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LATTICES OF ANNIHILATORS IN COMMUTATIVE ALGEBRAS OVER FIELDS

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Abstract. Let $\mathbb{K}$ be any field and L be any lattice. In this note we show that L is a sublattice of annihilators in an associative and commutative $\mathbb{K}$-algebra. If L is finite, then our algebra will be finite dimensional over $\mathbb{K}$.

1. Introduction

The problem of representing lattices as lattices of subalgebras or congruences in various abstract algebras is quite often investigated. In some cases it is natural to restrict considerations to finite lattices. We refer the reader to [11, 13, 12] for some results on this problem. Some papers are devoted to representing lattices as lattices of annihilators in associative rings, for example [8, 6], or annihilators in semigroups with zero (see [14, 9]).

In this paper $\mathbb{K}$ is any field. In [5] it is shown that every lattice L is embeddable in a lattice of left annihilators of a $\mathbb{K}$-algebra, denoted there by $\mathbb{K}\langle L\rangle$. If L is finite then this algebra is finite dimensional over $\mathbb{K}$. If L has at most 3 elements then the algebra $\mathbb{K}\langle L\rangle$ is commutative, but for greater number of elements it is noncommutative. Thus there is a natural question, whether every (finite) lattice is isomorphic to a sublattice of the lattice of annihilators in a commutative (finite dimensional) algebra over $\mathbb{K}$. This question was asked by some participants of conferences, where the results of [5] were presented. We solve this problem here by constructing suitable algebras over $\mathbb{K}$.

To make the paper more readable and self-contained, in Section 2 we recall some definitions and facts on algebras and some results from [5]. We

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use this opportunity to add a new consequence of Theorem 3.2 from that paper.

In Section 3, using ideas from [5], we show that for every lattice L there exists a local, commutative $\mathbb{K}$-algebra $\mathbb{K}\langle\langle L\rangle\rangle$ and a lattice embedding of L into the lattice of annihilators in $\mathbb{K}\langle\langle L\rangle\rangle$.

The cardinality of any set X we denote by $|X|$. All lattices considered here have the smallest element ω and the largest element $\Omega \neq \omega$. If P is any partially ordered set (poset), then by P^{op} we denote the set P, with the reverse order.

2. Lattices of annihilators

Here, by an algebra over $\mathbb{K}$ we mean a vector space A over $\mathbb{K}$ together with a bilinear associative multiplication. In other words, for arbitrary elements $a, b, c \in A$ and for arbitrary $\lambda \in \mathbb{K}$ the following equalities are satisfied:

1. $a(b + c) = ab + ac$;
2. $(b + c)a = ba + ca$;
3. $(ab)c = a(bc)$;
4. $(\lambda a)b = a(\lambda b) = \lambda(ab)$.

All algebras over $\mathbb{K}$ considered here, named simply algebras, are with $1 \neq 0$. An algeba A is called commutative if $ab = ba$ for any $a, b \in A$ and is finite dimensional if the space A is finite dimensional over $\mathbb{K}$. If A is an algebra, then by $\mathrm{J}(A)$ we denote the Jacobson radical of A. All other information about algebras used here one can find for example in [7, 3].

If $X \subseteq A$ is a subset of an algebra A then let $\mathrm{L}_A(X) = \mathrm{L}(X)$ be the left annihilator of X in A and let $\mathrm{R}_A(X) = \mathrm{R}(X)$ be the right annihilator of X in A :

$$\mathrm{L}(X) = \{a \in A : aX = 0\} \quad \text{and} \quad \mathrm{R}(X) = \{a \in A : Xa = 0\}.$$

Thus, by associativity of A, every left annihilator is a left ideal, and every right annihilator is a right ideal in A.

