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ONE-SIDED QUANTUM QUASIGROUPS AND LOOPS

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Abstract. Quantum quasigroups and quantum loops are self-dual objects providing a general framework for the nonassociative extension of quantum group techniques. This paper examines their one-sided analogues, which are not self-dual. Just as quantum quasigroups are the "quantum" version of quasigroups, so one-sided quantum quasigroups are the "quantum" version of left or right quasigroups.

1. Introduction

Hopf algebras (or "quantum groups") have been developed over the last few decades as an important extension of the concept of a group, from the category of sets with the Cartesian product to more general symmetric, monoidal categories, such as the category of vector spaces over a field with the tensor product [8]. Over the same time period, there has been an intensive parallel development of the theory of quasigroups and loops ("non-associative groups") [11]. Some work has also been done on extending Hopf algebras to non-associative products [1, 4, 7], and recently the self-dual concepts of quantum quasigroup and loop were introduced as a far-reaching unification of Hopf algebras and quasigroups [12].

The purpose of the current paper is to initiate investigation of one-sided (left or right) versions of quantum quasigroups and loops. The self-dual definition of a quantum quasigroup requires the invertibility of two dual morphisms: the left composite (4.1) and the right composite (4.2). The definition of a left quantum quasigroup requires only the invertibility of the left composite. Dually, the definition of a right quantum quasigroup requires only the invertibility of the right composite (Definition 4.1).

Section 2 recalls the combinatorial and equational approaches to traditional quasigroups and loops, both two-sided and one-sided. Section 3 presents

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the background on symmetric monoidal categories and Hopf algebras. The basic definitions of the one-sided quantum quasigroups and loops are given in Section 4, along with a brief discussion of their relation to other structures. It appears that the left Hopf algebras of Taft and his coauthors may not generally fit into the framework of left quantum quasigroups, although the question still awaits a fuller investigation.

Section 5 presents a study of one-sided quantum quasigroups and loops in the usual combinatorial setting of the category of sets (under the monoidal structure given by the Cartesian product). Here, left quantum loops reduce to the usual left loops, and counital left quantum quasigroups reduce to the usual left quasigroups (Theorem 5.3). General finite left quantum quasigroups are equivalent to left quasigroups with an automorphism and an endomorphism (Theorem 5.4). A full characterization of all left quantum quasigroups in the category of sets is still open.

Sections 6–8 discuss one-sided quantum quasigroups and loops in the category $\underline{\underline{S}}$ of modules over a commutative, unital ring S. Section 6 presents the one-sided left quasigroup and loop algebras, and their twisted versions (Remark 6.3), given by any left quasigroup or loop in a module category (or more general category of entropic algebras) under the tensor product. By contrast, Section 7 shows that there are no general strictly one-sided analogues of the two-sided quantum quasigroups of (ring-valued) functions on a finite left quasigroup, in the module category $\underline{\underline{S}}$ under the tensor product. On the other hand, Section 8 examines module categories under the direct sum, which also form symmetric monoidal categories. Theorem 8.7 characterizes the left quantum quasigroups in these categories, under appropriate finiteness assumptions. As for the category of sets, a full characterization of all left quantum quasigroups in these categories is still open.

For algebraic concepts and conventions that are not otherwise discussed in this paper, readers are referred to [13]. In particular, algebraic notation is used throughout the paper, with functions to the right of, or as superfixes to, their arguments. Thus compositions are read from left to right. These conventions serve to minimize the proliferation of brackets.

2. One-sided quasigroups and loops

2.1. Combinatorial or equational quasigroups. Quasigroups may be defined combinatorially or equationally. Combinatorially, a *quasigroup* $(Q, \cdot)$ is a set Q equipped with a binary *multiplication* operation denoted by $\cdot$ or simple juxtaposition of the two arguments, in which specification of any two of x, y, z in the equation $x \cdot y = z$ determines the third uniquely. A *loop* is a quasigroup Q with an *identity* element e such that $e \cdot x = x = x \cdot e$ for all x in Q.

Equationally, a quasigroup $(Q, \cdot, /, \backslash)$ is a set Q with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

$$\text{(2.1)} \qquad \begin{array}{ll} \text{(SL)}\ x \cdot (x \backslash z) = z\,; & \text{(SR)}\ z = (z/x) \cdot x\,; \\ \text{(IL)}\ \ x \backslash (x \cdot z) = z\,; & \text{(IR)}\ \ z = (z \cdot x)/x\,. \end{array}$$

If x and y are elements of a group $(Q, \cdot)$, the left division is given by $x \backslash y = x^{-1}y$, with $x/y = xy^{-1}$ as right division. For an abelian group considered as a combinatorial quasigroup under subtraction, the right division is addition, while the left division is subtraction.

2.2. Equational or combinatorial one-sided quasigroups. Equationally, a *left quasigroup* $(Q, \cdot, \backslash)$ is a set Q equipped with a multiplication and left division satisfying the identities (SL) and (IL) of (2.1). Dually, a *right quasigroup* $(Q, \cdot, /)$ is a set Q equipped with a multiplication and right division satisfying the identities (SR) and (IR) of (2.1). A *left loop* is a left quasigroup with an identity element. Dually, a *right loop* is a right quasigroup with an identity element.

