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LOCAL SPECTRAL THEORY OF ENDOMORPHISMS OF THE DISK ALGEBRA

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Abstract. Let $A(\mathbb{D})$ denote the disk algebra. Every endomorphism of $A(\mathbb{D})$ is induced by some $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$. In this paper, it is shown that if ϕ is not an automorphism of $\mathbb{D}$ and ϕ has a fixed point in the open unit disk then the endomorphism induced by ϕ is decomposable if and only if the fixed set of ϕ is singleton. Further, we determine the local spectra of the endomorphism induced by ϕ in the cases when the fixed set of ϕ either includes unit circle or is a singleton.

1. Introduction

Let X be a complex Banach space and $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators on X. For $T \in \mathcal{B}(X)$, the resolvent set $\rho(T)$ of T is the set $\{\lambda \in \mathbb{C} : (\lambda - T)^{-1} \in \mathcal{B}(X)\}$. The complement of $\rho(T)$ in $\mathbb{C}$ is called the spectrum of T and is usually denoted by $\sigma(T)$. The resolvent function $R : \rho(T) \longrightarrow \mathcal{B}(X)$, defined as, $R(\lambda) = (\lambda - T)^{-1}$ for each $\lambda \in \rho(T)$, is analytic on $\rho(T)$. Hence for $x \in X$, the function $R_x : \rho(T) \longrightarrow X$, defined as $R_x(\lambda) = (\lambda - T)^{-1}x$ for each $\lambda \in \rho(T)$, is also an analytic function on $\rho(T)$ and satisfies $(\lambda - T)R_x(\lambda) = x$ for each $\lambda \in \rho(T)$. The function $R_x(\lambda)$ may have an analytic extension to an open set G containing $\rho(T)$. If for a fixed $x \in X$, any two of such analytic extensions agree on their common domain, then we say that T has Single Valued Extension Property (abbreviated as SVEP). Equivalently, an operator $T \in \mathcal{B}(X)$ is said to have the SVEP if for every open set $G \subseteq \mathbb{C}$, the only analytic solution $f : G \longrightarrow X$ of the equation $(\lambda - T)f(\lambda) = 0$ for all $\lambda \in G$ is the zero function on G. It is clear that if the point spectrum $\sigma_p(T)$ of T has empty interior, then T has SVEP. The examples of operators without SVEP can be found among those which

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are surjective but not invertible ([5]). For $x \in X$, the local resolvent of T at x, denoted by $\rho_T(x)$, is defined as the union of all open subsets G of $\mathbb{C}$ for which there is an analytic function $f : G \longrightarrow X$ satisfying $(\lambda - T)f(\lambda) = x$ for all $\lambda \in G$. The complement of $\rho_T(x)$ in $\mathbb{C}$ is called the local spectrum of T at x and is denoted by $\sigma_T(x)$. For a subset $F \subseteq \mathbb{C}$, the local spectral subspace of T is defined by $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$. The subspace $X_T(F)$ is not necessarily closed but is invariant under each operator which commutes with T. If $X_T(F)$ is closed for every closed subset F of $\mathbb{C}$ then we say that T satisfies Dunford's property (C). An operator T is said to have Bishop's property (β) if for every open subset G of $\mathbb{C}$ and every sequence of analytic functions $f_n : G \longrightarrow X$ with the property that $(\lambda - T)f_n(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$, locally uniformly on G, then $f_n(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$, locally uniformly on G. For $T \in \mathcal{B}(X)$, it is well known that T has property $(\beta) \Longrightarrow$ T has property $(C) \Longrightarrow T$ has Single Valued Extension Property. The implications cannot be reversed in general, for suitable examples see [2] and [11]. An operator T is said to satisfy the decomposition property (δ) if for every open cover $\{U, V\}$ of $\mathbb{C}$, $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$, where $\mathcal{X}_T(\overline{U})$ is defined as the set of all $x \in X$ such that there is an analytic function $f : \mathbb{C}\backslash\overline{U} \longrightarrow X$ satisfying $(\lambda - T)f(\lambda) = x$. An operator T is called decomposable if for every open cover $\{U, V\}$ of $\mathbb{C}$ there exist T-invariant closed subspaces Y and Z of X such that $\sigma(T|Y) \subseteq U$, $\sigma(T|Z) \subseteq V$ and $X = Y + Z$. An operator is decomposable if and only if it has both properties (β) and (δ). For further study of local spectral theory we refer to [1, 2, 4, 9] and [12].