Let $\mathfrak{A}_l(A)$ be the set of all left annihilators in A and let $\mathfrak{A}_r(A)$ be the set of all right annihilators in A. Then $\mathfrak{A}_l(A)$ is a complete lattice with operations:

$$\bigvee_{s\in S} J_s = \mathrm{L}\Big(\mathrm{R}\Big(\sum_{s\in S} J_s\Big)\Big) \quad \text{and} \quad \bigwedge_{s\in S} J_s = \bigcap_{s\in S} J_s,$$

for every family $\{J_s\} \subseteq \mathfrak{A}_l(A)$. Similarly, $\mathfrak{A}_r(A)$ is a complete lattice with operations:

$$\bigvee_{s\in S} J_s = \mathrm{R}\Big(\mathrm{L}\Big(\sum_{s\in S} J_s\Big)\Big) \quad \text{and} \quad \bigwedge_{s\in S} J_s = \bigcap_{s\in S} J_s,$$

for every family $\{J_s\} \subseteq \mathfrak{A}_r(A)$. In every algebra, the lattices $\mathfrak{A}_l(A)$ and $\mathfrak{A}_r(A)$ have $\omega = 0$ and $\Omega = A$.

Between $\mathfrak{A}_l(A)$ and $\mathfrak{A}_r(A)$ we have a Galois correspondence

$$\mathfrak{A}_l(A) \xrightarrow{\mathrm{R}} (\mathfrak{A}_r(A))^{op} \quad \text{and} \quad (\mathfrak{A}_r(A))^{op} \xrightarrow{\mathrm{L}} \mathfrak{A}_l(A). \tag{2.1}$$

We need some notions related to monoids and their algebras. They are taken from [10]. Let M be a monoid and let I be an ideal in M. Then the Rees factor monoid M/I is equal to M/ρ, where ρ is the congruence on M given by $(s,t) \in \rho$ if either $s = t$ or $s, t \in I$. In this way we obtain a monoid with 0.

If M is a monoid, then a monoid algebra $\mathbb{K}[M]$ is a $\mathbb{K}$-space with the basis M and the multiplication induced by the multiplication in M. If M is a monoid with 0, then $\mathbb{K}0$ is an ideal in $\mathbb{K}[M]$. By contracted monoid algebra of M over $\mathbb{K}$, denoted by $\mathbb{K}_0[M]$, we mean the factor algebra $\mathbb{K}[M]/\mathbb{K}0$.

Now we recall the construction of an algebra $\mathbb{K}\langle L\rangle$ given in [5]. We also slightly extend some results about this algebra, proved there.

Example 2.1. ([5], Example 3.1) Let P be a nonempty poset. Then there exists a contracted monoid algebra $\mathbb{K}(P)$ such that $P \subset \mathbb{K}(P)$ and $\mathbb{K}(P)$ has a natural gradation given by:

$$\mathbb{K}(P) = \mathbb{K} \oplus V \oplus V^2, \tag{2.2}$$

where the natural base of V can be identified with P and the natural base of V^2 can be identified with $\{xy : x, y \in P,\ x \nleq y\}$. In this algebra $xy = 0$ for $x \leq y \in P$ and $V^3 = 0$. If $P = \emptyset$ then $\mathbb{K}(P) = \mathbb{K}$.

The algebra $\mathbb{K}(P)$ is a local algebra with the Jacobson radical $J = V \oplus V^2$ and with the residue field $\mathbb{K}(P)/J = \mathbb{K}$.

If L is a lattice, then $\mathbb{K}\langle L\rangle = \mathbb{K}(P)$, where $P = L\backslash\{\Omega, \omega\}$. Using the above notations we have

Theorem 2.2. ([5], Theorem 3.2) *Let P be any poset and let $\phi : P \longrightarrow \mathfrak{A}_l(\mathbb{K}(P))$ be given by $\phi(x) = \mathrm{L}_{\mathbb{K}(P)}(x)$ for $x \in P$. Then ϕ is an embedding and preserves all existing meets and joins.*

If L is a lattice, then ϕ extends uniquely to a lattice embedding of L into $\mathfrak{A}_l(\mathbb{K}\langle L\rangle)$.

If L is a complete lattice, then this extended ϕ is an isomorphism of L with the interval $[\phi(\omega), \phi(\Omega)] \subseteq \mathfrak{A}_l(\mathbb{K}\langle L\rangle)$.

