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OPEN SET LATTICES OF SUBSPACES OF SPECTRUM SPACES

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Abstract. We take a unified approach to study the open set lattices of various subspaces of the spectrum of a multiplicative lattice L. The main aim is to establish the order isomorphism between the open set lattice of the respective subspace and a sub-poset of L. The motivating result is the well known fact that the topology of the spectrum of a commutative ring R with identity is isomorphic to the lattice of all radical ideals of R. The main results are as follows: (i) for a given nonempty set S of prime elements of a multiplicative lattice L, we define the S-semiprime elements and prove that the open set lattice of the subspace S of $Spec(L)$ is isomorphic to the lattice of all S-semiprime elements of L; (ii) if L is a continuous lattice, then the open set lattice of the prime spectrum of L is isomorphic to the lattice of all m-semiprime elements of L; (iii) we define the pure elements, a generalization of the notion of pure ideals in a multiplicative lattice and prove that for certain types of multiplicative lattices, the sub-poset of pure elements of L is isomorphic to the open set lattice of the subspace $Max(L)$ consisting of all maximal elements of L.

1. Introduction

One classical result in commutative ring theory is that the open set lattice of the spectrum of a commutative ring R, endowed with the hull kernel topology, is isomorphic to the lattice of all radical ideals of R. It is then natural to consider the following general problem: given a multiplicative lattice L and a subspace S of the spectrum of L, can we find a subset of L which is order isomorphic to the open set lattice of S? In this paper, we first prove that for any nonempty subset S of the spectrum of L, the open set lattice of the subspace S is isomorphic to the sub-poset of all S-semiprime elements of L. Then we focus on two special subspaces of spectrum of L to give a more specific characterization of the corresponding S-semiprime elements.

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The outline of this paper is as follows: In Section 2, we state some basic definitions and results in lattice theory and multiplicative lattice theory. In Section 3, we introduce the notion of S-semiprime elements of a multiplicative lattice and prove some basic properties of such elements. One fundamental result proved here is that for any nonempty set S of prime elements of a multiplicative lattice L, the open set lattice of the hull kernel topology of S is isomorphic to the poset of all S-semiprime elements of L. In Section 4, we consider the largest subset S of prime elements of a multiplicative lattice L and characterize the S-semiprime elements for some special types of L. We show that if L is a continuous multiplicative lattice, then the open set lattice of the prime spectrum of L is isomorphic to the lattice of all m-semiprime elements of L. In Section 5, we generalize the notion of pure ideals in commutative ring theory by defining the pure elements in any multiplicative lattice. We also consider a special type of multiplicative lattice, called mp-multiplicative lattice that is principally generated with the top element compact and satisfying two additional conditions. We prove that if L is a reduced mp-multiplicative lattice and an r-lattice in which every prime element is beneath a unique maximal element, then the poset of pure elements of L is isomorphic to the open set lattice of the space of all maximal elements of L endowed with the hull kernel topology.

2. Preliminaries

By Dilworth [8], a multiplicative lattice is a complete lattice L together with a multiplication (the multiplication of x and y is simply denoted by xy) that is associative, commutative, distributive over arbitrary joins and has the greatest element 1_L as the multiplication identity. The complete lattices of ideals of commutative rings are the most important motivating examples of multiplicative lattices.

The following are some of the basic definitions we will frequently use in this paper:

(1) In a multiplicative lattice L, $xy \leq x$ and $xy \leq y$ hold for all $x, y \in L$.

(2) Let L be a multiplicative lattice. An element p of L is called a prime element if $p \neq 1_L$ and for any $a, b \in L$, $ab \leq p$ implies $a \leq p$ or $b \leq p$. The set of all prime elements of L will be denoted by $\mathrm{Spec}(L)$.

(3) An element p of a complete lattice L is a maximal (minimal) element if $p \neq 1_L$ ($p \neq 0_L$) and $p \leq m < 1_L$ ($0_L < m \leq p$) implies $m = p$ ($m = p$). Every maximal element of a multiplicative lattice is a prime element. In fact, suppose $ab \leq x$ where x is maximal, $a \nleq x$ and $b \nleq y$. Since x is maximal, $x \vee a = x \vee b = 1_L$. But $1_L = (x \vee a)(x \vee b) = x^2 \vee ax \vee xb \vee ab \leq x$ which is a contradiction.

(4) An element c of a complete lattice L is said to be compact if for any set $I \subseteq L$ with $c \leq \bigvee I$, then $c \leq \bigvee J$ for some finite subset $J \subseteq I$. The set of all compact elements of L will be denoted by $K(L)$.

A complete lattice L is algebraic, if each $x \in L$ is a supremum of compact elements of L. This is equivalent to that for each $x \in L$, $x = \bigvee\{y \in K(L) : y \leq x\}$.

If L is algebraic and 1_L is compact, then L is said to be compact.

(5) An element x of a multiplicative lattice L is called a principal element if $(x \wedge [y : a])a = xa \wedge y$ and $[(y \vee xa) : a] = [y : a] \vee x$ hold for any $x, y \in L$, where $[y : a] = \bigvee\{u \in L : ua \leq y\}$.

(6) A multiplicative lattice L is called principally generated if every element of L is a join of some principal elements.

A multiplicative lattice L is called an r-lattice if

(i) L is a modular lattice (that is, for any $a, b, c \in L$ with $c \leq a$, $a \wedge (b \vee c) = (a \wedge b) \vee c)$);
(ii) L is principally generated, and
(iii) L is compact.