Combinatorially, a left quasigroup $(Q, \cdot)$ is a set Q with a multiplication such that in the equation $a \cdot x = b$, specification of a and b determines x uniquely. In equational terms, the unique solution is $x = a \backslash b$. The combinatorial definition of right quasigroups is dual. If Q is a set, the right projection product $xy = y$ yields a left quasigroup structure on Q, while the left projection product $xy = x$ yields a right quasigroup structure.

3. Structures in symmetric monoidal categories

The general setting for the algebras studied in this paper is a symmetric monoidal category (or "symmetric tensor category" — compare [14, Ch. 11]) $(\mathbf{V}, \otimes, \mathbf{1})$. The standard example is provided by the category $\underline{\underline{K}}$ of vector spaces over a field K. More general concrete examples are provided by varieties $\mathbf{V}$ of entropic (universal) algebras, algebras on which each (fundamental and derived) operation is a homomorphism (compare [2]). These include the category **Set** of sets, the category of pointed sets, the category $\underline{\underline{R}}$ of (right) modules over a commutative, unital ring R, the category of commutative monoids, and the category of semilattices.

In a monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$, there is an object $\mathbf{1}$ known as the *unit object*. For example, the unit object of $\underline{\underline{K}}$ is the vector space K. For objects A and B in a monoidal category, a *tensor product* object $A \otimes B$ is defined. For example, if U and V are vector spaces over K with respective bases X and Y, then $U \otimes V$ is the vector space with basis $X \times Y$, written as

$\{x \otimes y \mid x \in X,\ y \in Y\}$. There are natural isomorphisms with components

$$\alpha_{A,B;C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C)\,,\ \rho_A \colon A \otimes \mathbf{1} \to A\,,\ \lambda_A \colon \mathbf{1} \otimes A \to A$$

satisfying certain *coherence* conditions guaranteeing that one may as well regard these isomorphisms as identities [14, p. 67]. Thus the bracketing of repeated tensor products is suppressed in this paper, although the natural isomorphisms ρ and λ are retained for clarity in cases such as the unitality diagram (3.1) below. In the vector space example, adding a third space W with basis Z, one has

$$\alpha_{U,V;W} \colon (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$$

for $z \in Z$, along with $\rho_U \colon x \otimes 1 \mapsto x$ and $\lambda_U \colon 1 \otimes x \mapsto x$ for $x \in X$.

A monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ is *symmetric* if there is a given natural isomorphism with *twist* components $\tau_{A,B} \colon A \otimes B \to B \otimes A$ such that $\tau_{A,B}\tau_{B,A} = 1_{A\otimes B}$ [14, pp. 67–8]. One uses $\tau_{U,V} \colon x \otimes y \mapsto y \otimes x$ with $x \in X$ and $y \in Y$ in the vector space example.

3.1. Diagrams. Let A be an object in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$. Consider the respective *associativity* and *unitality diagrams*

(3.1)
$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "1_A \otimes \nabla"] \arrow[d, "\nabla \otimes 1_A"'] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes A \arrow[dr, "\nabla"] & A \otimes \mathbf{1} \arrow[l, "1_A \otimes \eta"'] \arrow[d, "\rho_A"] \\ \mathbf{1} \otimes A \arrow[u, "\eta \otimes 1_A"] \arrow[r, "\lambda_A"'] & A \end{tikzcd}$$

in the category $\mathbf{V}$, the respective dual *coassociativity* and *counitality diagrams*

(3.2)
$$\begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "1_A \otimes \Delta"'] \\ A \otimes A \arrow[u, "\Delta \otimes 1_A"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes A \arrow[r, "1_A \otimes \varepsilon"] \arrow[d, "\varepsilon \otimes 1_A"'] & A \otimes \mathbf{1} \\ \mathbf{1} \otimes A & A \arrow[l, "\lambda_A^{-1}"] \arrow[ul, "\Delta"'] \arrow[u, "\rho_A^{-1}"'] \end{tikzcd}$$

in the category $\mathbf{V}$, the *bimagma diagram*

(3.3)
$$\begin{tikzcd} & A \arrow[dr, "\Delta"] & \\ A \otimes A \arrow[ur, "\nabla"] \arrow[d, "\Delta \otimes \Delta"'] \arrow[drr, dashed] & & A \otimes A \\ A \otimes A \otimes A \otimes A \arrow[rr, "1_A \otimes \tau \otimes 1_A"'] \arrow[urr, dashed] & & A \otimes A \otimes A \otimes A \arrow[u, "\nabla \otimes \nabla"'] \end{tikzcd}$$

in the category $\mathbf{V}$, the *biunital diagram*

(3.4)
$$\begin{tikzcd}
\mathbf{1}\otimes\mathbf{1} \arrow[r, "\nabla"] & \mathbf{1} & & \mathbf{1} \arrow[ll, "1"'] \arrow[r, "\Delta"] \arrow[dl, "\eta"] & \mathbf{1}\otimes\mathbf{1} \arrow[d, "\eta\otimes\eta"] \\
A\otimes A \arrow[u, "\varepsilon\otimes\varepsilon"] \arrow[rr, "\nabla"] & & A \arrow[ul, "\varepsilon"] \arrow[rr, "\Delta"] & & A\otimes A
\end{tikzcd}$$

in the category $\mathbf{V}$, and the *antipode diagram*

(3.5)
$$\begin{tikzcd}
& A\otimes A \arrow[rr, "S\otimes 1_A"] & & A\otimes A \arrow[dr, "\nabla"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \mathbf{1} \arrow[rr, "\eta"'] & & A \\
& A\otimes A \arrow[rr, "1_A\otimes S"'] & & A\otimes A \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

in the category $\mathbf{V}$, all of which are commutative diagrams. The arrow across the bottom of the bimagma diagram (3.3) makes use of the twist isomorphism $\tau_{A,A}$ or $\tau\colon A\otimes A \to A\otimes A$.