Let $\mathbb{D}$ be the open unit disk in the complex plane and $\overline{\mathbb{D}}$ denote its closure in the complex plane. Let $A(\mathbb{D})$ denote the Banach algebra of analytic functions on $\mathbb{D}$ which are continuous on $\overline{\mathbb{D}}$, equipped with supremum norm and pointwise addition and multiplication. An endomorphism $T : A(\mathbb{D}) \longrightarrow A(\mathbb{D})$ is a linear map which also satisfies $T(f_1 f_2) = T(f_1)T(f_2)$, for all $f_1, f_2 \in A(\mathbb{D})$. Every endomorphism T of $A(\mathbb{D})$ is induced by some $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$, in the sense that $T(f) = f \circ \phi$, for all $f \in A(\mathbb{D})$. In fact, if z is the identity function then $\phi = Tz$. Thus endomorphisms of $A(\mathbb{D})$ are nothing but the composition operators induced by some $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$. Therefore, throughout this paper we denote by C_ϕ, the endomorphism on $A(\mathbb{D})$ induced by $\phi \in A(\mathbb{D})$. For further details on the disk algebra we refer to [6].

2. Main results

Definition 2.1. Let $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$. For each positive integer n, let ϕ_n denote the n-th iterate of ϕ with itself. The fixed set of ϕ is defined as $\cap_{n\geq 1} range(\phi_n)$.

Note that the fixed set of ϕ is either singleton or infinite. Further, the fixed set of ϕ is connected and compact subset of $\overline{\mathbb{D}}$. The following lemma gives a necessary and sufficient condition for the fixed set of ϕ to be infinite.

Lemma 2.2. *Let $\phi \in A(\mathbb{D})$ with $\phi(0) = 0$ and $\|\phi\| \leq 1$. Suppose that ϕ is not an automorphism of $\mathbb{D}$. Then the fixed set of ϕ is infinite if and only if $\|\phi_n\| = 1$ for each positive integer n.*

Proof. Suppose that the fixed set of ϕ is infinite. If possible, let $N \in \mathbb{N}$ be such that $\|\phi_N\| < 1$. First we assume that $0 < |\phi'(0)| < 1$.

Choose $\epsilon > 0$ such that $(1+\epsilon)|\phi'(0)| < 1$. Then by differentiability of ϕ at 0, we can find a $\delta > 0$ such that

$$|\phi_{N+1}(z)| < (1+\epsilon)|\phi'(0)||\phi_N(z)| \text{ whenever } |\phi_N(z)| < \delta. \tag{1}$$

Since $|\phi_N(z)| < 1$ for all $z \in \overline{\mathbb{D}}$ therefore, by Schwarz's lemma $|\phi_{N+1}(z)| < |\phi_N(z)|$ for all $z \in \overline{\mathbb{D}}\backslash\{0\}$. Hence $|\phi_{N+1}(z)| < \delta$ whenever $|\phi_N(z)| < \delta$. This together with (1) gives

$$|\phi_{N+2}(z)| < [(1+\epsilon)|\phi'(0)|]^2|\phi_N(z)| \text{ whenever } |\phi_N(z)| < \delta.$$

By induction, it follows that for each $n \geq 1$,

$$|\phi_{N+n}(z)| < [(1+\epsilon)|\phi'(0)|]^n|\phi_N(z)| \text{ whenever } |\phi_N(z)| < \delta.$$

Let $F = \{\phi_N(z) : |\phi_N(z)| \leq \frac{\delta}{2}\}$. Then F is a compact subset of $\mathbb{D}$ with non-empty interior. Also, $\|\phi_n\| \leq 1$ for each $n \geq 1$. Hence, by Montel's theorem, $\{\phi_n : n \geq 1\}$ is a normal family. Therefore, there exist a subsequence $(\phi_{n_k})_{k=1}^{\infty}$ of $(\phi_n)_{n=1}^{\infty}$ and an analytic function f on $\mathbb{D}$ such that ϕ_{n_k} converges to f uniformly on F. Now

$$\|\phi_{n_k}\|_F = \sup_{w \in F} |\phi_{n_k}(w)| \leq [(1+\epsilon)|\phi'(0)|]^{n_k} \frac{\delta}{2} \to 0 \text{ as } k \to \infty.$$

Hence $\|f\|_F = 0$. Because F has non-empty interior and f is analytic, it follows that $f \equiv 0$ on $\mathbb{D}$.

Thus ϕ_{n_k} converges to 0 uniformly on each compact subset of $\mathbb{D}$.

Let $F_1 = \{z \in \overline{\mathbb{D}} : |z| \leq \|\phi_N\|\}$. Since $\|\phi_N\| < 1$, so $\phi_N(\overline{\mathbb{D}}) \subseteq F_1$. Now for each $n \geq 1$,

$$\begin{aligned} \|\phi_{N+n}\| &= \sup_{z \in \phi_N(\overline{\mathbb{D}})} |\phi_n(z)| \\ &\leq \sup_{z \in F_1} |\phi_n(z)| \\ &= \|\phi_n\|_{F_1}. \end{aligned}$$

As F_1 is a compact subset of $\mathbb{D}$, therefore $\|\phi_{n_k}\|_{F_1}$ converges to 0. This implies

that $\|\phi_{N+n_k}\|$ converges to 0. Hence $\bigcap_{k=1}^{\infty} range(\phi_{N+n_k}) = \{0\}$. Consequently, $\bigcap_{n=1}^{\infty} range(\phi_n) = \{0\}$, which contradicts the fact that the fixed set of ϕ is infinite. Hence $\|\phi_n\| = 1\ \forall n \geq 1$.