(7) A frame L is a complete lattice such that for any $S \subseteq L$ and $a \in L$,

$$a \wedge (\bigvee S) = \bigvee\{a \wedge s : s \in S\}.$$

Each frame can be regarded as a special multiplicative lattice in which the meet is the multiplication. A frame L is spatial if every element $x \in L$ is an infimum of prime elements of L.

A frame L is called coherent if L is algebraic, $1_L \in K(L)$ and $a, b \in K(L)$ implies $a \wedge b \in K(L)$ [6].

Lemma 2.1. [8] *If x and y are principal elements of a multiplicative lattice L, then xy is also a principal element of L.*

Lemma 2.2. [1] *If L is an r-lattice, then every principal element of L is compact.*

Lemma 2.3. *If L is an r-lattice, then the product of two compact elements of L is compact.*

Proof. Let a, b be compact elements of L. Then there exist sets P and Q of principal elements such that $a = \bigvee P, b = \bigvee Q$. Since a, b are compact, we can take P, Q to be finite sets. Now $ab = \bigvee\{xy : x \in P, y \in Q\}$ is a join of finite number of elements xy. As a product of principal elements, each xy is principal, hence compact. Since a finite join of compact elements is compact, ab is compact. ∎

For more details on lattices, multiplicative lattices and frames, see [5], [9], [8], [1] and [6].

3. S-semiprime elements

Let L be a multiplicative lattice and S be a nonempty set of prime elements of L.

For any element $a \in L$, define

$$p_S(a) = \bigwedge\{r \in S : a \leq r\}.$$

If $\{r \in S : a \leq r\} = \emptyset$, then $p_S(a) = 1_L$. Note that since L is a complete lattice, $p_S(a)$ exists for all $a \in L$. Thus $p_S : L \to L$ is a mapping.

Remark 3.1. It is clear that p_S has the following properties:

(i) $a \leq p_S(a)$ for all $a \in L$.
(ii) For $a, b \in L$, $a \leq b$ implies $p_S(a) \leq p_S(b)$.
(iii) For any $a \in L$, $p_S(p_S(a)) = p_S(a)$.

Thus p_S is a closure operator on L.

Lemma 3.2. *Let S be a nonempty set of prime elements of a multiplicative lattice L. Then for any $a, b \in L$,*

$$p_S(ab) = p_S(a \wedge b) = p_S(a) \wedge p_S(b).$$

Proof. Note that $ab \leq a$ holds for any $a, b \in L$. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, we have $p_S(a \wedge b) \leq p_S(a)$ and $p_S(a \wedge b) \leq p_S(b)$, which implies that $p_S(a \wedge b) \leq p_S(a) \wedge p_S(b)$. Since $ab \leq a \wedge b$, it follows that $p_S(ab) \leq p_S(a \wedge b)$.

Let $w \in S$ such that $w \geq ab$. Since w is prime, $a \leq w$ or $b \leq w$. So $w \geq p_S(a)$ or $w \geq p_S(b)$. In either case, $w \geq p_S(a) \wedge p_S(b)$. It follows that $p_S(ab) \geq p_S(a) \wedge p_S(b)$. Hence the equality follows. ∎

Definition 3.3. For a given nonempty set S of prime elements of a multiplicative lattice L, an element a of L is called *S-semiprime* if $p_S(a) = a$.

The set of all S-semiprime elements of L will be denoted by $\Omega_S(L)$.

Remark 3.4. (1) Clearly, for any nonempty set S of prime elements of a multiplicative lattice L, $S \subseteq \Omega_S(L)$.

(2) For any $a \in L$, $p_S(a) \in \Omega_S(L)$ since $p_S(p_S(a)) = p_S(a)$. Hence

$$\Omega_S(L) = \{p_S(a) : a \in L\}.$$

It thus follows that an element $a \in L$ is S-semiprime if and only if a is the meet of some elements in S.

Lemma 3.5. *Let S be a nonempty set of prime elements of a multiplicative lattice L. Then $(\Omega_S(L), \leq)$ is a complete lattice, where the partial order $\leq$ is* *inherited from L. Furthermore, for any $A \subseteq \Omega_S(L)$,*

(1) $$\bigvee_{\Omega_S(L)} A = p_S\Big(\bigvee_L A\Big)$$

(2) $$\bigwedge_{\Omega_S(L)} A = \bigwedge_L A$$

Proof. Let $A \subseteq \Omega_S(L)$. For any $x \in A$, $x \le \bigvee_L A \le p_S(\bigvee_L A)$. So $p_S(\bigvee_L A)$ is an upper bound for A in $\Omega_S(L)$.

If $c \in \Omega_S(L)$ is an upper bound of A in $\Omega_S(L)$, then $c \ge x$ for each $x \in A$ and so $c \ge \bigvee_L A$. Hence $c = p_S(c) \ge p_S(\bigvee_L A)$. Thus (1) holds.

For (2), first note that $\bigwedge_L A \le p_S(\bigwedge_L A)$. Secondly, for any $x \in A$, $p_S(\bigwedge_L A) \le p_S(x) = x$, so $p_S(\bigwedge_L A) \le \bigwedge_L A$. Hence $p_S(\bigwedge_L A) = \bigwedge_L A$ which means $\bigwedge_L A \in \Omega_S(L)$. Clearly, $\bigwedge_L A$ is a lower bound for A in $\Omega_S(L)$.

If $d \in \Omega_S(L)$ is a lower bound of A in $\Omega_S(L)$, then $d \le x$ for each $x \in A$ and so $d \le \bigwedge_L A$. Thus (2) holds. ∎

Remark 3.6. It follows from Lemma 3.5 that the infimum of a subset A of $\Omega_S(L)$ in $\Omega_S(L)$ is the same as the infimum of A in L.