3.2. Magmas and bimagmas. This paragraph and its successor collect a number of basic definitions of various structures and homomorphisms between them.

Definition 3.1. Let $\mathbf{V}$ be a symmetric monoidal category.

(a.1) A *magma* in $\mathbf{V}$ is a $\mathbf{V}$-object A with a $\mathbf{V}$-morphism
$$\nabla\colon A\otimes A \to A$$
known as *multiplication*.

(a.2) Let A and B be magmas in $\mathbf{V}$. Then a *magma homomorphism* $f\colon A \to B$ is a $\mathbf{V}$-morphism such that the diagram
$$\begin{tikzcd}
A \arrow[d, "f"'] & A\otimes A \arrow[l, "\nabla"'] \arrow[d, "f\otimes f"] \\
B & B\otimes B \arrow[l, "\nabla"]
\end{tikzcd}$$
commutes.

(b.1) A *comagma* in $\mathbf{V}$ is a $\mathbf{V}$-object A with a $\mathbf{V}$-morphism

$$\Delta \colon A \to A \otimes A$$

known as *comultiplication*.

(b.2) Let A and B be comagmas in $\mathbf{V}$. A *comagma homomorphism* $f \colon A \to B$ is a $\mathbf{V}$-morphism such that the diagram

$$\begin{tikzcd} B \arrow[r, "\Delta"] & B \otimes B \\ A \arrow[u, "f"] \arrow[r, "\Delta"'] & A \otimes A \arrow[u, "f\otimes f"'] \end{tikzcd}$$

commutes.

(c.1) A *bimagma* (A, ∇, Δ) in $\mathbf{V}$ is a magma (A, ∇) and comagma (A, Δ) in $\mathbf{V}$ such that the bimagma diagram (3.3) commutes.

(c.2) Let A and B be bimagmas in $\mathbf{V}$. Then a *bimagma homomorphism* $f \colon A \to B$ is a magma and comagma homomorphism between bimagmas A and B.

Remark 3.2. (a) Commuting of the bimagma diagram (3.3) in a bimagma (A, ∇, Δ) means that

$$\Delta \colon (A, \nabla) \to \big(A \otimes A, (1_A \otimes \tau \otimes 1_A)(\nabla \otimes \nabla)\big)$$

is a magma homomorphism (commuting of the upper-left solid and dotted quadrilateral), or equivalently, that

$$\nabla \colon \big(A \otimes A, (\Delta \otimes \Delta)(1_A \otimes \tau \otimes 1_A)\big) \to (A, \Delta)$$

is a comagma homomorphism (commuting of the upper-right solid and dotted quadrilateral).

(b) If $\mathbf{V}$ is an entropic variety of universal algebras, the comultiplication of a comagma in $\mathbf{V}$ may be written as

$$\Delta \colon A \to A \otimes A; a \mapsto \big((a^{L_1} \otimes a^{R_1}) \dots (a^{L_{n_a}} \otimes a^{R_{n_a}})\big) w_a \tag{3.6}$$

in a universal-algebraic version of the well-known *Sweedler notation*. In (3.6), the *tensor rank* of the image of a (or any such general element of $A \otimes A$) is the smallest arity n_a of the derived word w_a expressing the image (or general element) in terms of elements of the generating set $\{b \otimes c \mid b, c \in A\}$ for $A \otimes A$. A more compact but rather less explicit version of Sweedler notation, generally appropriate within any concrete monoidal category $\mathbf{V}$, is $a\Delta = a^L \otimes a^R$, with the understanding that the tensor rank of the image is not implied to be 1.

(c) As with quasigroups (§2), the magma multiplication on an object A of a concrete monoidal category is often denoted by juxtaposition, namely $(a \otimes b)\nabla = ab$, or with $a \cdot b$ as an infix notation, for elements a, b of A.

Definition 3.3. Suppose that A is an object in a symmetric monoidal category $\mathbf{V}$.

(a) A magma (A, ∇) is *commutative* if $\tau\nabla = \nabla$. Thus if $\mathbf{V}$ is concrete, this may be written in the usual form $ba = ab$ for $a, b \in A$.
(b) A comagma (A, Δ) is *cocommutative* if $\Delta\tau = \Delta$. In Sweedler notation: $a^R \otimes a^L = a^L \otimes a^R$ for $a \in A$.
(c) A magma (A, ∇) is *associative* if the associativity diagram (3.1) commutes. In the concrete case, one often writes $ab \cdot c = a \cdot bc$, with $\cdot$ binding less strongly than juxtaposition, for a, b, c in A.
(d) A comagma (A, Δ) is *coassociative* if the coassociativity diagram (3.2) commutes. Coassociativity takes the form

$$a^{LL} \otimes a^{LR} \otimes a^R = a^L \otimes a^{RL} \otimes a^{RR}$$

when written in Sweedler notation for $a \in A$.

Remark 3.4. In a bimagma (A, ∇, Δ), the concepts of Definition 3.3 may be applied to the respective magma and comagma reducts of A.