For finite sets we have

Theorem 2.3. ([5], Theorem 3.4) *Let P be a poset with $|P| = m < \infty$. Under the notation from the above theorem we have*

$$1 + \frac{m(m+1)}{2} \leq \mathrm{Dim}_{\mathbb{K}}(\mathbb{K}(P)) \leq 1 + m^2.$$

If L is a lattice with $|L| = n < \infty$, then

$$1 + \frac{(n-2)(n-1)}{2} \leq \mathrm{Dim}_{\mathbb{K}}(\mathbb{K}\langle L\rangle) \leq 1 + (n-2)^2.$$

From the construction of $\mathbb{K}\langle L\rangle$ and $\mathbb{K}(P)$ we obtain that, for $|L| \leq 3$, the algebra $\mathbb{K}\langle L\rangle$ is commutative, because for $|P| \leq 1$, the algebra $\mathbb{K}(P)$ is commutative. Thus, annihilators in these algebras are (two-sided) ideals. We can prove a more general result

Theorem 2.4. *If P is a poset then every annihilator in $\mathbb{K}(P)$ is an ideal. In particular, if L is a lattice, then every annihilator in the algebra $\mathbb{K}\langle L\rangle$ is an ideal.*

Proof. Let us denote $\mathbb{K}(P)$ by A. Using arguments as in the proof of Theorem 3.2 in [5], for every nonempty subset $X \subseteq V \oplus V^2$ we have $V^2 \subseteq \mathrm{L}(X)$ and $\mathrm{L}(X)A \subseteq \mathrm{L}(X) + V^2$. Hence $\mathrm{L}(X)A \subseteq \mathrm{L}(X)$. If $X \nsubseteq V \oplus V^2$ then X contains an invertible element and $\mathrm{L}(X) = 0$. Thus any left annihilator in A is a right ideal, hence an ideal in A. Every right annihilator in A is also an ideal, as a right annihilator of its left annihilator. ∎

The lattice of ideals in any algebra is isomorphic to the lattice of congruences in this algebra, so we have

Corollary 2.5. Let L be a lattice. Then ϕ is a natural embedding of L into the lattice of all congruences of the algebra $\mathbb{K}\langle L\rangle$. This is a lower semilattice embedding.

This, in general, is only a semilattice embedding, because any algebraic sum of annihilators need not be an annihilator.

3. Commutative case

Let A be a commutative algebra. Then $\mathrm{R}(X) = \mathrm{L}(X)$ for any subset $X \subseteq A$. Thus $\mathfrak{A}_l(A) = \mathfrak{A}_r(A)$. Hence, we put in this section $\mathrm{R}(X) = \mathrm{L}(X) = \mathcal{A}(X)$ and $\mathfrak{A}_l(A) = \mathfrak{A}_r(A) = \mathfrak{A}(A)$. We point out that, as a consequence of the Formula (2.1), the mapping $X \longrightarrow \mathcal{A}(X)$ is an antiautomorphism of the lattice $\mathfrak{A}(A)$.

Each commutative, finite dimensional algebra is uniquely (up to isomorphism) a finite direct product of local, commutative algebras ([1], Theorem 8.7.). It is obvious that, for a local algebra A, the lattice $\mathfrak{A}(A)$ has a unique atom and a unique coatom. Moreover, if A and B are algebras, then we have:

$$\mathfrak{A}(A \oplus B) = \mathfrak{A}(A) \times \mathfrak{A}(B). \tag{3.1}$$

From these facts, it can be deduced that, for an algebra A, the lattice $\mathfrak{A}(A)$ is indecomposable (is not a direct product of two nontrivial lattices) if and only if A is local. In this section, due to Formula (3.1), we restrict to local algebras.

As a modification of Example 2.1 we can consider

Example 3.1. Let $P \neq \emptyset$ be a poset and let P' be the set such that $|P| = |P'|$. Let $f : P \longrightarrow P'$ be a bijection given by $f(x) = x' \in P'$. Put $P^* = P \cup P'$. Let $\mathrm{S}(P^*)$ be the free, commutative monoid with the set P^* of free generators.

Consider in $\mathrm{S}(P^*)$ an ideal I generated by all products xyz where $x, y, z \in P^*$ and by all elements of the set $\{yx' \mid x, y \in P \text{ and } x \leq y\}$. Put $\overline{S} = \mathrm{S}(P^*)/I$, the Rees factor monoid.