Theorem 3.7. *Let S be a nonempty set of prime elements of a multiplicative lattice. Then $\Omega_S(L)$ is a spatial frame.*

Proof. For any $a, b_i \in \Omega_S(L)(i \in I)$,

$$\begin{aligned} a \wedge \textstyle\bigvee_{\Omega_S(L)}\{b_i : i \in I\} &= p_S(a) \wedge p_S(\textstyle\bigvee_L\{b_i : i \in I\}) \\ &= p_S(a(\textstyle\bigvee_L\{b_i : i \in I\})) \\ &= p_S(\textstyle\bigvee_L\{ab_i : i \in I\}) \\ &= p_S(\textstyle\bigvee_L\{a \wedge b_i : i \in I\}) \\ &= \textstyle\bigvee_{\Omega_S(L)}\{a \wedge b_i : i \in I\}). \end{aligned}$$

The second last equation holds because for every $i \in I$, $p_S(\bigvee_L\{ab_i : i \in I\}) \ge p_S(ab_i) = p_S(a) \wedge p_S(b_i)$ (by Lemma 3.2) $= a \wedge b_i$, and for each $i \in I$, $p_S(\bigvee_L\{a \wedge b_i : i \in I\}) \ge p_S(a \wedge b_i) \ge ab_i$. The last equation holds by Lemma 3.5. It follows that $\Omega_S(L)$ is distributive over arbitrary joins and so $\Omega_S(L)$ is a frame.

To show that $\Omega_S(L)$ is a spatial frame, we need to show that the set of prime elements of $\Omega_S(L)$ is meet dense in $\Omega_S(L)$. As mentioned in Remark 3.4(1), $S \subseteq \Omega_S(L)$. Also, for any $p \in S$, $a, b \in \Omega_S(L)$, if $a \wedge b \le p$, then $ab \le a \wedge b \le p$, thus $a \le p$ or $b \le p$.

Hence any element in S is a prime element of the lattice $\Omega_S(L)$. Also, the infimum of a subset A of $\Omega_S(L)$ in $\Omega_S(L)$ is equal to its infimum in L, thus S is meet dense in $\Omega_S(L)$, which further implies that the set of prime elements of $\Omega_S(L)$ is meet dense in $\Omega_S(L)$. Hence $\Omega_S(L)$ is a spatial frame. ∎

Proposition 3.8. *Let L be a multiplicative lattice. For any nonempty set S of prime elements of L and any $b \in L$, define*

$$D_S(b) = \{p \in S : b \nleq p\}.$$

Then

(i) $D_S(0_L) = \emptyset$;
(ii) $D_S(bc) = D_S(b \wedge c) = D_S(b) \cap D_S(c)$;
(iii) $D_S(\bigvee b_i) = \bigcup D_S(b_i), i \in I$.

Proof. (i) is trivial.

(ii) For any $w \in D_S(b) \cap D_S(c)$, $b \nleq w$ and $c \nleq w$. So $bc \nleq w$ since w is prime and so $w \in D_S(bc)$. Hence $D_S(b) \cap D_S(c) \subseteq D_S(bc)$.

For any $x \notin D_S(b \wedge c)$, $x \geq b \wedge c$ which implies that $x \geq bc$. Hence $x \notin D_S(bc)$ and so $D_S(bc) \subseteq D_S(b \wedge c)$.

To prove $D_S(b \wedge c) \subseteq D_S(b) \cap D_S(c)$, we suppose $y \notin D_S(b) \cap D_S(c)$. Then $y \notin D_S(b)$ or $y \notin D_S(c)$. So $y \geq b$ or $y \geq c$ which implies that in either case, $y \geq b \wedge c$. It follows that $y \notin D_S(b \wedge c)$. So $D_S(b \wedge c) \subseteq D_S(b) \cap D_S(c)$.

Combining the three inclusions, we have $D_S(b) \cap D_S(c) \subseteq D_S(bc) \subseteq D_S(b \wedge c) \subseteq D_S(b) \cap D_S(c)$ and so the equality follows.

To prove (iii), it is enough to note that for any $y \in S$, $\bigvee b_i \nleq y$ if and only if $b_{i_0} \nleq y$ for some $i_0 \in I$. ∎

It follows from above that sets of the form $D_S(b)$ $(b \in L)$ are the open sets of a topology on S. This topology τ on S will be called the hull-kernel topology and so (S, τ) is a topological space.

Theorem 3.9. *For any nonempty set S of prime elements of a multiplicative lattice L, the open set lattice of the hull kernel topology on S is isomorphic to $\Omega_S(L)$.*

Proof. Let $\mathcal{O}(S) = \{D_S(b) : b \in L\}$. Define $\phi : \mathcal{O}(S) \to \Omega_S(L)$ by $\phi(D_S(b)) = p_S(b)$, $b \in L$. We first show that ϕ is well-defined. Suppose $D_S(b_1) = D_S(b_2)$ for $b_1, b_2 \in L$. Then

$$\{p \in S : b_1 \nleq p\} = \{p \in S : b_2 \nleq p\}$$

which implies that

$$\{p \in S : b_1 \leq p\} = \{p \in S : b_2 \leq p\}.$$

It follows that

$$p_S(b_1) = \bigwedge\{p \in S : b_1 \leq p\} = \bigwedge\{p \in S : b_2 \leq p\} = p_S(b_2).$$

If $D_S(b_1) \subseteq D_S(b_2)$ for $b_1, b_2 \in L$, then

$$p_S(b_1) = \bigwedge\{p \in S : p \geq b_1\} \leq \bigwedge\{p \in S : p \geq b_2\} = p_S(b_2).$$

Hence $\phi(D_S(b_1)) \leq \phi(D_S(b_2))$ and so ϕ is monotone.

