qt \\
&\quad +\|g'\|\sum_{k=1}^{n+1} p_{n+1,k}(q;x)q^{n+1-k}\frac{[k]}{[n+1]}\int_0^1 [n+2]q^{-k}p_{n+1,k}(q;qt)\frac{2}{t}|t-x_0|d_qt \\
&\le \frac{2-q}{1-q}\|g'\| \sum_{k=0}^{n} p_{n+1,k}(q;x)\frac{[n+1-k]}{[n+1]}q^{n+1-k}\int_0^1 [n+2]q^{-k}p_{n+1,k}(q;qt)|t-x_0|d_qt \\
&\quad +2\|g'\|\sum_{k=1}^{n+1} p_{n+1,k}(q;x)q^{n+1-k}\frac{[k]}{[n+1]}[n+2] \\
&\quad \times \int_0^1 [n+1]q^{-k+1}p_{n,k-1}(q;qt)|t-x_0|d_qt \\
&\le \frac{2-q}{1-q}\|g'\| \sum_{k=0}^{n} p_{n+1,k}(q;x)\frac{[n+1-k]}{[n+1]}q^{n+1-k} \\
&\quad \times \left\{\int_0^1 [n+2]q^{-k}p_{n+1,k}(q;qt)(t-x_0)^2d_qt\right\}^{1/2} \\
&\quad +2\|g'\|\sum_{k=1}^{n+1} p_{n+1,k}(q;x)q^{n+1-k}\frac{[k]}{[n+1]}[n+2] \\
&\quad \times \left\{\int_0^1 [n+1]q^{-k+1}p_{n,k-1}(q;qt)(t-x_0)^2d_qt\right\}^{1/2}.
\end{aligned}
$$

Hence, in view of $\int_0^1 t^s p_{n,k}(q;qt)\,d_qt = \frac{q^k[n]![k+s]!}{[n+s+1]![k]!}$, $s=0,1,2,\dots$ (see [3, p. 173, (3)]), we obtain

$$
\begin{aligned}
&|(D_{n,q}(g))(x)-(D_{n+1,q}(g))(x)| \\
&\quad \le \frac{2-q}{1-q}\|g'\| \sum_{k=0}^{n} p_{n+1,k}(q;x)\frac{[n+1-k]}{[n+1]}q^{n+1-k}\left(\frac{[k+1][k+2]}{[n+4][n+3]}\right. \\
&\quad\quad \left.-2x_0\frac{[k+1]}{[n+3]}+x_0^2\right)^{1/2} + 2\|g'\|\sum_{k=1}^{n+1} p_{n+1,k}(q;x)q^{n+1-k}\frac{[k]}{[n+1]} \\
&\quad\quad \times [n+2]\left(\frac{[k][k+1]}{[n+3][n+2]}-2x_0\frac{[k]}{[n+2]}+x_0^2\right)^{1/2}.
\end{aligned}
$$

Choosing $x_0 = \frac{[k+1]}{[n+3]}$, $k=0,1,\ldots,n+1$, and taking into account $(B_{n+1,q}(e_0))(x) = 1$, we have, for all $n \geq 1$, that

$$
\begin{aligned}
(3.12)\qquad &|(D_{n,q}(g))(x) - (D_{n+1,q}(g))(x)| \\
&\leq \frac{2-q}{1-q}\|g'\| \sum_{k=0}^{n} p_{n+1,k}(q;x) \frac{[n+1-k]}{[n+1]} q^{n+1-k} \\
&\quad \times \left(\frac{[k+1]}{[n+3]} \frac{q^{k+1}[n+2-k]}{[n+3][n+4]}\right)^{1/2} + 2\|g'\| \sum_{k=1}^{n+1} p_{n+1,k}(q;x) \\
&\quad \times q^{n+1-k} \frac{[k]}{[n+1]} [n+2] \left(\frac{[k+1]}{[n+3]} \frac{q^k[n+2-k]}{[n+2][n+3]}\right)^{1/2} \\
&\leq \frac{2-q}{1-q}\|g'\| q^{(n+3)/2} + 2\|g'\| q^{(n+1)/2}[n+2]^{1/2} \\
&\leq \frac{2-q}{1-q}\|g'\| q^{n/2} + 2\|g'\| q^{n/2}(3[n])^{1/2} \\
&\leq \left(\frac{2-q}{1-q} + 2\sqrt{3}\right) \|g'\| (q^n[n])^{1/2}.
\end{aligned}
$$

We set $\beta_n = (q^n[n])^{1/2}$, $n \geq 1$. Then, for all $n, p \geq 1$, we get

$$
\begin{aligned}
&\beta_n + \beta_n + \ldots + \beta_{n+p-1} \\
&\quad = (q^n[n])^{1/2} + (q^{n+1}[n+1])^{1/2} + \ldots + (q^{n+p-1}[n+p-1])^{1/2} \\
&\quad = \left(q^n \frac{1-q^n}{1-q}\right)^{1/2} + \left(q^{n+1} \frac{1-q^{n+1}}{1-q}\right)^{1/2} + \ldots + \left(q^{n+p-1} \frac{1-q^{n+p-1}}{1-q}\right)^{1/2} \\
&\quad \leq \left(\frac{q^n}{1-q}\right)^{1/2} \{1 + \sqrt{q} + \ldots + (\sqrt{q})^{p-1}\} \leq \frac{q^{n/2}}{\sqrt{1-q}(1-\sqrt{q})}.
\end{aligned}
$$

For $\alpha_n = \frac{q^{n/2}}{\sqrt{1-q}(1-\sqrt{q})}$, $n \geq 1$, we obtain $\alpha_n \to 0$ as $n \to \infty$. Thus, by Corollary 2.1, *there exists a positive linear operator* $D_{\infty,q} : C[0,1] \to C[0,1]$ *such that*

$$
\|D_{n,q}(f) - D_{\infty,q}(f)\| \leq \left(\frac{2-q}{1-q} + 2\sqrt{3}\right) \omega\left(f, \frac{q^{n/2}}{\sqrt{1-q}(1-\sqrt{q})}\right),
$$

for every $f \in C[0,1]$ *and* $q \in (0,1)$ *fixed.*

Remark 3.1. Similar results can be obtained for further q-parametric operators as Lupaş q-analogue of the Bernstein operator [6], q-Bernstein operators [9], q-Meyer-König and Zeller operators [10]. We mention only the following result: *there exists a positive linear operator* $B_{\infty,q} : C[0,1] \to C[0,1]$ *such that*

$$\|B_{n,q}(f) - B_{\infty,q}(f)\| \leq 2\,\omega_2\left(f, \frac{q^{n/2}}{1-q^n}\right),$$

for all $n \geq 1$, $f \in C[0,1]$ *and* $q \in (0,1)$ *fixed.*

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