If $\phi'(0) = 0$, then put ϵ in place of $(1+\epsilon)|\phi'(0)|$ and rest of the proof goes same as above.

Conversely, suppose that $\|\phi_n\| = 1$ for each positive integer n. Then for each positive integer n, there is a real number θ_n such that $|\phi_n(e^{i\theta_n})| = 1$. Let $\phi_n(e^{i\theta_n}) = e^{i\alpha_n}$, α_n is real. Since $(e^{i\alpha_n})_{n=1}^{\infty}$ is a bounded sequence, therefore there is a real number β and a subsequence $(e^{i\alpha_{n_k}})_{k=1}^{\infty}$ such that $e^{i\alpha_{n_k}}$ converges to $e^{i\beta}$ as k goes to infinity. We claim that $e^{i\beta}$ is in $range(\phi_n)$ for each positive integer n. On contrary, suppose that there is a natural number N such that $e^{i\beta}$ is not in $range(\phi_N)$. Since $range(\phi_N)$ is a compact subset of $\overline{\mathbb{D}}$ therefore, $dist(e^{i\beta}, range(\phi_N)) = \delta > 0$. For each $n_k \geq N$, $e^{i\alpha_{n_k}} \in range(\phi_N)$. Therefore, $|e^{i\alpha_{n_k}} - e^{i\beta}| \geq \delta > 0\ \forall n_k \geq N$, which contradicts the fact that $e^{i\alpha_{n_k}}$ converges to $e^{i\beta}$. Therefore, $e^{i\beta}$ is in the fixed set of ϕ. Since 0 and $e^{i\beta}$ are in the fixed set of ϕ and the fixed set is connected, we conclude that the fixed set of ϕ is infinite. ∎

Remark 2.3. If ϕ has a fixed point $z_0 \in \mathbb{D}$ other than 0, then by the transform $\eta(z) = \frac{z_0 - z}{1 - \bar{z}_0 z}$, we see that $\psi = \eta \circ \phi \circ \eta$ has 0 as its fixed point. Also, $\psi_n = \eta \circ \phi_n \circ \eta$ for each $n \geq 1$. Now it can be easily seen that $\|\psi_n\| = 1$ if and only if $\|\phi_n\| = 1$ and the cardinality of the fixed sets of ϕ and ψ are the same. Hence Lemma 2.2 also holds in this case.

Remark 2.4. The fixed set being infinite does not necessarily imply that ϕ has a fixed point on the boundary. For example, $\phi(z) = \frac{z^3 - z}{2}$ has infinite fixed set as $0, i, -i$ are in range of each ϕ_n but ϕ has no fixed point on the boundary.

The following proposition is an improvement over the Lemma 3 of [7].

Proposition 2.5. *Suppose that $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$ and $\phi(0) = 0$. Then we have following cases.*

(1) *If ϕ is an automorphism of $\mathbb{D}$, then $\sigma_p(C_\phi) = \{\phi'(0)^n : n > 0\} \cup \{1\}$.*
(2) *If ϕ is not an automorphism of $\mathbb{D}$ and the fixed set of ϕ is infinite, then $\sigma_p(C_\phi) = \{1\}$.*
(3) *If the fixed set of ϕ is $\{0\}$ then*

$$\sigma_p(C_\phi) = \begin{cases} \{0, 1\}, & \text{if } \phi \equiv 0, \\ \{1\}, & \text{if } \phi \neq 0 \text{ but } \phi'(0) = 0, \\ \{\phi'(0)^n : n > 0\} \cup \{1\}, & \text{otherwise.} \end{cases}$$

Proof. From Lemma 3 of [7], we know that if $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$ and $\phi(0) = 0$ then

$$\sigma_p(C_\phi) \subseteq \{\phi'(0)^n : n \geq 1\} \cup \{0, 1\}. \tag{2}$$

(1). If ϕ is an automorphism of $\mathbb{D}$, then by Schwarz's lemma, $\phi(z) = \phi'(0)z$. In this case, for each $n \geq 0$, we have

$$C_\phi z^n = \phi(z)^n = \phi'(0)^n z^n.$$

Hence $\phi'(0)^n$ is an eigenvalue of C_ϕ for each $n \geq 0$. Since $C_\phi f(z) = 0$ for all $z \in \overline{\mathbb{D}}$ implies that $f \equiv 0$, we have $\sigma_p(C_\phi) = \{\phi'(0)^n : n \geq 0\}$.

