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DERIVABLE MAPS AND GENERALIZED DERIVATIONS
ON NEST AND STANDARD ALGEBRAS*Communicated by E. Weber*

Abstract. For an algebra \mathcal{A} , an \mathcal{A} -bimodule \mathcal{M} , and $m \in \mathcal{M}$, define a relation on \mathcal{A} by $\mathcal{R}_{\mathcal{A}}(m, 0) = \{(a, b) \in \mathcal{A} \times \mathcal{A} : amb = 0\}$. We show that generalized derivations on unital standard algebras on Banach spaces can be characterized precisely as derivable maps on these relations. More precisely, if \mathcal{A} is a unital standard algebra on a Banach space X then $\Delta \in L(\mathcal{A}, B(X))$ is a generalized derivation if and only if Δ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$, for some $M \in B(X)$. We give an example to show this is not the case in general for nest algebras. On the other hand, for an idempotent P in a nest algebra $\mathcal{A} = \text{alg}\mathcal{N}$ on a Hilbert space H such that P is either left-faithful to \mathcal{N} or right-faithful to \mathcal{N}^\perp , if $\delta \in L(\mathcal{A}, B(H))$ is derivable on $\mathcal{R}_{\mathcal{A}}(P, 0)$ then δ is a generalized derivation.

1. Introduction

For vector spaces \mathcal{U} and \mathcal{V} , we use $L(\mathcal{U}, \mathcal{V})$ to denote the set of all linear maps from \mathcal{U} to \mathcal{V} . For a unital algebra \mathcal{A} and an \mathcal{A} -bimodule \mathcal{M} , $\delta \in L(\mathcal{A}, \mathcal{M})$ is called a *derivation* if for all $a, b \in \mathcal{A}$, $\delta(ab) = \delta(a)b + a\delta(b)$ and δ is called a *generalized derivation* if for all $a, b \in \mathcal{A}$, $\delta(ab) = \delta(a)b + a\delta(b) - a\delta(1)b$. Fix any $u, v \in \mathcal{M}$, then $\delta_{uv}(a) = ua - av$, $\forall a \in \mathcal{A}$ is a generalized derivation; the study of such maps dates back at least to [13]. By a relation on \mathcal{A} , we mean a nonempty subset $\mathcal{R}_{\mathcal{A}} \subseteq \mathcal{A} \times \mathcal{A}$. We say $\delta \in L(\mathcal{A}, \mathcal{M})$ is *derivable on* $\mathcal{R}_{\mathcal{A}}$ if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $(a, b) \in \mathcal{R}_{\mathcal{A}}$. Derivable maps have garnered interests of many researchers, for example, authors of [2], [4], [8], and [10] have studied maps that are derivable on $\mathcal{R}_{\mathcal{A}} = \{(a, b) \in \mathcal{A} \times \mathcal{A} : a = b\}$, such maps are called *Jordan derivations*. In [3], [6–7], [9], [11], and [14–16], the authors have studied derivable maps on relations $\mathcal{R}_{\mathcal{A}}(c) = \{(a, b) \in \mathcal{A} \times \mathcal{A} : ab = c\}$, for some $c \in \mathcal{A}$. Not all derivable maps are derivations. If a derivable map is not a derivation, it is natural to ask whether it is close to being a derivation, e.g. whether it is a generalized derivation. Every generalized derivation is

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inherently derivable on a relation $\mathcal{R}_{\mathcal{A}}(m, 0) = \{(a, b) \in \mathcal{A} \times \mathcal{A} : amb = 0\}$, for some $m \in \mathcal{M}$, which leads to the study of maps preserving the derivation structure on these relations, i.e. derivable maps on $\mathcal{R}_{\mathcal{A}}(m, 0)$, see [12] for more details. In this regard, it is more natural to consider maps that are derivable on relations $\mathcal{R}_{\mathcal{A}}(m, 0)$. It should be noted that many of the techniques used to study derivable maps on $\mathcal{R}_{\mathcal{A}}(c)$ cannot be adapted to $\mathcal{R}_{\mathcal{A}}(m, 0)$. For example, consider the following technique for derivable maps on $\mathcal{R}_{\mathcal{A}}(c)$: If $(a, b) \in \mathcal{R}_{\mathcal{A}}(c)$, then both (t^{-1}, tc) , $(at^{-1}, tb) \in \mathcal{R}_{\mathcal{A}}(c)$, for any invertible $t \in \mathcal{A}$. From this, one can try to construct t based on the structure of \mathcal{A} to achieve desired properties. This would not work on $\mathcal{R}_{\mathcal{A}}(m, 0)$, because m blocks a and b from directly acting with each other with the multiplication operation of \mathcal{A} , e.g. $(a, b) \in \mathcal{R}_{\mathcal{A}}(m, 0)$ does not imply $(at^{-1}, tb) \in \mathcal{R}_{\mathcal{A}}(m, 0)$.