3.3. Unital structures and Hopf algebras.

Definition 3.5. Let $\mathbf{V}$ be a symmetric monoidal category.

(a.1) A magma (A, ∇) in $\mathbf{V}$ is *unital* if it has a $\mathbf{V}$-morphism $\eta \colon \mathbf{1} \to A$ such that the unitality diagram (3.1) commutes.
(a.2) Let A and B be unital magmas in $\mathbf{V}$. Then a *unital magma homomorphism* $f \colon A \to B$ is a magma homomorphism such that the diagram

$$\begin{tikzcd} A \arrow[d, "f"'] & \mathbf{1} \arrow[l, "\eta"'] \arrow[d, "1"] \\ B & \mathbf{1} \arrow[l, "\eta"] \end{tikzcd}$$

commutes.
(b.1) A comagma (A, Δ) in $\mathbf{V}$ is *counital* if it has a $\mathbf{V}$-morphism $\varepsilon \colon A \to \mathbf{1}$ such that the counitality diagram (3.2) commutes.
(b.2) Let A and B be comagmas in $\mathbf{V}$. Then a *counital comagma homomorphism* $f \colon A \to B$ is a comagma homomorphism such that the diagram

$$\begin{tikzcd} B \arrow[r, "\varepsilon"] & \mathbf{1} \\ A \arrow[u, "f"] \arrow[r, "\varepsilon"'] & \mathbf{1} \arrow[u, "1"'] \end{tikzcd}$$

commutes.

(c.1) A *biunital bimagma* $(A, \nabla, \Delta, \eta, \varepsilon)$ is a unital magma (A, ∇, η) and counital comagma (A, Δ, ε) such that (A, ∇, Δ) is a bi-magma, and the biunital diagram (3.4) commutes.

(c.2) A *biunital bimagma homomorphism* $f\colon A \to B$ is a unital magma and counital comagma homomorphism between biunital bimagmas A and B.

Remark 3.6. Joint commuting of the bimagma diagram (3.3) and biunital diagram (3.4) in a biunital bimagma $(A, \nabla, \Delta, \eta, \varepsilon)$ means that the comultiplication $\Delta\colon A \to A \otimes A$ is a unital magma homomorphism, or equivalently, that $\nabla\colon A \otimes A \to A$ is a counital comagma homomorphism.

Definition 3.7. Let $\mathbf{V}$ be a symmetric monoidal category.

(a) A *monoid* in $\mathbf{V}$ is an associative unital magma in $\mathbf{V}$.
(b) A *comonoid* in $\mathbf{V}$ is a coassociative counital comagma in $\mathbf{V}$.
(c) A *bimonoid* in $\mathbf{V}$ is defined as an associative, coassociative, and biunital bimagma.
(d) A *Hopf algebra* in $\mathbf{V}$ is a bimonoid A in $\mathbf{V}$ that is equipped with a $\mathbf{V}$-morphism $S\colon A \to A$ known as the *antipode*, such that the antipode diagram (3.5) commutes.

4. One-sided quantum quasigroups and loops

Definition 4.1. Consider a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) A *left quantum quasigroup* (A, ∇, Δ) in $\mathbf{V}$ is a bimagma in $\mathbf{V}$ for which the *left composite* morphism

$$A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A \tag{4.1}$$

is invertible.

(b) A *right quantum quasigroup* (A, ∇, Δ) in $\mathbf{V}$ is a bimagma in $\mathbf{V}$ for which the *right composite* morphism

$$A \otimes A \xrightarrow{1_A \otimes \Delta} A \otimes A \otimes A \xrightarrow{\nabla \otimes 1_A} A \otimes A \tag{4.2}$$

is invertible.

Definition 4.2. Suppose that $(A, \nabla, \Delta, \eta, \varepsilon)$ is a biunital bimagma in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) Suppose that (A, ∇, Δ) is a left quantum quasigroup in $\mathbf{V}$. Then $(A, \nabla, \Delta, \eta, \varepsilon)$ is said to be a *left quantum loop*.
(b) Suppose that (A, ∇, Δ) is a right quantum quasigroup in $\mathbf{V}$. Then $(A, \nabla, \Delta, \eta, \varepsilon)$ is said to be a *right quantum loop*.

Remark 4.3. (a) Unlike the concepts of quantum quasigroup and quantum loop, the concepts of left and right quantum quasigroups and loops are not self-dual.

(b) In the bimonoid context, the left and right composites are often described as *fusion operators* or *Galois operators*.

4.1. Relations with other structures. The *quantum quasigroups* considered in [12] are structures that are simultaneously left and right quantum quasigroups. Similarly, the *quantum loops* considered there are structures that are simultaneously left and right quantum loops. Both quantum quasigroups and quantum loops are self-dual structures. Hopf algebras (include reducts that) are quantum loops [12].

Taft and his co-authors have investigated *left Hopf algebras* [3, 6, 9]. These structures satisfy all of the requirements for a Hopf algebra listed in Definition 3.7(d), except for the commuting of the lower pentagon in the antipode diagram (3.5). In this situation, the $\mathbf{V}$-morphism S is known as a *left antipode*. Since the proof in [12] that Hopf algebras are quantum loops uses the commuting of both pentagons in the antipode diagram to show that Hopf algebras are left quantum quasigroups, it appears that general left Hopf algebras may not necessarily form left quantum quasigroups. This is a topic for further investigation.