(2). Suppose that ϕ is not an automorphism of $\mathbb{D}$ and the fixed set K of ϕ is infinite. Let $|\lambda| < 1$ and $f \in A(\mathbb{D})$ be such that

$$(\lambda - C_\phi)f(z) = 0, \text{ for all } z \in \overline{\mathbb{D}}.$$

Then

$$|\lambda|\|f\|_K = \|f\|_K,$$

where $\|f\|_K = \sup_{z \in K} |f(z)|$. Hence we have $\|f\|_K = 0$. Since the fixed set K is a connected subset of $\overline{\mathbb{D}}$ containing more than one point, it follows that $f \equiv 0$. Now as $1 \in \sigma_p(C_\phi)$, so from (2), we conclude that $\sigma_p(C_\phi) = \{1\}$.

(3). If the fixed set K of ϕ is $\{0\}$, then from Theorem 9 of [7], we have $\sigma(C_\phi) = \{\phi'(0)^n : n \geq 1\} \cup \{0, 1\}$. If $\phi \equiv 0$, then $C_\phi(z) = \phi(z) = 0$. Since $C_\phi(1) = 1$, therefore $\sigma_p(C_\phi) = \{0, 1\}$. Also, by preceding Lemma 2.2, there is a natural number N such that $\|\phi_N\| < 1$. Hence by Theorem 1.2 of [8], C_ϕ^N is compact and so C_ϕ is a Riesz operator. As the spectrum of a Riesz operator consists of countable eigenvalues clustering possibly at 0, so for each $n \geq 0$, $\phi'(0)^n$ is an eigenvalue of C_ϕ, provided $\phi'(0) \neq 0$. If $\phi \neq 0$ and $\phi'(0) = 0$, then it is easy to see that $\sigma_p(C_\phi) = \{1\}$. Thus the result follows. ∎

Remark 2.6. If $\phi(z_0) = z_0$ for some non-zero $z_0 \in \mathbb{D}$ then it can be easily shown that the above proposition is also true with $\phi'(0)$ replaced by $\phi'(z_0)$.

We now state following lemma which will be used in our subsequent results. The proof of this lemma can be found in [13].

Lemma 2.7. *Let X be a complex Banach space and $T \in \mathcal{B}(X)$. Suppose that $\sigma(T)$ is not a singleton and $\bigcap_{\substack{x \in X \\ x \neq 0}} \sigma_T(x) \neq \emptyset$ then T has Single Valued Extension Property but T does not have decomposition property (δ) and hence, T is not decomposable.*

Theorem 2.8. *Let $\phi \in A(\mathbb{D})$, $\|\phi\| \leq 1$, $\phi(0) = 0$ and C_ϕ be the endomorphism induced by ϕ. If ϕ is not an automorphism and the fixed set of ϕ is infinite, then C_ϕ has Single Valued Extension Property but does not have decomposition property (δ). Hence, in this case, C_ϕ is not decomposable.*

Proof. The Single Valued Extension Property follows directly from the Proposition 2.5 as $\sigma_p(C_\phi) = \{1\}$, in this case. If f is a non-zero function in $A(\mathbb{D})$ such that $f(0) = 0$, then from Theorem 7 of [7], it follows that $\partial\mathbb{D} \subseteq \sigma_{C_\phi}(f)$. Also, it is easy to see that $\sigma_{C_\phi}(\alpha) = \{1\}$ for each non-zero $\alpha \in \mathbb{C}$. Now, let $f \in A(\mathbb{D})$ be a non-constant function such that $f(0) \neq 0$. Then we can write $f(z) = f(0) + g(z)$, where $g(0) = 0$. We claim that $1 \in \sigma_{C_\phi}(f)$. For, if $1 \notin \sigma_{C_\phi}(f)$ then there exist a neighborhood G_1 of 1 and an analytic function $F : G_1 \longrightarrow A(\mathbb{D})$ defined by $F(\lambda) = F_\lambda$ and satisfying

$$(\lambda - C_\phi)F_\lambda = f \text{ for each } \lambda \in G_1.$$

This implies that $F_\lambda(0) = \frac{f(0)}{\lambda - 1}$, which contradicts the fact that F is analytic on G_1.

Hence $1 \in \bigcap_{\substack{f \in A(\mathbb{D}) \\ f \neq 0}} \sigma_{C_\phi}(f)$. Now by preceding Lemma 2.7, C_ϕ does not have decomposition property (δ). ∎

Remark 2.9. If ϕ has a fixed point $z_0 \in \mathbb{D}$ other than 0, then as in Remark 2.3, C_ϕ is similar to C_ψ. Therefore, by above theorem, C_ψ is not decomposable. Since decomposability is preserved under similarity, so C_ϕ is not decomposable.

Definition 2.10. Let X be a complex Banach space and $T \in \mathcal{B}(X)$. The analytic core of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(y_n)_{n=1}^{\infty} \subset X$ and $\delta > 0$ for which:

(1) $x = y_0$, and $Ty_{n+1} = y_n$ for every $n \geq 0$.
(2) $\|y_n\| \leq \delta^n \|x\|$ for every $n \in \mathbb{N}$.