If \mathcal{A} is an algebra with unit 1 and \mathcal{M} is a left \mathcal{A} -module, \mathcal{M} is called a *unital left \mathcal{A} -module* if $1m = m$, $\forall m \in \mathcal{M}$. In this case a map $\psi \in L(\mathcal{A}, \mathcal{M})$ is called *right-annihilator-preserving* if $a\psi(b) = 0$, $\forall a, b \in \mathcal{A}$ with $ab = 0$ and ψ is called a *right multiplier* if $\psi(a) = a\psi(1)$, $\forall a \in \mathcal{A}$. Similarly, if \mathcal{M} is a right \mathcal{A} -module, \mathcal{M} is called a *unital right \mathcal{A} -module* if $m1 = m$, $\forall m \in \mathcal{M}$. In this case a map $\psi \in L(\mathcal{A}, \mathcal{M})$ is called *left-annihilator-preserving* if $\psi(a)b = 0$, $\forall a, b \in \mathcal{A}$ with $ab = 0$ and ψ is called a *left multiplier* if $\psi(a) = \psi(1)a$, $\forall a \in \mathcal{A}$. Clearly, left multipliers are left-annihilator-preserving and right multipliers are right-annihilator-preserving.

Let X be a separable complex Banach space and let $B(X)$ be the set of all bounded linear operators on X . By a *subspace lattice* on X , we mean a collection \mathcal{L} of subspaces of X with 0 and X in \mathcal{L} such that for every family $\{M_t\}$ of elements of \mathcal{L} , both $\cap M_t$ and $\vee M_t$ belong to \mathcal{L} . For a subspace lattice \mathcal{L} of X , we use $\text{alg}\mathcal{L}$ to denote the algebra of all operators in $B(X)$ that leave members of \mathcal{L} invariant; and for a subalgebra \mathcal{A} of $B(X)$, we use $\text{lat}\mathcal{A}$ to denote the lattice of all subspaces of X that are invariant under all operators in \mathcal{A} . A lattice \mathcal{L} is called *reflexive* if $\mathcal{L} = \text{lat}(\text{alg}\mathcal{L})$. A totally ordered subspace lattice \mathcal{N} is called a *nest* and the corresponding algebra $\text{alg}\mathcal{N}$ is called a *nest algebra*. It is well known that nests are reflexive, see [1] for more on nest algebras. For any $x \in X$ and $f \in X^*$, the rank-one operator $x \otimes f$ is defined by $x \otimes f y = f(y)x$, $\forall y \in X$. When X is a Hilbert space we change it to H . For a Hilbert space orthogonal projection, we use the same letter to denote the projection and its range space.

While we use lower case letters for elements in algebras and modules in abstract settings, we will use capital letters for operators in operator algebras on Banach and Hilbert spaces. For a nest algebra $\mathcal{A} = \text{alg}\mathcal{N}$ on a Hilbert space H and $T \in B(H)$, we say T is *left-faithful* to \mathcal{N} if $\forall N \in \mathcal{N}, TN = 0$ iff $N = 0$ and T is *right-faithful* to \mathcal{N}^\perp if $\forall N \in \mathcal{N}, N^\perp T = 0$ iff $N^\perp = 0$. In Section 2, we show that if $P \in \mathcal{A}$ is an idempotent, then every left-

annihilator-preserving map from PAP to $B(H)P$ is a left multiplier and every right-annihilator-preserving map from PAP to $PB(H)$ is a right multiplier. Using this we show that if P is either left-faithful to \mathcal{N} or right-faithful to \mathcal{N}^\perp then every derivable map on $\mathcal{R}_{\mathcal{A}}(P, 0)$ is a generalized derivation.