4.2. Symmetric monoidal functors. Since right and left quantum quasigroups and loops are formulated entirely in the language of symmetric monoidal categories, one immediately has the following result. (Compare [14, p. 86] for the concept of a symmetric monoidal functor.)

Proposition 4.4. *Suppose that* $(\mathbf{V}, \otimes, \mathbf{1_V})$ *and* $(\mathbf{W}, \otimes, \mathbf{1_W})$ *are symmetric monoidal categories. Let* $F\colon \mathbf{V} \to \mathbf{W}$ *be a symmetric monoidal functor.*

(a) *If* (A, ∇, Δ) *is a left (right) quantum quasigroup in* $\mathbf{V}$*, then the structure* (AF, ∇^F, Δ^F) *is a left (right) quantum quasigroup in* $\mathbf{W}$*.*
(b) *Suppose that* $(A, \nabla, \Delta, \eta, \varepsilon)$ *is a left (right) quantum loop in* $\mathbf{V}$*. Then* $(AF, \nabla^F, \Delta^F, \eta^F, \varepsilon^F)$ *is a left (right) quantum loop in* $\mathbf{W}$*.*

Noting that the conditions of (co)commutativity and (co)associativity are also formulated entirely in the language of symmetric monoidal categories, one obtains the following.

Corollary 4.5. *In the context of Proposition 4.4, validity of any one of the commutativity, cocommutativity, associativity, or coassociativity conditions for the left or right quantum quasigroup* (A, ∇, Δ) *implies validity of the corresponding condition for the left or right quantum quasigroup* (AF, ∇^F, Δ^F)*.*

5. Combinatorial examples

In this section, the basic symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ is taken to be $(\mathbf{Set}, \times, \top)$, the category of sets with the Cartesian product $\times$ and singleton set $\top = \{1\}$ (a terminal object of $\mathbf{Set}$), with the twist symmetry

$$\tau\colon A \times B \to B \times A; (a, b) \mapsto (b, a)$$

and identifications such as $\rho_A\colon A \times \top \to A; (a, 1) \mapsto a$. In order to facilitate reference to the diagrams of §3.1, the direct product of two sets A and B will be written in monoidal category notation as $A \otimes B$, while an ordered pair $(a, b) \in A \times B$ will be written as an element $a \otimes b$ of $A \otimes B$, of tensor rank 1. In this case, the Sweedler notation $\Delta\colon A \to A \otimes A; a \mapsto a^L \otimes a^R$ introduced in Remark 3.2(b) corresponds directly with a pair of functions $L\colon A \to A; a \mapsto a^L$ and $R\colon A \to A; a \mapsto a^R$.

To avoid tedious repetition, from now on the discussion will be explicitly restricted to left quasigroups and loops. The corresponding results for right quasigroups and loops are readily formulated and proved in dual fashion (reversal of arrows in diagrams, along with a syntactical left/right switch).

Lemma 5.1. *If (A, Δ, ε) is a counital comagma in* **Set***, then the comultiplication is the diagonal embedding $\Delta\colon a \mapsto a \otimes a$. Conversely, the diagonal embedding on each set A yields a cocommutative, coassociative counital comagma (A, Δ, ε) in* **Set**.

Corollary 5.2. *Left quantum loops and counital left quantum quasigroups in $(\mathbf{Set}, \times, \top)$ are cocommutative and coassociative.*

5.1. Left quantum loops and counital quasigroups. Lemma 5.1 leads to a direct identification of left quantum loops and counital left quantum quasigroups in **Set**.

Theorem 5.3. *Consider the category* **Set** *of sets and functions, with the symmetric monoidal category structure $(\mathbf{Set}, \times, \top)$.*

(a) *Counital left quantum quasigroups in* **Set** *are equivalent to left quasigroups.*
(b) *Left quantum loops in* **Set** *are equivalent to left loops.*

Proof. (a): Let $(A, \nabla, \Delta, \varepsilon)$ be a counital left quantum quasigroup in **Set**. By Lemma 5.1, the left composite function (4.1) takes the form

$$a \otimes b \xmapsto{\Delta \otimes 1_A} a \otimes a \otimes b \xmapsto{1_A \otimes \nabla} a \otimes (a \cdot b) \tag{5.1}$$

for $a, b \in A$. Thus the inverse function may be written as

$$c \otimes (c \backslash d) \longleftarrow\!\!\shortmid\ c \otimes d \tag{5.2}$$

for $c, d \in A$ and a binary operation $(c, d) \mapsto c\backslash d$ on A. The mutual inverse relationship between (5.1) and (5.2) yields the identities (SL) and (IL) of (2.1) on A.

Conversely, given a left quasigroup $(A, \cdot, \backslash)$, one may define

$$\nabla\colon a \otimes b \mapsto a \cdot b\,.$$

The left quasigroup identities (SL) and (IL) of (2.1) yield an inverse (5.2) to (4.1), and a dual inverse to (4.2). The remaining structure is provided by Lemma 5.1, and verification of the bimagma condition (3.3) is immediate.

(b): If $(A, \nabla, \Delta, \eta, \varepsilon)$ is a left quantum loop in **Set**, the counital left quantum quasigroup reduct $(A, \nabla, \Delta, \varepsilon)$ yields a left quasigroup $(A, \cdot, \backslash)$ by (a). The unit $\eta\colon \top \to A$ selects an element e of A, which then becomes an identity element for $(A, \cdot, \backslash)$ by virtue of the unitality.