We now show that if $\phi(D_S(b_1)) \leq \phi(D_S(b_2))$ for $b_1, b_2 \in L$, then $D_S(b_1) \subseteq D_S(b_2)$. Let $w \in D_S(b_1)$. Then $b_1 \nleq w$. Since $p_S(b_1) = \phi(D_S(b_1)) \leq \phi(D_S(b_2)) = p_S(b_2)$, it follows that $b_2 \nleq w$. Hence $w \in D_S(b_2)$ and so $D_S(b_1) \subseteq D_S(b_2)$.

We proceed to show that ϕ is injective. Suppose $\phi(D_S(b_1)) = \phi(D_S(b_2))$, for $b_1, b_2 \in L$. Then $\phi(D_S(b_1)) \leq \phi(D_S(b_2))$ and $\phi(D_S(b_2)) \leq \phi(D_S(b_1))$. It follows that $D_S(b_1) \subseteq D_S(b_2)$ and $D_S(b_2) \subseteq D_S(b_1)$. Hence $D_S(b_1) = D_S(b_2)$.

For any $b \in \Omega_S(L)$, $\phi(D_S(b)) = p_S(b) = b$. It follows that ϕ is surjective.

All these show that ϕ is an isomorphism and so the open set lattice of the hull kernel topology on S is isomorphic to $\Omega_S(L)$. ∎

4. Prime spectrum of a multiplicative lattice

In this section, we consider the largest subset S of prime elements of a multiplicative lattice L and characterize the S-semiprime elements for some special types of L.

An element a of a multiplicative lattice L is called m-semiprime [10] if $a \neq 1_L$ and for any $x \in L$,

$$x^2 \leq a \text{ implies } x \leq a.$$

Let $mSprime(L)$ denote the set of all m-semiprime elements of L. Clearly, $Spec(L) \subseteq mSprime(L)$.

Lemma 4.1. *If A is a nonempty subset of $mSprime(L)$, then $\bigwedge A \in mSprime(L)$.*

Proof. If $x^2 \leq \bigwedge A$, then $x^2 \leq a$ for each $a \in A$. Since $A \subseteq mSprime(L)$, so $x \leq a$ for each $a \in A$ and so $x \leq \bigwedge A$. ∎

The next question we want to address is for what L, every m-semiprime element of L is a meet of some prime elements of L?

An element x of a complete lattice L is way-below an element y in L, written $x \ll y$, if for any directed set $D \subseteq L$, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$ [4].

A complete lattice L is continuous if for any $a \in L$,

$$a = \bigvee \{x \in L : x \ll a\}.$$

Every algebraic lattice is continuous.

A subset U of a complete lattice A is Scott open iff $U = \uparrow U$ and for any directed subset $D \subseteq A$, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. The complements of Scott open sets are called Scott closed sets [4].

A Scott open set U of a multiplicative lattice L is called *m-filtered* if $x \in U$ and $y \in U$ imply $xy \in U$.

Lemma 4.2. *If U is an m-filtered Scott open set of L, then*

(1) *for any $a \in U^c = L - U$, there is a maximal element b of U^c such that $a \leq b$;*
(2) *each maximal element of U^c is a prime element.*

Proof. (1) Let $a \in U^c$. Take any maximal chain C in U^c such that $a \in C$. Since U^c, as a Scott closed set, is closed under supremum of directed sets, $\bigvee C \in U^c$. Clearly $\bigvee C$ is a maximal element of U^c lying above a.

(2) Let p be any maximal element of U^c. Let $d_1, d_2 \in L$ such that $d_1 d_2 \leq p$. If $d_1 \nleq p$ and $d_2 \nleq p$, then $d_1 \vee p \in U$ and $d_2 \vee p \in U$ since p is maximal in U^c. It follows that $(d_1 \vee p)(d_2 \vee p) = d_1 d_2 \vee d_1 p \vee p d_2 \vee p^2 \leq p$ which implies that $(d_1 \vee p)(d_2 \vee p) \in U^c$, contradicting the fact that U is m-filtered. Thus p is a prime element of L. ∎

Theorem 4.3. *Let a be any element of a multiplicative lattice L. Consider the following statements:*

(1) *For any $b \in L$ with $b \nleq a$, there is an m-filtered Scott open set U such that $b \in U$ and $a \notin U$.*
(2) $a = \bigwedge \{p \in Spec(L) : a \leq p\}$.
(3) *a is m-semiprime.*

Then (1) *is equivalent to* (2), *and* (2) *implies* (3).

If L is a continuous lattice, then (3) *implies* (2). *Thus all the statements are equivalent.*

Proof. (1) implies (2): Let $w = \bigwedge \{p \in Spec(L) : a \leq p\}$. Clearly, $a \leq w$. Suppose $w \nleq a$. Then since (1) holds, there is an m-filtered Scott open set U such that $w \in U$ and $a \notin U$. So $a \in U^c$. By the previous lemma, there is a maximal element $b \in U^c$ such that $a \leq b$ and that b is prime. So $b \geq w$ by definition of w, which implies $b \in U$ since U is Scott open. This contradicts $b \in U^c$. Hence $w \leq a$ and so $w = a$.

(2) implies (1): Assume that $a = \bigwedge \{p \in Spec(L) : a \leq p\}$. If $b \nleq a$, then there is $p \in Spec(L)$ such that $a \leq p$ and $b \nleq p$. Let $U = L - \downarrow p$. Clearly, U is a Scott open set and $b \in U, a \notin U$. We prove that U is m-filtered. Let $d_1 \in U$ and $d_2 \in U$. Then $d_1 \nleq p$ and $d_2 \nleq p$. So $d_1 d_2 \nleq p$ which implies that $d_1 d_2 \in U$.