In Section 3, we show that if \mathcal{A} is a unital standard algebra on a Banach space X , then $\Delta \in L(\mathcal{A}, B(X))$ is a generalized derivation iff Δ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$, for some $M \in B(X)$. In particular, the above conclusion holds when $\mathcal{A} = B(X)$, which generalizes the main result of [12].

2. Derivable maps and generalized derivations on nest algebras

The following is the main result of this section.

THEOREM 2.1. *Let $\mathcal{A} = \text{alg}\mathcal{N}$ be a nest algebra on a Hilbert space H and $P \in \mathcal{A}$ be an idempotent. If P is either left-faithful to \mathcal{N} or right-faithful to \mathcal{N}^\perp , and $\delta \in L(\mathcal{A}, B(H))$ is derivable on $\mathcal{R}_{\mathcal{A}}(P, 0)$ then δ is a generalized derivation; in this case $\delta(I) \in \mathbb{C}P$.*

We will proceed first with some lemmas, starting with the ones that hold in abstract settings of modules over algebras, which should be of more general interest. Note that these can be easily adapted to the context of modules over rings and additive maps, with essentially the same proofs.

Let \mathcal{A} be a unital algebra. If \mathcal{M} is a left \mathcal{A} -module and $\psi \in L(\mathcal{A}, \mathcal{M})$, define $\mathcal{R}_\psi = \{a \in \mathcal{A} : \psi(a) = a\psi(1)\}$. Clearly ψ is a right multiplier iff $\mathcal{R}_\psi = \mathcal{A}$.

Similarly, if \mathcal{M} is a right \mathcal{A} -module and $\psi \in L(\mathcal{A}, \mathcal{M})$, define $\mathcal{L}_\psi = \{a \in \mathcal{A} : \psi(a) = \psi(1)a\}$. Then ψ is a left multiplier iff $\mathcal{L}_\psi = \mathcal{A}$.

LEMMA 2.2. *Let \mathcal{A} be a unital algebra.*

- (i) *Let \mathcal{M} be a unital left \mathcal{A} -module and $\psi \in L(\mathcal{A}, \mathcal{M})$. If ψ is right-annihilator-preserving then \mathcal{R}_ψ contains the subalgebra of \mathcal{A} generated by the idempotents of \mathcal{A} .*
- (ii) *Let \mathcal{M} be a unital right \mathcal{A} -module and $\psi \in L(\mathcal{A}, \mathcal{M})$. If ψ is left-annihilator-preserving then \mathcal{L}_ψ contains the subalgebra of \mathcal{A} generated by the idempotents of \mathcal{A} .*

Proof. To prove (i), for any idempotent $u \in \mathcal{A}$, $(1-u)u = u(1-u) = 0$. Since ψ is right-annihilator-preserving, we have $(1-u)\psi(u) = 0$ and $u\psi(1-u) = 0$. It follows that $\psi(u) = u\psi(1)$, i.e. $u \in \mathcal{R}_\psi$.

For any $b \in \mathcal{A}$, define $\psi_b \in L(\mathcal{A}, \mathcal{M})$ by $\psi_b(a) = \psi(ab)$, $\forall a \in \mathcal{A}$. If $a, c \in \mathcal{A}$ satisfy $ac = 0$ then $acb = 0$. Since ψ is right-annihilator-preserving, $a\psi(cb) = 0$. Thus $a\psi_b(c) = 0$, i.e. ψ_b is right-annihilator-preserving also. For any idempotents $a, b \in \mathcal{A}$, applying the previous paragraph to ψ and ψ_b

yields $\psi(ab) = \psi_b(a) = a\psi_b(1) = a\psi(b) = ab\psi(1)$. Thus $ab \in \mathcal{R}_\psi$. Now (i) follows from induction and linearity of ψ .