The analytic core $K(T)$ of T is a subspace of X and $TK(T) = K(T)$. Mbekhta [10] and Vrbova [14] proved that for each $\lambda \in \mathbb{C}$, $K(\lambda - T) = X_T(\mathbb{C}\backslash\{\lambda\})$. The proof of above facts can be found in [1], [10] or [14].

Remark 2.11. Let $T, S \in \mathcal{B}(X)$ be similar operators. Then $T = ASA^{-1}$ for some $A \in \mathcal{B}(X)$. It can be easily seen that $A^{-1}K(T) \subseteq K(S)$ and $AK(S) \subseteq K(T)$.

Lemma 2.12. *Suppose that $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$, $\phi(0) = 0$ and ϕ is not an automorphism of $\mathbb{D}$. If the fixed set of ϕ includes $\partial\mathbb{D}$, then $K(C_\phi) = \mathbb{C}$.*

Proof. It is easy to see that $\mathbb{C} \subseteq K(C_\phi)$. We shall show that if $f \in K(C_\phi)$ with $f(0) = 0$, then $f \equiv 0$. From this it will follow that $K(C_\phi) = \mathbb{C}$. To this end, let $f \in K(C_\phi)$ and f has a zero of multiplicity n at 0. Then by definition of analytic core, there is a sequence $(f_n)_{n=1}^{\infty} \subseteq A(\mathbb{D})$ and $\delta > 0$ such that

$$f_0 = f,\ C_\phi f_{n+1} = f_n \text{ and } \|f_n\| \leq \delta^n \|f\| \text{ for each } n \geq 1.$$

This implies that $\|f\| = \|f_n \circ \phi_n\| \leq \|f_n\| \leq \delta^n \|f\|$ for each $n \geq 1$. Hence $\delta \geq 1$. Since $\phi(\partial\mathbb{D}) \supseteq \partial\mathbb{D}$, therefore $\|f_n \circ \phi_n\| = \|f_n\| = \|f\|$ for each $n \geq 1$. Hence we can take $\delta = 1$. Thus

$$F(\lambda) = -\sum_{n=1}^{\infty} \lambda^{n-1} f_n$$

is an analytic function on $\mathbb{D}$ satisfying

$$(\lambda - C_\phi)F(\lambda) = f \text{ for each } \lambda \in \mathbb{D}.$$

Also from Theorem 7 of [7], the function

$$G(\lambda)(z) = \lambda^{-1} \sum_{k=0}^{\infty} f(\phi_k(z))\lambda^{-k} \text{ for all } z \in \mathbb{D},$$

is analytic in $\{\lambda : |\phi'(0)| < |\lambda| < 1\}$ and satisfies

$$(\lambda - C_\phi)G(\lambda) = f.$$

As Theorem 2.8 ensures us that C_ϕ has SVEP, we must have

$$F(\lambda) = G(\lambda) \text{ for each } \lambda \text{ with } |\phi'(0)| < |\lambda| < 1.$$

This implies that

$\sum_{k=0}^{\infty} f(\phi_k(z))\lambda^{-(k+1)} + \sum_{k=1}^{\infty} \lambda^{k-1} f_k(z) = 0$ for $|\phi'(0)| < |\lambda| < 1$ and $|z| < 1$.

The above expression is a Laurent expansion of an analytic function which is zero on $\{\lambda : |\phi'(0)|^n < |\lambda| < 1\}$, therefore $f(\phi_k(z)) = 0$ for each $k \geq 1$ and $|z| < 1$. Fix $z_0 \in \mathbb{D}$, then by Schwarz's lemma, $\phi_k(z_0) \longrightarrow 0$ as $k \longrightarrow \infty$. Now, since $f(\phi_k(z_0)) = 0$ for each $k \geq 1$ and $f(0) = 0$, therefore $f \equiv 0$. Thus $K(C_\phi) = \mathbb{C}$. ∎

Corollary 2.13. *Suppose that $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$, $\phi(z_0) = z_0$ for some non-zero $z_0 \in \mathbb{D}$ and ϕ is not an automorphism of $\mathbb{D}$. If the fixed set of ϕ includes $\partial\mathbb{D}$, then $K(C_\phi) = \mathbb{C}$.*

Proof. If $\phi(z_0) = z_0$ for some non-zero $z_0 \in \mathbb{D}$, then from Remark 2.3, $C_\psi = C_\eta C_\phi C_\eta$ and from above lemma $K(C_\psi) = \mathbb{C}$. Further, by Remark 2.11, $\mathbb{C} = C_\eta(\mathbb{C}) = C_\eta(K(C_\psi)) \subseteq K(C_\phi)$ and $C_\eta(K(C_\phi)) \subseteq K(C_\psi) = \mathbb{C}$. Hence $K(C_\phi) = \mathbb{C}$. ∎