(2) implies (3): Let $x \in L$ such that $x^2 \leq a$. Hence $x^2 \leq p$ for all $p \in Spec(L)$. Since p is prime, $x \leq p$ for all $p \in Spec(L)$ which implies that $x \leq a$ and so a is m-semiprime.

For the proof of (3) implies (2) when L is a continuous lattice, we refer the reader to [13]. ∎

If L is a continuous distributive lattice, then L is a multiplicative lattice where the multiplication is the meet. Then every element is $m-$semiprime, so by the above theorem we deduce the following result which appeared in [4].

Corollary 4.4. *If L is a continuous distributive lattice, then every element of L is a meet of meet prime elements.*

Corollary 4.5. *Let L be a continuous multiplicative lattice. Then the open set lattice of the prime spectrum of L is isomorphic to the lattice $mSprime(L)$.*

For any commutative ring R, an ideal I is a radical ideal if for any $x \in R$, $x^n \in I$ for some positive integer n implies $x \in I$. The set of all radical ideals of R is denoted by $RIdl(R)$.

One can easily verify that $I \in RIdl(R)$ if and only if I is an $m-$semiprime element of the multiplicative lattice $Idl(R)$ of all ideals of R.

Thus Corollary 4.5 deduces the following well known nice fact.

Corollary 4.6. *For any commutative ring R, the open set lattice of $Spec(R)$ is isomorphic to the lattice $RIdl(R)$ of all radical ideals of R.*

Example 4.7. Let $L = [0, 1]$ be the set of all real numbers between 0 and 1. Then $(L, \leq, \times)$ is a multiplicative lattice, which is continuous but not algebraic. Hence, there is no commutative ring R such that $Idl(R)$ is isomorphic to $L = [0, 1]$.

In general, the Cartesian product of any collection of $[0, 1]$ is a continuous multiplicative lattice which is not algebraic.

5. Pure elements and the space of maximal elements

An ideal I in a commutative ring R is called a pure ideal [7] if for each $a \in I$, there exists $b \in I$ such that $ab = a$. There are a number of papers that characterize pure ideals in certain special types of rings, such as reduced Gelfand rings.

By a theorem of G. De Marco [7], for any commutative ring R with identity in which every prime ideal is contained in a unique maximal ideal, the lattice of all pure ideals of R is isomorphic to the open set lattice of the space $Max(R)$ of all maximal ideals of R. In this section, we first define and study pure elements of multiplicative lattices; we then generalize G. De Marco's result to some types of multiplicative lattices.

Definition 5.1. Let L be a multiplicative lattice. An element $a \in L$ is called a *pure element* if for any $x \leq a$, there exists $y \leq a$ such that $x = xy$.

Proposition 5.2. *Let a be any element of a multiplicative lattice L. Then the following statements are equivalent:*

(i) *a is a pure element.*
(ii) *For any $x \in L$, $x \leq a$ implies $x = xa$.*
(iii) *For any $b \in L$, $ab = a \wedge b$.*

Proof. (i) implies (ii): Since a is a pure element and $x \leq a$ for any $x \in L$, so there exists $y \leq a$ such that $x = xy \leq xa$. Clearly, $xa \leq x$ for any $x \in L$. Hence $x = xa$.

(ii) implies (iii): We only need to show that $a \wedge b \leq ab$. Let $w = a \wedge b$. Since $w \leq a$, by (ii), $w = wa$. Also $w \leq b$ which implies that $wa \leq ba = ab$. Thus $w = wa \leq ab$.

(iii) implies (i): Let $x \leq a$. Then $x = x \wedge a$ and by (iii), $x = xa$. Choose y to be a. Hence there exists $y = a \leq a$ such that $xy = xa = x$. It follows that a is a pure element. ■

Clearly, 0_L and 1_L are pure elements of L. We shall denote the set of all pure elements of L by $Pur(L)$.

Corollary 5.3. *If $a, b \in Pur(L)$, then $ab \in Pur(L)$.*

Proof. Let $a, b \in L$. For any $x \in L$ such that $x \leq ab$, we have $x \leq a$ and since a is pure, $xa = x$. Similarly, since $x \leq b$ and b is pure, $xb = x$. It then follows that $xab = (xa)b = xb = x$, thus ab is pure. ■

Lemma 5.4. *If L is a principally generated multiplicative lattice, then $a \in L$ is a pure element of L if and only if for any principal element $y \leq a$,*

$$y = ya.$$

Proof. The necessity is trivial. Now let a satisfy the given condition. For any element $x \in L$ with $x \leq a$, since L is principally generated, $x = \bigvee\{y \in L : y$ is principal, $y \leq x\}$. So

$$\begin{aligned} xa &= \Big(\bigvee\{y \in L : y \leq x, y \text{ is principal}\}\Big)a \\ &= \bigvee\{ay : y \leq x, y \text{ is principal}\} \\ &= \bigvee\{y : y \leq x, y \text{ is principal}\} \text{ (if } y \leq x \text{ then } y \leq a) \\ &= x. \ ■ \end{aligned}$$

Definition 5.5. Let L be a principally generated multiplicative lattice, with the top element 1_L compact. For any $a \in L$, define:

$$D(a) = \{p \in Spec(L) : a \nleq p\};$$

$$V(a) = \{p \in Spec(L) : a \leq p\};$$

$$Supp(a) = \bigcup\{V([0_L : x]) : x \text{ is principal}, x \leq a\},$$

$$\text{where } [0_L : x] = \bigvee\{y \in L : yx = 0_L\}.$$

Let $L = Idl(R)$ be the multiplicative lattice of all ideals of a commutative ring R with identity. If $I = (x)$ is a principal ideal such that $IJ = I$ for an ideal J, then there is a principal ideal $K \subseteq J$ such that $IK = I$. In addition, if $I = (x), J = (y)$ are two principal ideals such that $IJ = I$, then there is $j \in J$ such that $xj = x$. Take $K = (1_R - j)$. Then $IK = \{0_R\}$ and $J + K = R$. Noting that principal elements are the abstraction of principal ideals in multiplicative lattices; these facts motivate us to define the following.