The proof of (ii) is similar. ■

Suppose \mathcal{A} is an algebra and \mathcal{A}_1 is a subset of \mathcal{A} . If \mathcal{M} is a left \mathcal{A} -module and \mathcal{M}_1 is a subset of \mathcal{M} , we say \mathcal{A}_1 is *left-faithful* to \mathcal{M}_1 if for any $m \in \mathcal{M}_1$, the condition $\mathcal{A}_1 m = \{0\}$ implies $m = 0$. If \mathcal{M} is a right \mathcal{A} -module and \mathcal{M}_1 is a subset of \mathcal{M} , we say \mathcal{A}_1 is *right-faithful* to \mathcal{M}_1 if the condition $m\mathcal{A}_1 = \{0\}$ implies $m = 0$.

LEMMA 2.3. *Let \mathcal{A} be a unital algebra.*

- (i) *If \mathcal{M} is a unital left \mathcal{A} -module and \mathcal{A} has a right ideal \mathcal{I} generated as an algebra by idempotents of \mathcal{A} such that \mathcal{I} is left-faithful to \mathcal{M} , then $\forall \psi \in L(\mathcal{A}, \mathcal{M})$, ψ is a right multiplier iff ψ is right-annihilator-preserving.*
- (ii) *If \mathcal{M} is a unital right \mathcal{A} -module and \mathcal{A} has a left ideal \mathcal{I} generated as an algebra by idempotents of \mathcal{A} such that \mathcal{I} is right-faithful to \mathcal{M} , then $\forall \psi \in L(\mathcal{A}, \mathcal{M})$, ψ is a left multiplier iff ψ is left-annihilator-preserving.*

Proof. For (i), suppose ψ is right-annihilator-preserving. For any $e \in \mathcal{I}$ and $b \in \mathcal{A}$, $eb \in \mathcal{I}$. Define $\psi_b(a) = \psi(ab)$, $\forall a \in \mathcal{A}$, then ψ_b is also right-annihilator-preserving. Since \mathcal{I} is generated by idempotents of \mathcal{A} , by Lemma 2.2 we have $eb\psi(1) = \psi(eb) = \psi_b(e) = e\psi_b(1) = e\psi(b)$. Thus $e[b\psi(1) - \psi(b)] = 0$, and $b\psi(1) - \psi(b) = 0$, since \mathcal{I} is left-faithful to \mathcal{M} .

The other direction of (i) is clear and the proof of (ii) is similar. ■

For a unital algebra \mathcal{A} , a unital \mathcal{A} -bimodule \mathcal{M} , and an idempotent $p \in \mathcal{A}$, we define \mathcal{A}_{ij} and \mathcal{M}_{ij} , the Peirce decompositions of \mathcal{A} and \mathcal{M} with respect to p as follows: Let $p_1 = p$, $p_2 = 1 - p_1$, $\mathcal{A}_{ij} = p_i \mathcal{A} p_j$, and $\mathcal{M}_{ij} = p_i \mathcal{M} p_j$, $i, j = 1, 2$.

LEMMA 2.4. *Let \mathcal{A} be a unital algebra, \mathcal{M} be a unital \mathcal{A} -bimodule, $p \in \mathcal{A}$ be an idempotent, \mathcal{A}_{ij} and \mathcal{M}_{ij} be the Peirce decompositions of \mathcal{A} and \mathcal{M} with respect to p . If $d \in L(\mathcal{A}, \mathcal{M})$ is derivable on $\mathcal{R}_{\mathcal{A}}(m, 0)$ for some $m \in \mathcal{M}_{11}$, then there exists a $\delta \in L(\mathcal{A}, \mathcal{M})$ such that $d - \delta$ is an inner derivation, $\delta(1 - p) = 0$, and $\delta(\mathcal{A}_{ij}) \subseteq \mathcal{M}_{ij}$.*

Proof. Let $q = 1 - p$, $u = pd(q)q - qd(q)p$ and $\delta_u(a) = ua - au, \forall a \in \mathcal{A}$. Let $\delta = d - \delta_u$. It follows that δ is derivable on $\mathcal{R}_{\mathcal{A}}(m, 0)$ and one can verify $p\delta(q)q = q\delta(q)p = 0$. Therefore,

$$\delta(q) = p\delta(q)p + p\delta(q)q + q\delta(q)p + q\delta(q)q = p\delta(q)p + q\delta(q)q.$$