Theorem 2.14. *Suppose that $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$, $\phi(z_0) = z_0$ for some $z_0 \in \mathbb{D}$ and ϕ is not an automorphism of $\mathbb{D}$. If the fixed set of ϕ includes $\partial\mathbb{D}$ then C_ϕ has property (β). Moreover,*

$$\sigma_{C_\phi}(f) = \begin{cases} \{1\}, & \text{if } f \text{ is a non-zero constant,} \\ \overline{\mathbb{D}}, & \text{if } f \text{ is non-constant.} \end{cases}$$

Proof. If ϕ is not an automorphism and the fixed set of ϕ includes $\partial\mathbb{D}$ then C_ϕ is a non-invertible isometry. Hence by Proposition 1.6.7 of [9], C_ϕ has property (β) and, for each $f \in A(\mathbb{D})$, $\sigma_{C_\phi}(f)$ is either contained in $\partial\mathbb{D}$ or is equal to $\overline{\mathbb{D}}$. But from the Corollary 2.13, we have

$$K(C_\phi) = A_{C_\phi}(\mathbb{C}\backslash\{0\}) = \mathbb{C},$$

where $A_{C_\phi}(\mathbb{C}\backslash\{0\}) = \{f \in A(\mathbb{D}) : 0 \notin \sigma_{C_\phi}(f)\}$. Hence

$$\sigma_{C_\phi}(f) = \overline{\mathbb{D}}, \text{ for each non-constant } f \in A(\mathbb{D}).$$

It is easy to see that $\sigma_{C_\phi}(\alpha) = \{1\}$ for each non-zero $\alpha \in \mathbb{C}$. Thus the result follows. ∎

Definition 2.15. Let X be a complex Banach space and $T \in \mathcal{B}(X)$. The quasi-nilpotent part of T is defined to be the set

$$H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Note that $H_0(T)$ is a subspace of X. Further, if T has Single Valued Extension Property, then for each $\lambda \in \mathbb{C}$, $H_0(\lambda - T) = X_T(\{\lambda\})$. For proof, see [1], [10] or [14].

Definition 2.16. An operator $T \in \mathcal{B}(X)$, where X is a complex Banach space, is super-decomposable if, for every open cover $\{U, V\}$ of $\mathbb{C}$ there is an operator $S \in \mathcal{B}(X)$ which commutes with T and for which

$$\sigma(T|\overline{SX}) \subseteq U \text{ and } \sigma(T|\overline{(I-S)X}) \subseteq V.$$

It is easy to see that every super-decomposable operator is decomposable.

Now we determine the local spectra of C_ϕ at each $f \in A(\mathbb{D})$ when the fixed set of ϕ is a singleton.

Proposition 2.17. *Let $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$ and $\phi(0) = 0$. If the fixed set of ϕ is $\{0\}$ and $\phi'(0) = 0$, then C_ϕ is super-decomposable and for each non-zero $f \in A(\mathbb{D})$,*

$$\sigma_{C_\phi}(f) = \begin{cases} \{1\}, & \text{if } f(0) \neq 0 \text{ and } g \equiv 0, \\ \{0\}, & \text{if } f(0) = 0 \text{ and } g \neq 0, \\ \{0, 1\}, & \text{if } f(0) \neq 0 \text{ and } g \neq 0, \end{cases}$$

where $f(z) = f(0) + g(z)$.

Proof. If $\phi'(0) = 0$, then by Theorem 9 of [7], we have $\sigma(C_\phi) = \{0, 1\}$ and hence by Proposition 1.4.5 of [9], C_ϕ is super-decomposable. Also, for the closed sets $\{0\}$ and $\{1\}$, we have

$$A(\mathbb{D}) = A_{C_\phi}(\{0\}) \oplus A_{C_\phi}(\{1\}). \tag{3}$$

It is easy to see that $\mathbb{C} \subseteq A_{C_\phi}(\{1\})$. Let $f \in A(\mathbb{D})$ has a zero of multiplicity n at 0. Then $f(z) = z^n g(z)$, $g \in A(\mathbb{D})$. Hence,

$$\lim_{k\to\infty} \|C_\phi^k f\|^{\frac{1}{k}} \leq \lim_{k\to\infty} (\|\phi_k\|^n \|g\|)^{\frac{1}{k}} = \lim_{k\to\infty} \|\phi_k\|^{\frac{n}{k}}.$$

Since the fixed set of ϕ is $\{0\}$, therefore by Lemma 2.2, there is a natural number N such that $\|\phi_N\| < 1$. Hence by Lemma 6 of [7], for a given $\epsilon > 0$ and $r = \|\phi_N\|$, there is a $B > 0$ such that $|\phi_k(\phi_N(z))| \leq B\epsilon^k$ for all $z \in \overline{\mathbb{D}}$. This implies that $\|\phi_{k+N}\| \leq B\epsilon^k$. Hence $\lim_{k\to\infty} \|\phi_{k+N}\|^{\frac{1}{k+N}} \leq \epsilon$. Thus

$$\lim_{k\to\infty} \|C_\phi^k f\|^{\frac{1}{k}} \leq \epsilon.$$

Therefore $f \in H_0(C_\phi) = A_{C_\phi}(\{0\})$. If f is a non-zero function in $A(\mathbb{D})$ then $f(z) = f(0) + g(z)$, $g \in A(\mathbb{D})$. Now using equation (3), it can be easily seen that

$$\sigma_{C_\phi}(f) = \begin{cases} \{1\}, & \text{if } f(0) \neq 0 \text{ and } g \equiv 0, \\ \{0\}, & \text{if } f(0) = 0 \text{ and } g \neq 0, \\ \{0, 1\}, & \text{if } f(0) \neq 0 \text{ and } g \neq 0. \end{cases}$$