Definition 5.6. A principally generated multiplicative lattice L with the top element 1_L compact is called an *mp*-multiplicative lattice if L satisfies the following two conditions:

(1) If $x \in L$ is a principal element of L and $x = xa$, then there exists a principal element $c \in L$, $c \leq a$ such that $x = xc$.
(2) If $x, y \in L$ are principal elements in L such that $x = xy$, there exists a principal element $c \in L$ such that $c \vee y = 1_L$ and $cx = 0_L$.

Lemma 5.7. *Let L be an mp-multiplicative lattice and a be any element of L. Then the following statements are equivalent:*

(1) *a is pure.*
(2) *$Supp(a) = D(a)$.*

Proof. (1) implies (2): For any $a \in L$, if $p \in D(a)$, then $a \nleqslant p$. So there exists a principal element $x \in L$ such that $x \leq a$ but $x \nleqslant p$. Now, $p \geq [0_L : x]x = 0_L$ and since p is prime, $p \geq [0_L : x]$ which implies that $p \in V([0_L : x]) \subseteq Supp(a)$. So $D(a) \subseteq Supp(a)$.

Since a is pure, for any principal $x \leq a$, $xa = x$. As L is a *mp*-multiplicative lattice, there exists a principal element $y \in L$ such that $y \leq a$ and $xy = x$. Also there exists a principal element $c \in L$ such that $c \vee y = 1_L$ and $cx = 0_L$.

Now, $1_L = c \vee y \leq [0_L : x] \vee a \leq 1_L$. Thus $[0_L : x] \vee a = 1_L$. Hence $V([0_L : x]) \cap V(a) = \emptyset$. Then $V([0_L : x]) \subseteq (V(a))^c = D(a)$. Thus $Supp(a) \subseteq D(a)$. Therefore $Supp(a) = D(a)$.

(2) implies (1): Let $x \leq a$ be any principal element. Since $Supp(a) = D(a)$, then $V([0_L : x]) \subseteq D(a) = (V(a))^c$. So $V([0_L : x]) \cap V(a) = \emptyset$. There is no prime element $p \in L$ satisfying $p \geq a$ and $p \geq [0_L : x]$. Then $[0_L : x] \vee a = 1_L$. In fact, if $[0_L : x] \vee a < 1_L$, then using the assumption that 1_L is compact, we can deduce that there exists a maximal element p such that $[0_L : x] \vee a < p$ (for instance take p to be the supremum of a maximal chain C with $[0_L : x] \vee a \in C$ and $1_L \notin C$). Since p is maximal, p is prime and so $p \in V([0_L : x]) \cap V(a)$ which is a contradiction.

It follows that $x = x \cdot 1_L = x([0_L : x] \vee a) = x[0_L : x] \vee xa = 0_L \vee xa = xa$. Hence a is pure. ∎

Proposition 5.8. *Let L be an mp-multiplicative lattice satisfying the following condition:*

For any principal element $x \in L$ such that $x \leq a \vee b$, there exist principal elements $e_1, e_2 \in L$ such that

$$e_1 \leq a \text{ and } e_2 \leq b \text{ such that } [0_L : e_1] \wedge [0_L : e_2] \leq [0_L : x].$$

Then for any $a, b \in Pur(L)$, $a \vee b \in Pur(L)$.

Proof. Let $a, b \in Pur(L)$. By Lemma 5.7, we just need to verify that $V([0_L : x]) \subseteq D(a \vee b)$ holds for any principal element x with $x \leq a \vee b$.

By the condition given, there exist principal elements $e_1 \leq a$ and $e_2 \leq b$ such that $[0_L : e_1] \wedge [0_L : e_2] \leq [0_L : x]$. For any $p \in V([0_L : x])$, $p \geq [0_L : x]$. So $p \geq [0_L : e_1]$ or $p \geq [0_L : e_2]$ which implies that $p \in V([0_L : e_1]) \subseteq Supp(a)$ or $p \in V([0_L : e_2]) \subseteq Supp(b)$. Since a and b are pure, $Supp(a) = D(a) \subseteq D(a \vee b)$ and $Supp(b) = D(b) \subseteq D(a \vee b)$. Thus $p \in D(a \vee b)$. It follows that $V([0_L : x]) \subseteq D(a \vee b)$ and so $a \vee b \in Pur(L)$. ∎

Remark 5.9. Let $L = Idl(R)$ be the multiplicative lattice of all ideals of a commutative ring R with identity. If $I = (x) \subseteq J \vee K$ (which equals $J + K$) for some $x \in R$ and $J, K \in L$, then there are $j \in J, k \in K$ such that $x = j + k$. Then $J' = (j) \subseteq J, K' = (k) \subseteq K$ and if $A \in L$ such that $AJ' = 0_L, AK' = 0_L$ then $AI = 0_L$, so $[0_L : J'] \wedge [0_L : K'] \subseteq [0_L : I]$. Thus, if every principal element of $L = Idl(R)$ is a principal ideal, then L satisfies the condition in Proposition 5.8.