Since $m \in \mathcal{M}_{11}$, we see $aqmb = 0$ and $amqb = 0$, for all $a, b \in \mathcal{A}$. Since δ is

derivable on $\mathcal{R}_{\mathcal{A}}(m, 0)$,

$$(2.1) \quad \delta(aqb) = \delta(aq)b + aq\delta(b)$$

and

$$(2.2) \quad \delta(aqb) = \delta(a)qb + a\delta(qb).$$

Setting $a = q$ and $b = p$ in Eq. 2.1, we get $0 = \delta(q)p + q\delta(p)$. It follows $p\delta(q)p = 0$. Setting $a = b = q$ in Eq. 2.2, we get $\delta(q) = \delta(q)q + q\delta(q)$. It follows $q\delta(q)q = 0$. Thus $\delta(q) = 0$.

For any $a_{11} \in \mathcal{A}_{11}$, setting $a = q$ and $b = a_{11}$ in Eq. 2.1, we get $0 = q\delta(a_{11})$; setting $a = a_{11}$ and $b = q$ in Eq. 2.2, we get $0 = \delta(a_{11})q$. Thus $\delta(a_{11}) \in \mathcal{M}_{11}$.

For any $a_{12} \in \mathcal{A}_{12}$, setting $a = q$ and $b = a_{12}$ in Eq. 2.1, we get $0 = q\delta(a_{12})$; setting $a = a_{12}$ and $b = p$ in Eq. 2.1, we get $0 = \delta(a_{12})p$. Thus $\delta(a_{12}) \in \mathcal{M}_{12}$.

For any $a_{21} \in \mathcal{A}_{21}$, setting $a = a_{21}$ and $b = q$ in Eq. 2.2, we get $0 = \delta(a_{21})q$; setting $a = p$ and $b = a_{21}$ in Eq. 2.2, we get $0 = p\delta(a_{21})$. Thus $\delta(a_{21}) \in \mathcal{M}_{21}$.

For any $a_{22} \in \mathcal{A}_{22}$, setting $a = p$ and $b = a_{22}$ in Eq. 2.2, we get $0 = p\delta(a_{22})$; setting $a = a_{22}$ and $b = p$ in Eq. 2.1, we get $0 = \delta(a_{22})p$. Thus $\delta(a_{22}) \in \mathcal{M}_{22}$. ■

If \mathcal{A} is a subalgebra of a unital algebra \mathcal{M} such that $1 \in \mathcal{A}$ then \mathcal{M} is a unital \mathcal{A} -bimodule with the inherited algebraic operations. Moreover, if $p \in \mathcal{A}$ is an idempotent, then $p\mathcal{A}p$ is a unital algebra with unit p , $p\mathcal{M}$ is a unital left $p\mathcal{A}p$ -module, and $\mathcal{M}p$ is a unital right $p\mathcal{A}p$ -module. If \mathcal{M} is an algebra and $\mathcal{S} \subseteq \mathcal{M}$, we define the commutant of \mathcal{S} by $\mathcal{S}' = \{t \in \mathcal{M} : st = ts, \forall s \in \mathcal{S}\}$.

LEMMA 2.5. *Let \mathcal{A} be a subalgebra of a unital algebra \mathcal{M} such that $1 \in \mathcal{A}$ and $p \in \mathcal{A}$ be an idempotent. Suppose that the only right-annihilator-preserving maps from $p\mathcal{A}p$ to $p\mathcal{M}$ are the right multipliers and the only left-annihilator-preserving maps from $p\mathcal{A}p$ to $\mathcal{M}p$ are the left multipliers. If $\delta \in L(\mathcal{A}, \mathcal{M})$ is derivable on $\mathcal{R}_{\mathcal{A}}(p, 0)$ then for any $x \in p\mathcal{A}p$ and $y \in \mathcal{A}$*

- (i) $p\delta(xy) = p\delta(x)y + x\delta(py) - x\delta(p)y$.
- (ii) $\delta(yx)p = y\delta(x)p + \delta(yx)p - y\delta(p)x$.