∎

Remark 2.18. Suppose that ϕ has a fixed point $z_0 \in \mathbb{D}$ other than 0 and $\phi'(z_0) = 0$. From Remark 2.3, $C_\psi = C_\eta C_\phi C_\eta$. Then it is easy to see that for each $f \in A(\mathbb{D})$,

$$\sigma_{C_\psi}(f) = \sigma_{C_\phi}(C_\eta f). \tag{4}$$

Let $f \in A(\mathbb{D})$, then there exists a unique $g \in A(\mathbb{D})$ such that $f(z) = g(\eta(z))$ for each $z \in \overline{\mathbb{D}}$. Let $g(z) = g(0) + h(z)$, where $h \in A(\mathbb{D})$ and $h(0) = 0$. Then

$$f(z) = g(\eta(z)) = g(0) + h(\eta(z)) \text{ for each } z \in \overline{\mathbb{D}}.$$

Here $h(\eta(z_0)) = h(0) = 0$. Thus each $f \in A(\mathbb{D})$ can be uniquely expressed as

$$f(z) = \alpha + f_1(z), \tag{5}$$

where $f_1 \in A(\mathbb{D})$ and $f_1(z_0) = 0$.

Now, if $g \in A(\mathbb{D})$ is such that $g(z_0) = 0$, then $g = h \circ \eta$ for some $h \in A(\mathbb{D})$ with $h(0) = 0$. Hence from Proposition 2.17 and equation (4), we get

$$\{0\} = \sigma_{C_\psi}(h) = \sigma_{C_\phi}(C_\eta h) = \sigma_{C_\phi}(g).$$

Now using equation (5) and the arguments used in Proposition 2.17, it can be easily shown that for each non-zero $f \in A(\mathbb{D})$,

$$\sigma_{C_\phi}(f) = \begin{cases} \{1\}, & \text{if } \alpha \neq 0 \text{ and } f_1 \equiv 0, \\ \{0\}, & \text{if } \alpha = 0 \text{ and } f_1 \neq 0, \\ \{0,1\}, & \text{if } \alpha \neq 0 \text{ and } f_1 \neq 0. \end{cases}$$

In the subsequent results, we assume that $\phi'(0) \neq 0$.

Definition 2.19. (1). Let $(F_n)_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{C}$. The limit inferior, $\liminf F_n$ is the set of $\lambda \in \mathbb{C}$ such that every neighbourhood of λ has non-empty intersection with all but finitely many of the F_n.
(2). Let ξ be a mapping defined on a subset F of a metric space into the subsets of $\mathbb{C}$. Then ξ is said to be lower semicontinuous at $x \in F$ if for each sequence $(x_n)_{n=1}^{\infty} \subseteq F$ such that $x_n \longrightarrow x$, we have $\xi(x) \subseteq \liminf \xi(x_n)$.

For $x \in X$ and $T \in \mathcal{B}(X)$, the local spectrum $\sigma_T(x)$ is a function of two variables x and T. In [3], Dollinger and Oberai studied the continuity of $\sigma_T(x)$ as a function of T and as a function of x. In particular, they proved that if T has Dunford's property (C), then the map $x \longmapsto \sigma_T(x)$ is lower semicontinuous at each $x \in X$.

If $\phi(z) = \frac{z}{2}$, then it is easy to see that the set of eigenfunctions of C_ϕ forms a Schauder basis of $A(\mathbb{D})$. We feel that this is the general nature of the eigenfunctions of C_ϕ in the case when ϕ is not an automorphism, $\phi'(0) \neq 0$ and the fixed set of ϕ is $\{0\}$. Motivated by this simple example, we give the following result.

Theorem 2.20. *Let $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$ and $\phi(0) = 0$. If the fixed set of ϕ is $\{0\}$ then C_ϕ is super-decomposable. Moreover, if f_n, $n \geq 0$ is an eigenfunction corresponding to the eigenvalue $\phi'(0)^n$ and span$\{f_n : n \geq 0\}$ is dense in $A(\mathbb{D})$, then for each non-zero $f \in A(\mathbb{D})$*

$$\sigma_{C_\phi}(f) = \overline{\{\phi'(0)^n : n \geq 0 \text{ and } a_n \neq 0\}},$$

where $f(z) = \sum_{n=0}^{\infty} a_n f_n(z)$.