Conversely, given a left loop $(A, \cdot, \backslash, e)$, the left quasigroup reduct $(A, \cdot, \backslash)$ specifies a counital left quantum quasigroup $(A, \nabla, \Delta, \varepsilon)$ by (a). Defining $\eta\colon 1 \mapsto e$ with the identity element e then makes $(A, \nabla, \Delta, \eta, \varepsilon)$ a biunital bimagma. ■

5.2. Finite left quantum quasigroups.

Theorem 5.4. *In the category* $(\mathbf{FinSet}, \times, \top)$, *left quantum quasigroups are equivalent to triples* (A, L, R) *consisting of a left quasigroup A with an automorphism L and endomorphism R.*

Proof. Let (A, ∇, Δ) be a left quantum quasigroup in **FinSet**, with comagma $\Delta\colon a \mapsto a^L \otimes a^R$. Commuting of the bimagma diagram (3.3) shows that the functions $L\colon A \to A$ and $R\colon A \to A$ are endomorphisms of the magma (A, ∇). The left composite function (4.1) takes the form

$$a \otimes b \xmapsto{\Delta\otimes 1_A} a^L \otimes a^R \otimes b \xmapsto{1_A\otimes\nabla} a^L \otimes (a^R \cdot b) \tag{5.3}$$

for $a, b \in A$. Its invertibility implies that L is surjective. Since A is finite, it follows that L is invertible.

The inverse function to the left composite (5.3) may now be written as

$$c^{L^{-1}} \otimes (c^{L^{-1}R}\backslash d) \longleftarrow\!\!\shortmid\; c \otimes d \tag{5.4}$$

for $c, d \in A$ and a binary operation $(x, y) \mapsto x\backslash y$ on A. The mutual inverse relationship between (5.3) and (5.4) yields the identities (SL) and (IL) of (2.1) on A. Thus $(A, \cdot, \backslash)$ is a left quasigroup equipped with an automorphism L and endomorphism R.

Conversely, given a left quasigroup $(A, \cdot, \backslash)$ with automorphism L and endomorphism R, define a multiplication

$$\nabla\colon A \otimes A \to A; a \otimes b \mapsto ab$$

and comultiplication

$$\Delta\colon A \to A \otimes A; a \mapsto a^L \otimes a^R .$$

It is then straightforward to verify that (A, ∇, Δ) is a left quantum quasigroup in **Set**. ∎

Since finiteness of the underlying set A was not assumed in the concluding paragraph of the proof of Theorem 5.4, one may immediately observe the following.

Corollary 5.5. *Given a left quasigroup $(A, \cdot, \backslash)$ equipped with an automorphism L and endomorphism R, define $\nabla\colon A \otimes A \to A; a \otimes b \mapsto ab$ as a multiplication and $\Delta\colon A \to A \otimes A; a \mapsto a^L \otimes a^R$ as a comultiplication. Then (A, ∇, Δ) is a left quantum quasigroup in* **Set**.

Corollary 5.6. *Let (A, ∇, Δ) be a left quantum quasigroup in* **FinSet**, *with corresponding triple (A, L, R).*

(a) *The left quantum quasigroup (A, ∇, Δ) is commutative if and only if the left quasigroup A is commutative.*
(b) *The left quantum quasigroup (A, ∇, Δ) is associative if and only if the left quasigroup A is associative.*
(c) *The left quantum quasigroup (A, ∇, Δ) is cocommutative if and only if the endomorphisms L and R coincide.*

6. Left quasigroup and loop algebras

For simplicity, the results of this section are presented within the category $\underline{\underline{S}}$ of modules over a commutative, unital ring S, construed as a symmetric tensor category $(\underline{\underline{S}}, \otimes, S)$ under the tensor product of modules. Discussion of extensions to more general entropic varieties is confined to Remark 6.3.

Proposition 6.1. *Let Q be a left quasigroup. Suppose that QS is the free S-module over Q. Define a magma (QS, ∇) by the free extension of the quasigroup multiplication $\nabla\colon Q \otimes Q \to Q; x \otimes y \mapsto xy$. Define a comagma (QS, Δ) by the free extension of the diagonal $\Delta\colon q \mapsto q \otimes q$ for q in Q. Then (QS, ∇, Δ) is a cocommutative, coassociative left quantum quasigroup in $\underline{\underline{S}}$.*

Proof. The left composite (4.1) takes the form

$$x \otimes y \xmapsto{\Delta \otimes 1_{QS}} x \otimes x \otimes y \xmapsto{1_{QS} \otimes \nabla} x \otimes (x \cdot y)$$

for $x, y \in Q$. The inverse is given by

$$u \otimes (u \backslash v) \longleftarrow\!\!\!\!\longleftarrow\!\!\!\dashv u \otimes v$$

for $u, v \in Q$. Verification of the commuting of the bimagma diagram (3.3) is straightforward. ∎

Definition 6.2. The left quantum quasigroup QS of Proposition 6.1 is known as the *left quasigroup algebra* of Q over the ring S.