Moreover, $p\delta(p)p \in \{p\mathcal{A}p\}'$.

Proof. For any $a, b \in p\mathcal{A}p$ such that $ab = 0$ then $apb = ab = 0$ and $apby = 0$, $\forall y \in \mathcal{A}$. Since δ is derivable on $\mathcal{R}_{\mathcal{A}}(p, 0)$, we have

$$\delta(a)b + a\delta(b) = \delta(ab) = 0$$

and

$$\delta(a)by + a\delta(by) = \delta(aby) = 0.$$

The above two equations yield

$$(2.3) \quad a\delta(by) - a\delta(b)y = 0.$$

For any $x \in p\mathcal{A}p$, define $\psi(x) = p\delta(xy) - p\delta(x)y$. By Eq. 2.3, ψ is a right-annihilator-preserving map from $p\mathcal{A}p$ to $p\mathcal{M}$, thus ψ is a right multiplier, so $\psi(x) = x\psi(p)$, i.e. $p\delta(xy) - p\delta(x)y = x\delta(py) - x\delta(p)y$ and Part (i) follows.

Similarly, for any $a, b \in p\mathcal{A}p$ such that $ab = 0$ then $yapb = yab = 0$, $\forall y \in \mathcal{A}$. Since δ is derivable on $\mathcal{R}_{\mathcal{A}}(p, 0)$, we have

$$\delta(ya)b + ya\delta(b) = \delta(yab) = 0.$$

Combining with $\delta(a)b + a\delta(b) = 0$ we get

$$(2.4) \quad \delta(ya)b - y\delta(a)b = 0.$$

For any $x \in p\mathcal{A}p$, define $\phi(x) = \delta(yx)p - y\delta(x)p$. By Eq. 2.4, ϕ is a left-annihilator-preserving map from $p\mathcal{A}p$ to $\mathcal{M}p$, thus ϕ is a left multiplier, so $\phi(x) = \phi(p)x$, i.e. $\delta(yx)p - y\delta(x)p = \delta(y)p x - y\delta(p)x$ and Part (ii) follows.

For any $x, y \in p\mathcal{A}p$ such that $xy = 0 = xpy$, since δ is derivable on $\mathcal{R}_{\mathcal{A}}(p, 0)$,

$$0 = \delta(xy) = \delta(x)y + x\delta(y).$$

On the other hand, by Part (i),

$$p\delta(xy) = p\delta(x)y + x\delta(py) - x\delta(p)y = p\delta(x)y + px\delta(y) - x\delta(p)y.$$

Thus $x\delta(p)y = 0$, which implies $\Phi(t) = p\delta(p)t$, $\forall t \in p\mathcal{A}p$ is a right-annihilator-preserving map from $p\mathcal{A}p$ to $p\mathcal{M}$, so Φ is a right multiplier and $p\delta(p)pt = p\delta(p)t = \Phi(t) = t\Phi(p) = tp\delta(p)p$, i.e. $p\delta(p)p \in \{p\mathcal{A}p\}'$. ■

LEMMA 2.6. *Let \mathcal{A} be a subalgebra of $B(H)$ containing rank-one operators $x \otimes f$, $y \otimes f$, $x \otimes g$, and $y \otimes g$. If $g(y) \neq 0$ then $x \otimes f$ can be written as a linear combination of idempotents in \mathcal{A} .*

Proof. Note that if a rank-one operator $z \otimes h \in \mathcal{A}$ satisfies $h(z) \neq 0$ then $z \otimes h$ is a scalar multiple of an idempotent, so we only need to show $x \otimes f$ can be written as a linear combination of such operators. Without loss of generality, suppose $f(x) = 0$.

If $f(y) \neq 0$, write $x \otimes f = \frac{1}{2}[(x+y) \otimes f + (x-y) \otimes f]$.

If $g(x) \neq 0$, write $x \otimes f = \frac{1}{2}[x \otimes (f+g) + x \otimes (f-g)]$.