Clearly $P^* \subseteq \overline{S}$ in a natural way and $(P^*)^2 = \{0\} \cup \{xy \mid x, y \in P\} \cup \{x'y' \mid x', y' \in P'\} \cup \{yx' \mid x, y \in P, x \nleq y\}$. Moreover, $\overline{S} = \{1\} \cup P^* \cup (P^*)^2$.

Now let $\mathbb{K}(\!(P)\!) = \mathbb{K}_0[\overline{S}]$ be the contracted monoid algebra.

Thus $P \subset \mathbb{K}(\!(P)\!)$ and $\mathbb{K}(\!(P)\!)$ has the natural gradation given by:

$$\mathbb{K}(\!(P)\!) = \mathbb{K} \oplus W \oplus W^2, \tag{3.2}$$

where the natural base of W can be identified with P^* and the natural base of W^2 can be identified with $(P^*)^2 \backslash \{0\}$.

Our algebra $\mathbb{K}(\!(P)\!)$ is a local, commutative algebra with the Jacobson radical $J = W \oplus W^2$ and with the residue field $\mathbb{K}(\!(P)\!)/J = \mathbb{K}$.

It is easily seen from the above construction that, for every nonempty poset P, the lattice of annihilators in algebra $\mathbb{K}(\!(P)\!)$ contains a unique atom $W^2 = J^2$ and a unique coatom $W \oplus W^2 = J$. Moreover, we observe that, for $x, y, z, t \in P^*$, we have

$$\text{If } xy = zt \neq 0, \text{ then either } x = z \text{ and } y = t, \text{ or } \ x = t \text{ and } y = z. \tag{3.3}$$

If P is finite set then the algebra $\mathbb{K}(\!(P)\!)$ is finite dimensional.

Using the above notations we have

Theorem 3.2. *Let P be a nonempty poset and let $\psi : P \longrightarrow \mathfrak{A}(\mathbb{K}(\!(P)\!))$ be given by $\psi(x) = \mathcal{A}_{\mathbb{K}(\!(P)\!)}(x)$ for $x \in P$. Then ψ is an embedding and preserves all existing meets and joins.*

Proof. Observe that for every element $x \in P$ we have

$$W^2 \subseteq \psi(x) = \mathcal{A}(x) = P'_x \oplus W^2,$$

where $P'_x \subseteq W$ is a subspace spanned by elements of the set $\{y' \in P' \mid y \leq x, \text{ where } x, y \in P\}$. In particular, $x' \in \psi(x)$ and ψ is an order preserving embedding.

Now let $P_x \subseteq W$ be a subspace spanned by elements $\{y \in P \mid x \leq y \text{ where } x, y \in P\}$. Note that $\mathcal{A}(x') = P_x \oplus W^2$.

Assume that, for $S \subseteq P$ there exists $\bigvee S \in P$. We'll show that $\psi(\bigvee S) = \bigvee_{s\in S} \psi(s) = \mathcal{A}(\mathcal{A}(\sum_{s\in S} \psi(s)))$.

It turns out that only annihilators of elements in P^* are important. Let $z = \sum_i \alpha_i z_i$ and $w = \sum_j \beta_j w_j$ where $\alpha_i, \beta_j \in \mathbb{K}\backslash 0$ and $\{z_i\}, \{w_j\}$ are finite subsets of P^*. Assume that $zw = \sum_{i\,j} \alpha_i\beta_j z_i w_j = 0$. If, for example, $w_1 = z_1$ then, by the independence of suitable elements of $(P^*)^2$, we obtain that $\alpha_1\beta_1 = 0$, a contradiction. Hence $\{z_i\} \cap \{w_j\} = \emptyset$, and then $z_i w_j = 0$ for every pair i, j. In view of this, every annihilator in the algebra $\mathbb{K}(\!(P)\!)$ properly contained in J is the intersection of annihilators of elements in P^*. Moreover, we can see that for any $J^2 \subsetneq X \subsetneq J$ we have $\mathcal{A}(X) = B \oplus W^2$, where B is a subspace spanned by some elements in P^*.