Remark 6.3. If $\mathbf{V}$ is an entropic variety, with corresponding free algebra functor $V\colon \mathbf{Set} \to \mathbf{V}$, then an analogous left quasigroup algebra in $\mathbf{V}$ may be constructed on the free $\mathbf{V}$-algebra QV over the set Q. It is obtained by applying Proposition 4.4, with the free $\mathbf{V}$-algebra functor V, to the left quantum quasigroup in $\mathbf{Set}$ corresponding under Theorem 5.4 to the triple $(Q, 1_Q, 1_Q)$. Note that Proposition 6.1 actually represents the special case where $\mathbf{V} = \underline{\underline{S}}$. More generally, taking a triple (Q, L, R) with an automorphism L and endomorphism R of Q yields a *twisted left quasigroup algebra* in $\mathbf{V}$.

Corollary 6.4. *If $(Q, \cdot, e)$ is a left loop, then the left quasigroup algebra (QS, ∇, Δ) of Q over the ring S admits an augmentation to a left quantum loop $(QS, \nabla, \Delta, \eta, \varepsilon)$ in $\underline{\underline{S}}$.*

Proof. The counit $\varepsilon\colon QS \to S$ is the free extension of $\varepsilon\colon Q \to S; x \mapsto 1$. The unit $\eta\colon S \to QS$ is the free extension of $\eta\colon \{1\} \to QS; 1 \mapsto e$. Verification of the unitality, counitality, and biunitality conditions is straightforward. ∎

Definition 6.5. The left quantum loop QS of Corollary 6.4 is known as the *left loop algebra* of Q over the ring S.

Example 6.6. If Q is a group, then the left loop algebra of Q over a field K is (a reduct of) the usual group Hopf algebra (compare [5, Ex. 1.6]).

7. Dual quasigroup algebras

For a commutative, unital ring S, let $\underline{\underline{S}}$ be the category of modules over S, taken as a symmetric tensor category under the tensor product of modules. For a finite set Q, recall that the free S-module over Q is modeled by the set S^Q of functions from Q to S, under the pointwise module structure. A basis is provided by the *delta functions* $\delta_q\colon Q \to S$ with

$$x\delta_q = \begin{cases} 1 & \text{if } x = q; \\ 0 & \text{otherwise} \end{cases} \tag{7.1}$$

for elements x, q of Q. If Q is a two-sided quasigroup, the set S^Q carries a quantum quasigroup structure in $\underline{\underline{S}}$ known as a *dual quasigroup algebra*:

Proposition 7.1. [12] *Let Q be a finite quasigroup. Define a magma (S^Q, ∇) by pointwise multiplication of S-valued functions. Define a comagma (S^Q, Δ) by the free extension of the factorization*

$$\Delta\colon \delta_q \mapsto \sum_{q^L q^R = q} \delta_{q^L} \otimes \delta_{q^R}$$

for an element q of Q. Then (S^Q, ∇, Δ) is a commutative, associative quantum quasigroup in $\underline{\underline{S}}$.

The following example shows that if the commutative, unital ring S is non-trivial, there need be no analogous left quantum quasigroup structure in $\underline{\underline{S}}$ when Q is just a left quasigroup.

Example 7.2. Let Q be the two-element set $\{a, b\}$, construed as a left quasigroup with the projection product $xy = y$. Mimicking the construction of Proposition 7.1, the left composite (4.1) takes the form

$$\delta_x \otimes \delta_y \overset{\Delta\otimes 1_{S^Q}}{\longmapsto} \sum_{x^L x^R = x} \delta_{x^L} \otimes \delta_{x^R} \otimes \delta_y \overset{1_{S^Q}\otimes\nabla}{\longmapsto} \sum_{zy=x} \delta_z \otimes \delta_y$$

for $x, y \in Q$. In particular, it maps $\delta_a \otimes \delta_b$ to 0, as there is no element z in Q with $zb = a$. But since the ring S is nontrivial, $\delta_a \otimes \delta_b \neq 0$ in $S^Q \otimes S^Q$, so the linear left composite (4.1) is not injective, and thus certainly not invertible.

8. Linear one-sided quasigroups

In this section, let S be a commutative, unital ring. The category $\underline{\underline{S}}$ of S-modules is taken as a symmetric tensor category $(\underline{\underline{\mathbb{S}}}, \oplus, \{0\})$ under the direct sum (biproduct) $\oplus$ of modules.

8.1. Linear bimagmas.

Lemma 8.1. *Let (A, ∇) be a magma in $(\underline{\underline{S}}, \oplus, \{0\})$. Then*

$$\nabla\colon A \oplus A \to A; x \oplus y \mapsto x^\rho + y^\lambda \tag{8.1}$$

for endomorphisms ρ, λ of the module A.

Proof. Note that $A \oplus A$ is the coproduct of two copies of A in $\underline{\underline{S}}$. ∎

Lemma 8.2. *Let (A, Δ) be a comagma in $(\underline{\underline{S}}, \oplus, \{0\})$. Then*

$$\Delta\colon A \to A \oplus A; x \mapsto x^L \oplus x^R \tag{8.2}$$

for endomorphisms L, R of the module A.

Proof. Note that $A \oplus A$ is the product of two copies of A in $\underline{\underline{S}}$. ∎

Remark 8.3. The expression (8.2) serves as a model for the general abbreviated version of Sweedler notation established in Remark 3.2(b).