If $f(y) = g(x) = 0$, write

$$x \otimes f = \frac{1}{4}[(x+y) \otimes (f+g) + (x+y) \otimes (f-g) + (x-y) \otimes (f+g) + (x-y) \otimes (f-g)].$$

■

LEMMA 2.7. *If $\mathcal{A} = \text{alg}\mathcal{N}$ is a nest algebra on a Hilbert space H and $P \in \mathcal{A}$ is an idempotent, then PAP has ideals \mathcal{I} and \mathcal{J} , both generated as algebras by idempotents in PAP , such that \mathcal{I} is right-faithful to $B(H)P$ and \mathcal{J} is left-faithful to $PB(H)$.*

Proof. For any $N \in \mathcal{N}$, clearly PNP and $P(N - NTN^\perp)P$ are idempotents in PAP , thus $PNTN^\perp P$ is a linear combination of idempotents in PAP .

Let E be the orthogonal projection of H on to $\vee\{NH : N \in \mathcal{N}, NH \not\supseteq PH\}$. Then $E \in \text{lat}\mathcal{A} = \mathcal{N}$, in particular, $E \in \mathcal{A}$. We will construct \mathcal{I} in two separate cases: $EH \supseteq PH$ and $EH \not\supseteq PH$

If $EH \supseteq PH$, take $\mathcal{I} = \text{span}\{PNTN^\perp P : N \in \mathcal{N}, NH \not\supseteq PH, T \in B(H)\}$. The previous paragraph shows \mathcal{I} is generated by idempotents in PAP . Clearly \mathcal{I} is an ideal of PAP . Take any $S \in B(H)P$ such that $S\mathcal{I} = \{0\}$. If $N \in \mathcal{N}$ and $NH \not\supseteq PH$, then $N^\perp P \neq 0$. Thus if $SPNTN^\perp P = 0$, $\forall T \in B(H)$ then $SPN = 0$, so $SPE = 0$. Since $EH \supseteq PH$, $SP = 0$, i.e. \mathcal{I} is right-faithful to $B(H)P$.

If $EH \not\supseteq PH$, take $\mathcal{I} = \text{span}\{x \otimes fE^\perp P : \forall x \in PH, f \in H^*\}$. For any $x \in PH, f \in H^*$, one can easily check that $x \otimes fE^\perp P \in PAP$, \mathcal{I} is an ideal of PAP , and \mathcal{I} is right-faithful to $B(H)P$. Since $EH \not\supseteq PH$, $E^\perp P \neq 0$. Thus there exist $y \in PH$ and $g \in H^*$ such that $g(E^\perp Py) = 1$. Applying Lemma 2.6 to $x \otimes fE^\perp P$, $y \otimes fE^\perp P$, $x \otimes gE^\perp P$, and $y \otimes gE^\perp P$, we see $x \otimes fE^\perp P$ can be written as a linear combination of idempotents in PAP .

To construct our \mathcal{J} , note that \mathcal{A}^* is also a nest algebra and P^* is an idempotent in \mathcal{A}^* . From the previous paragraphs, $P^*\mathcal{A}^*P^*$ has an ideal \mathcal{J}^* generated by idempotents of $P^*\mathcal{A}^*P^*$ and \mathcal{J}^* is right-faithful to $B(H)P^*$. Let \mathcal{J} be the adjoint of \mathcal{J}^* , it follows that \mathcal{J} is an ideal of PAP generated by idempotents of PAP , and \mathcal{J} is left-faithful to $PB(H)$. ■

The following theorem follows immediately from Lemmas 2.3 and 2.7.

THEOREM 2.8. *If \mathcal{A} is a nest algebra on a Hilbert space H and $P \in \mathcal{A}$ is an idempotent, then any $\psi \in L(PAP, PB(H))$ is a right multiplier iff it is right-annihilator-preserving and any $\psi \in L(PAP, B(H)P)$ is a left multiplier iff it is left-annihilator-preserving.*