A multiplicative lattice L is said to be reduced if for any $x \in L$, $\bigwedge\{r : r \in Spec(L)\} = \{0_L\}$. Clearly, if L is reduced, then for any $x \in L$, $x^n = 0_L$ for some positive integer n implies $x = 0_L$.

Proposition 5.10. *Let L be a reduced mp-multiplicative lattice. Then $Pur(L) \subseteq mSprime(L)$.*

Proof. Let $a \in Pur(L)$. Suppose $x \in L$ such that $x^2 \leq a$. For any principal element $y \in L$ such that $y \leq x$, we have $y^2 \leq a$. Then since a is pure, $ay^2 = y^2$. As y^2 is a principal element and L is an *mp*-multiplicative lattice, there exists a principal element $b \in L$ such that $b \leq a$ and $by^2 = y^2$. Also, there exists a principal element $c \in L$ such that $c \vee b = 1_L$ and $cy^2 = 0_L$. It then follows that $(cy)^2 = 0_L$ (noting that $0_L \leq (cy)^2 \leq cy^2$), which implies $cy = 0_L$ as L is reduced. Hence $y = (c \vee b)y = cy \vee by = by$ and so $y \leq b \leq a$. It follows that $x \leq a$, hence $a \in mSprime(L)$. ∎

Recall that an element a in a frame H, is *regular* [11] if

$$a = \bigvee\{x \in H : x^{\perp} \vee a = 1_H\},$$

where $x^{\perp}$ is the largest $y \in H$ such that $x \wedge y = 0_H$.

The set of all regular elements of H will be denoted by $Reg(H)$.

Recall that if L is a continuous multiplicative lattice, then $mSprime(L)$ is a frame by Corollary 4.5.

PROPOSITION 5.11. *Let L be a reduced, continuous mp-multiplicative lattice such that for any principal element x, $p(x) = \bigwedge\{r \in Spec(L) : x \leq r\}$ is a compact element of $M = mSprime(L)$. Then $a \in M$ is a regular element of M if and only if a is a pure element of L.*

Proof. First, $0_L \in M$ because L is reduced.

Let a be a pure element of L. For any principal element $x \leq a$, $ax = x$ as a is pure. Since L is an mp-multiplicative lattice, there exists a principal element $b \in L$, $b \leq a$ such that $bx = x$. Also there exists a principal element $c \in L$ such that $b \vee_L c = 1_L$ and $cx = 0_L$.

Then $p(x) = \bigwedge_L\{u \in Spec(L) : x \leq u\} \in mSprime(L)$ by Theorem 4.3. From $cx = 0_L$, we have $p(c) \wedge_L p(x) = p(cx) = p(0_L)$ by Lemma 3.2 (here $S = \mathrm{Spec}(L)$). Also L is reduced, $p(0_L) = 0_L$. Note that, by Lemma 3.2 again, $M = mSprime(L)$ is a meet sub-semilattice of L. Thus $p(c) \wedge_M p(x) = 0_L$ so $p(c) \leq (p(x))^{\perp}$. It follows that $a \vee_M (p(x))^{\perp} \geq a \vee_L p(c) \geq b \vee_L c = 1_L$.

Note that $a \vee_M y^{\perp} = 1_L$ implies $y \leq a$ for any $y \in M$. Hence

$$\begin{aligned} a \geq \bigvee_M \{y \in M : a \vee_M y^{\perp} = 1_L\} &\geq \bigvee_M \{p(x) : a \vee_M (p(x))^{\perp} = 1_L\} \\ &\geq \bigvee_M \{p(x) : x \leq a \text{ and } x \text{ is principal}\} \\ &\geq \bigvee_L \{x : x \leq a \text{ and } x \text{ is principal}\} = a. \end{aligned}$$

Thus $a = \bigvee_M\{y \in M : a \vee_M y^{\perp} = 1_L\}$, implying that a is a regular element of M.

Conversely, assume that a is a regular element of M. Let $x \leq a$ be any principal element of L. Then $p(x) \leq p(a) = a = \bigvee_M\{y \in M : a \vee_M y^{\perp} = 1_L\}$. Note that $y_1, y_2 \in \{y \in M : a \vee_M y^{\perp} = 1_L\}$ imply $y_1 \vee_M y_2 \in \{y \in M : a \vee_M y^{\perp} = 1_L\}$. Since $p(x)$ is a compact element of M, there exist $y_1, y_2, \cdots, y_n \in \{y \in M : a \vee_M y^{\perp} = 1_L\}$ such that $p(x) \leq y_1 \vee_M y_2 \vee_M \cdots \vee_M y_n$. Let $y = y_1 \vee_M y_2 \vee_M \cdots \vee_M y_n$. Then $y \in M, p(x) \leq y$ and $y^{\perp} \vee_M a = 1_L$. Then as 1_L is a compact element of L, we must have $y^{\perp} \vee_L a = 1_L$ (otherwise there is a maximal element q such that $y^{\perp} \vee_L a \leq q < 1_L$, so $y^{\perp} \vee_M a \leq q$). As $p(x) \wedge y^{\perp} = 0_L$, so $p(x)y^{\perp} = 0_L$ because $p(x)y^{\perp} \leq p(x) \wedge y^{\perp}$. Now $x = x1_L = x(y^{\perp} \vee_L a) = xy^{\perp} \vee_L xa = 0_L \vee xa = xa$. The last equation holds because $xy^{\perp} \leq p(x)y^{\perp}$. Thus by Lemma 5.4, a is pure. ∎

Let $Max(L)$ be the set of all maximal elements of the multiplicative lattice L. We have indicated earlier that $Max(L) \subseteq Spec(L)$.