Proposition 8.4. *Suppose that an S-module A carries a magma structure (8.1) and a comagma structure (8.2). Then (A, ∇, Δ) is a bimagma in $(\underline{\underline{S}}, \oplus, \{0\})$ if and only if the commutation relations*

$$\lambda L = L\lambda\,, \quad \rho L = L\rho\,, \quad \lambda R = R\lambda\,, \quad \rho R = R\rho \tag{8.3}$$

are satisfied by the endomorphisms λ, L, ρ, R of the module A.

Proof. The bimagma diagram (3.3) takes the form

$$\begin{tikzcd}
x \oplus y \arrow[r, mapsto, "\Delta\otimes\Delta"] \arrow[d, mapsto, "\nabla"'] & x^L \oplus x^R \oplus y^L \oplus y^R \arrow[ddd, mapsto, "1_A\oplus\tau\oplus 1_A"] \\
x^\rho + y^\lambda \arrow[d, mapsto, "\Delta"'] & \\
(x^\rho + y^\lambda)^L \oplus (x^\rho + y^\lambda)^R \arrow[d, equal] & \\
x^{L\rho} + y^{L\lambda} \oplus x^{R\rho} + y^{R\lambda} & x^L \oplus y^L \oplus x^R \oplus y^R \arrow[l, mapsto, "\nabla\oplus\nabla"]
\end{tikzcd}$$

for $x, y \in A$. Thus the relations (8.3) are equivalent to commutativity of the diagram. ∎

8.2. Linear quantum left quasigroups.

DEFINITION 8.5. A combinatorial left quasigroup $(A, \cdot)$ is (S-)*linear* if there is an S-module structure $(A, +, 0)$, with automorphism λ and endomorphism ρ, such that

$$x \cdot y = x^\rho + y^\lambda \tag{8.4}$$

for x, y in A.

REMARK 8.6. In Definition 8.5, the unique solution $x = a\backslash b$ to the equation $a \cdot x = b$ for given a and b is $x = (b - a^\rho)^{\lambda^{-1}}$.

The following theorem may be regarded as a linear version of the combinatorial Theorem 5.4, in a sense made precise by Corollary 8.8 below.

THEOREM 8.7. *Let S be a commutative, unital ring.*

(a) *Finite left quantum quasigroups in the symmetric, monoidal category $(\underline{S}, \oplus, \{0\})$ are equivalent to triples (A, L, R) consisting of a finite linear left quasigroup A, along with an automorphism L and endomorphism R of the left quasigroup A.*

(b) *Suppose that S is a field. Suppose that $\mathbf{V}$ is the category of finite-dimensional vector spaces over S. Then left quantum quasigroups in $(\mathbf{V}, \oplus, \{0\})$ are equivalent to triples (A, L, R) consisting of a finite-dimensional linear left quasigroup A with an automorphism L and endomorphism R of the left quasigroup A.*

Proof. Let (A, ∇, Δ) be a left quantum quasigroup in $(\underline{S}, \oplus, \{0\})$, with multiplication as in (8.1) and comultiplication as in (8.2). For case (a), assume that A is finite. For case (b), assume that A is finite-dimensional.

By Proposition 8.4, the commutation relations (8.3) are satisfied by the endomorphisms λ, L, ρ, R of the module A. The invertible left composite morphism (4.1) takes the form

$$x \oplus y \xmapsto{\Delta \oplus 1_A} x^L \oplus x^R \oplus y \xmapsto{1_A \oplus \nabla} x^L \oplus (x^{R\rho} + y^{\lambda}) \tag{8.5}$$

for $x, y \in A$. Its invertibility implies that L is surjective. Then by the assumptions on A in each case, it follows that L is invertible. The equation $(x \oplus y)(\Delta \oplus 1_A)(1_A \oplus \nabla) = 0 \oplus y^{\lambda}$ implies $x = 0$, so $0 \oplus y^{\lambda}$ can only be the image of $0 \oplus y$. Thus λ is surjective, and again by the assumptions on A in each case, it follows that λ is invertible.

The inverse to the left composite (8.5) is given by

$$u^{L^{-1}} \oplus (v - u^{L^{-1}R\rho})^{\lambda^{-1}} \longleftarrow\!\dashv u \oplus v \tag{8.6}$$

for $u, v \in A$, noting that L and λ are automorphisms. Thus the multiplication (8.1) on A yields a linear left quasigroup; the commutation relations (8.3) imply that L is a left quasigroup automorphism and R is a left quasigroup endomorphism.

Conversely, a linear left quasigroup $(A, \cdot)$ with $x \cdot y = x^{\rho} + y^{\lambda}$, automorphism L, and endomorphism R yields a bimagma (A, ∇, Δ) with multiplication (8.1) and comultiplication (8.2) by Proposition 8.4. Invertibility of the left composite (4.1) follows as illustrated above for (8.5) by means of (8.6). ∎

Since no assumptions were placed on the underlying module A in the concluding paragraph of the proof of Theorem 8.7, one may observe the following.

Corollary 8.8. *Let S be a commutative, unital ring. Consider a triple (A, L, R) comprising an S-linear left quasigroup A with an automorphism L and endomorphism R of A. The triple yields a left quantum quasigroup in $(\mathbf{Set}, \times, \top)$ by Corollary 5.5, and also in $(\underline{\underline{S}}, \oplus, \{0\})$. Then the former is obtained by applying Proposition 4.4 to the latter, with the underlying set functor $\underline{\underline{S}} \to \mathbf{Set}$.*

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