Let $z \in P^*$ and $z \in \mathcal{A}(\sum_{s\in S} \psi(s))$. Then, in particular, $zs' = 0$ for each $s \in S$ and hence $z \geq s$ for all $s \in S$. This implies $z \geq \bigvee S$. Therefore $\mathcal{A}(\sum_{s\in S} \psi(s)) \subseteq P_{\bigvee S} \oplus W^2$. Obviously $P_{\bigvee S} \oplus W^2 \subseteq \mathcal{A}(\sum_{s\in S} \psi(s))$ and thus $P_{\bigvee S} \oplus W^2 = \mathcal{A}(\sum_{s\in S} \psi(s))$. It is easy to see that $\mathcal{A}(P_{\bigvee S} \oplus W^2) = \mathcal{A}(\bigvee S)$. Consequently ψ preserves all existing joins.

It is easy to show that ψ preserves all existing meets. ∎

Corollary 3.3. Every finite lattice can be represented as a sublattice of a lattice of annihilators in a commutative $\mathbb{K}$-algebra.

Similarly as in [5], with use of the result 16.7 in [2], we have

Corollary 3.4. There is no nontrivial lattice identity satisfied in lattices of annihilators of all finite dimensional, commutative algebras.

If P is a finite poset then we can estimate the dimension of the algebra $\mathbb{K}(\!(P)\!)$.

Theorem 3.5. *Let $|P| = m < \infty$. Then*

$$\frac{2+5m+3m^2}{2} \leq \mathrm{Dim}_{\mathbb{K}}(\mathbb{K}(\!(P)\!)) \leq 1+2m+2m^2.$$

Proof. Under notation from Formula (3.2), we have

$$\mathrm{Dim}_{\mathbb{K}}(W) = 2m \quad \text{and} \quad \frac{3m^2+m}{2} \leq \mathrm{Dim}_{\mathbb{K}}(W^2) \leq 2m^2. \tag{3.4}$$

Hence

$$1+2m+\frac{3m^2+m}{2} \leq \mathrm{Dim}_{\mathbb{K}}(\mathbb{K}(\!(P)\!)) \leq 1+2m+2m^2. \tag{3.5}$$

The first inequality can be checked by induction on m, while the second is evident. ∎

In the case when L is a lattice, the dimension of the algebra $\mathbb{K}(\!(L)\!)$ is relatively big. Due to the fact that every annihilator in an algebra A is a subspace in A, we have that the length of any chain in $\mathfrak{A}(A)$ is less or equal to the dimension of A.

Let L be a lattice and A be an algebra such that L is a sublattice of $\mathfrak{A}(A)$. Keeping in mind that, if A is local then $\mathfrak{A}(A)$ has unique atom and unique coatom, one can indicate a lower bound of dimension of A. For some special lattices L we give commutative algebras A of the least possible dimension with $L \subseteq \mathfrak{A}(A)$.

Example 3.6. Let L be a chain with n elements and $A = \mathbb{K}[x]/(x^{n-1})$. It is easy to check that L is isomorphic to $\mathfrak{A}(A)$ and A is an algebra with the least possible dimension.

Example 3.7. Let $\mathbb{K}$ be an infinite field, $n > 3$ and let M_n be the lattice of length 2 with n elements. Let us take the commutative polynomial algebra $\mathbb{K}[x, y]$ and $A = \mathbb{K}[x, y]/(x^2, y^2)$. Then the lattice M_n is embeddable in $\mathfrak{A}(A)$ and A is an algebra with the least possible dimension. Indeed, if B is an algebra of dimension at most 3, then $\mathfrak{A}(B)$ is either a chain or a product of chains.

Example 3.8. Let B_{2^n} be the boolean algebra of cardinality 2^n. Obviously B_{2^n} is isomorphic to the lattice of annihilators in the non local algebra $\mathbb{K}^n$ and this is an algebra with the least possible dimension. Indeed, B_{2^n} contains a chain of n+1 elements.

Let L be a finite lattice. Just as in [5] one can find a general way to construct an algebra A with $L \subseteq \mathfrak{A}(A)$ and $\mathrm{Dim}_{\mathbb{K}}(A) < \mathrm{Dim}_{\mathbb{K}}(\mathbb{K}(\!(L)\!))$.

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