In the next part we show that, for certain types of multiplicative lattices L, the lattice $Pur(L)$ is isomorphic to the open set lattice of $Max(L)$ with the hull kernel topology.

The following result can be proved in a similar way as for the lattice $Idl(R)$ of ideals of commutative rings R with identity (see [2]).

Proposition 5.12. *Let L be an r-lattice. Then*

(1) *for any $x \in L$,*

$$\begin{aligned} p(x) &= \bigwedge \{y \in Spec(L) : x \leq y\} \\ &= \bigvee \{z \in L : z^n \leq x \text{ for some positive integer } n\}; \end{aligned}$$

(2) *an element a is a compact element of $mSprime(L)$ if and only if*

$$a = p(c_1 \vee c_2 \vee \cdots \vee c_n)$$

for some compact elements c_i of L;

(3) *$mSprime(L)$ is a coherent frame.*

To prove the main result of this section we need the following definitions and results from [12].

(1) Let L be a compact frame (that is, 1_L is compact). For each $a \in L$, let $s(a) = \{z \in L : z \vee y = 1_L \text{ implies } a \vee y = 1_L\}$. Then $s(a) \vee y = 1_L$ implies $a \vee y = 1_L$, by the compactness of 1_L. We shall denote $SL = \{a \in L : s(a) = a\}$.

(2) A frame L is *normal* [6] if whenever $x \vee y = 1_L$, there exist $u, v \in L$ such that $u \wedge v = 0_L$ and $x \vee v = u \vee y = 1_L$.

Proposition 5.13. [12] *For any compact normal frame L, $a \in SL$ if and only if a is an infimum of maximal elements of L.*

Theorem 5.14. [12] *Let L be a compact normal frame. Then lattices $Reg(L)$ and SL are isomorphic.*

Lemma 5.15. *The prime elements of a multiplicative lattice L are the same as the prime elements of the lattice $mSprime(L)$.*

Proof. Let r be a prime element of L. Then clearly, $r \in mSprime(L)$. Suppose $a \wedge_{mSprime(L)} b \leq r$ for $a, b \in mSprime(L)$. Since $mSprime$ is closed under meets, $a \wedge_{mSprime(L)} b = a \wedge_L b$. Thus $ab \leq a \wedge_L b \leq r$, implying $a \leq r$ or $b \leq r$. So r is a prime element of the lattice $mSprime(L)$.

Conversely, suppose r is a prime element of $mSprime(L)$, and $ab \leq r$, where $a, b \in L$. Then $p(ab) \leq p(r) = r$, so $p(a) \wedge p(b) \leq r$. But $p(a) \wedge p(b) = p(a) \wedge_{mSprime(L)} p(b)$, thus $p(a) \leq r$ or $p(b) \leq r$ because $p(a), p(b) \in mSprime(L)$. As $a \leq p(a)$ and $b \leq p(b)$, so $a \leq r$ or $b \leq r$. Hence r is a prime element of L. ∎

Since every maximal element of L is in $mSprime(L)$, so the following is true.

Lemma 5.16. *The maximal elements of a multiplicative lattice L are exactly the maximal elements of $mSprime(L)$.*

Proposition 5.17. *Let L be a multiplicative lattice. Then the hull kernel topology of $Max(L)$ is the same as the hull kernel topology of $Max(mSprime(L))$.*

Proof. We need to show that any open set of the space $Max(L)$ under the hull kernel topology is also an open set of the space $Max(mSprime(L))$ and vice versa. Let $x \in W_{p(a)} = \{m \in Max(L) : p(a) \nleq m\}$. Then $x \in Max(L)$ and $p(a) \nleq x$. Then as $x \in Spec(L)$, $a \nleq x$. Thus $x \in W_a$.

Now let $v \in W_a$. Then $v \in Max(L)$ and $a \nleq v$. Suppose $v \notin W_{p(a)}$. Then $p(a) \leq v$ which implies that $a \leq v$, a contradiction. Hence $v \in W_{p(a)}$. ∎

Remark 5.18. By [12], in a coherent normal frame, every prime element is below a unique maximal element. Conversely, if every prime element in a coherent frame is below a unique maximal element, then the frame is normal.

Theorem 5.19. *Let L be a reduced mp-multiplicative lattice and an r-lattice in which every prime element is beneath a unique maximal element. Then $Pur(L)$ is isomorphic to the open set lattice, $Max(L)$, of all maximal elements of L endowed with the hull kernel topology.*

Proof. By Proposition 5.12, $mSprime(L)$ is a coherent frame. Since every prime element of $mSprime(L)$ is beneath a unique maximal element, $mSprime(L)$ is a normal coherent frame. Since L is a reduced mp-multiplicative lattice and an r-lattice, by Proposition 5.11, $Reg(mSprime(L)) = Pur(L)$. By Theorem 5.14, $Reg(mSprime(L))$ is isomorphic to $S(mSprime(L)) = \{a \in mSprime(L) : s(a) = a\}$. By Proposition 5.13, $a \in S(mSprime(L))$ if and only if a is the meet of all maximal elements above a, if and only if $a \in \Omega_S(L)$ where $S = Max(L)$ (see Theorem 3.9). Hence by Theorem 3.9, $Pur(L)$ is isomorphic to the open set lattice of $Max(L)$. ∎

One remaining problem that interests us is the following: Let L be a multiplicative lattice and $Spec(\Omega_S(L))$ be the set of prime elements of $\Omega_S(L)$. For what subset S of $Spec(L)$, $Spec(\Omega_S(L)) = S$?

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