Proof. If the fixed set of ϕ is $\{0\}$, then by Theorem 9 of [7], $\sigma(C_\phi)$ is countable. Hence by Proposition 1.4.5 of [9], C_ϕ is super-decomposable.

It can be easily seen that $f_0 = 1$, $f_n^{(k)}(0) = 0 \ \forall k,\ 0 \leq k \leq n-1$ and $f_n^{(n)}(0) \neq 0$. Hence $\{f_n : n \geq 0\}$ is linearly independent and from our hypothesis $\{f_n : n \geq 0\}$ is a Schauder basis of $A(\mathbb{D})$. From Proposition 1.3.2 of [9], we have

$$\sigma_{C_\phi}(f_n) = \{\phi'(0)^n\}, \text{ for each } n \geq 0.$$

Let $f \in A(\mathbb{D})$. Then there exists a sequence of scalars $(a_n)_{n=1}^{\infty}$ such that $f = \sum_{n=0}^{\infty} a_n f_n$. For each $n \geq 0$, put $g_n = \sum_{m=0}^{n} a_m f_m$ and $F_n = \{\phi'(0)^m : 0 \leq m \leq n \text{ and } a_m \neq 0\}$. We claim that $\sigma_{C_\phi}(g_n) = F_n$. The inclusion $\sigma_{C_\phi}(g_n) \subseteq F_n$ follows easily from the definition. For the reverse inclusion, suppose that for a non-negative integer k, $\phi'(0)^k \in F_n$ but $\phi'(0)^k \notin \sigma_{C_\phi}(g_n)$. Consider $F = F_n \backslash \{\phi'(0)^k\}$. Then $f_k \notin A_{C_\phi}(F)$. Since $\sigma_{C_\phi}(g_n) \subseteq F$, $\sigma_{C_\phi}(\sum_{m=0, m\neq k}^{n} a_m f_m) \subseteq F$ and $A_{C_\phi}(F)$ is a subspace of $A(\mathbb{D})$, therefore $g_n - \sum_{m=0, m\neq k}^{n} a_m f_m \in A_{C_\phi}(F)$. This implies that $f_k \in A_{C_\phi}(F)$, which is a contradiction. Thus $\sigma_{C_\phi}(g_n) = F_n$. It is easy to observe that

$$F_0 \subseteq F_1 ... \subseteq F_n \subseteq F_{n+1} ...$$

Hence, $\liminf F_n = \overline{\{\phi'(0)^n : n \geq 0 \text{ and } a_n \neq 0\}}$. Since $g_n \longrightarrow f$ and the map $g \longmapsto \sigma_{C_\phi}(g)$ is lower semicontinuous [3], therefore

$$\sigma_{C_\phi}(f) \subseteq \overline{\{\phi'(0)^n : n \geq 0 \text{ and } a_n \neq 0\}}.$$

To prove that above inclusion goes other way round, suppose that for some $k \geq 0$, $a_k \neq 0$ and $\phi'(0)^k \notin \sigma_{C_\phi}(f)$. Consider

$$F = \overline{\{\phi'(0)^n : n \geq 0,\ n \neq k \text{ and } a_n \neq 0\}}.$$

Then $f_k \notin A_{C_\phi}(F)$. Since $\sigma_{C_\phi}(f) \subseteq F$, $\sigma_{C_\phi}(\sum_{n=0, n\neq k}^{\infty} a_n f_n) \subseteq F$ and $A_{C_\phi}(F)$ is a subspace of $A(\mathbb{D})$, hence $f - \sum_{n=0, n\neq k}^{\infty} a_n f_n \in A_{C_\phi}(F)$. This implies that $f_k \in A_{C_\phi}(F)$, which is a contradiction. Thus

$$\{\phi'(0)^n : n \geq 0 \text{ and } a_n \neq 0\} \subseteq \sigma_{C_\phi}(f).$$

Since $\sigma_{C_\phi}(f)$ is compact, therefore

$$\overline{\{\phi'(0)^n : n \geq 0 \text{ and } a_n \neq 0\}} \subseteq \sigma_{C_\phi}(f).$$

Thus we get

$$\sigma_{C_\phi(f)} = \overline{\{\phi'(0)^n : n \geq 0 \text{ and } a_n \neq 0\}}. \qquad \blacksquare$$

Remark 2.21. If ϕ has a fixed point $z_0 \in \mathbb{D}$ other than 0, then above theorem is still true with $\phi'(0)$ replaced by $\phi'(z_0)$.

Combining Theorem 2.8 and Theorem 2.20, we have the following result.

Theorem 2.22. *Let $\phi \in A(\mathbb{D})$ with $\|\phi\| \leq 1$ and $\phi(z_0) = z_0$, where $|z_0| < 1$. Suppose that ϕ is not an automorphism of $\mathbb{D}$. Then C_ϕ is decomposable if and only if the fixed set of ϕ is $\{z_0\}$.*

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