For any $N \in \mathcal{N}$, define $N_- = \vee\{FH : F \in \mathcal{N}, F \subsetneq N\}$. It is well known that for a nest algebra $\text{alg}\mathcal{N}$, a rank-one operator $x \otimes f \in \text{alg}\mathcal{N}$ iff there exists an $N \in \mathcal{N}$ such that $x \in N$ and $f \in (N_-)^\perp$. For vector spaces \mathcal{U} and \mathcal{V} , take any $S \in L(\mathcal{U}, \mathcal{V})$, the one-dimensional subspace $\mathbb{C}S \subseteq L(\mathcal{U}, \mathcal{V})$ is *algebraically reflexive* in the sense that if $T \in L(\mathcal{U}, \mathcal{V})$ such that $Tx \in \mathbb{C}Sx$, $\forall x \in \mathcal{U}$ then $T \in \mathbb{C}S$; a more general case for n -dimensional subspaces of $L(\mathcal{U}, \mathcal{V})$ can be found in [5]. This will be used in the proofs of Lemmas 2.9 and 3.2.

LEMMA 2.9. *Let $\mathcal{A} = \text{alg}\mathcal{N}$ be a nest algebra on a Hilbert space H and $\mathcal{M} = B(H)$. If $P \in \mathcal{A}$ is an idempotent with \mathcal{A}_{ij} and \mathcal{M}_{ij} as the Peirce decompositions of \mathcal{A} and \mathcal{M} with respect to P , respectively, then $\{\mathcal{A}_{11}\}' = \mathbb{C}P + \mathcal{M}_{22}$.*

Proof. Let E be the orthogonal projection from H onto $\cap\{NH : N \in \mathcal{N}, NH \supseteq PH\}$. Then clearly $EH \supseteq PH$, and $E \in \text{lat}\mathcal{A} = \mathcal{N}$. Taking any $T \in \{\mathcal{A}_{11}\}'$, since $TP = PT$, we see $(I - P)TP = 0 = PT(I - P)$, thus $T \in \mathcal{M}_{11} + \mathcal{M}_{22}$.

If $E_- \neq E$, then $E_-H \not\supseteq PH$. For any $x \in E$, we can choose $f \in (E_-)^\perp$ such that $x \otimes fP \neq 0$. Since $TPx \otimes fP = Px \otimes fPT$, $TPx \in \mathbb{C}Px$. Since $EH \supseteq PH$, it follows that for any $y \in H$, $TPy = TPPy \in \mathbb{C}PPy = \mathbb{C}Py$, thus $TP \in \mathbb{C}P$.

If $E_- = E$, take any $F \in \mathcal{N}$ such that $F \subsetneq E$, then $F_-H \not\supseteq PH$. For any $x \in F$, take $f \in (F_-)^\perp$ such that $x \otimes fP \neq 0$. Since $TPx \otimes fP = Px \otimes fPT$, $TPx \in \mathbb{C}Px$. It follows $TPF \in \mathbb{C}PF$. Note that $E_- = \vee\{FH : F \in \mathcal{N}, F \subsetneq E\}$ and $E_- = E$, it follows $TPE \in \mathbb{C}PE$, thus $TP \in \mathbb{C}P$. ■

LEMMA 2.10. *Let \mathcal{L} be a reflexive lattice on a Hilbert space H , $\mathcal{A} = \text{alg}\mathcal{L}$, $P \in \mathcal{A}$ be an idempotent, $Q = I - P$, $\mathcal{M} = B(H)$, and \mathcal{A}_{ij} be the corresponding Peirce decomposition of \mathcal{A} with respect to P . Then*

- 1) \mathcal{A}_{12} is left-faithful to QM iff P is left-faithful to \mathcal{L} .
- 2) \mathcal{A}_{21} is right-faithful to MQ iff P is right-faithful to \mathcal{L}^\perp .

Proof. To see 1): For any $T \in QM$ such that $PAQT = 0$, $\forall A \in \mathcal{A}$, let $N = \vee\{AQTH : A \in \mathcal{A}\}$, then $PN = 0$. Since \mathcal{L} is reflexive, $N \in \text{lat}\mathcal{A} = \mathcal{L}$. If P is left-faithful to \mathcal{L} , then $N = 0$, $T = 0$.

For any $N \in \mathcal{L}$ such that $PN = 0$, then $PAQN = PNAQN = 0$, $\forall A \in \mathcal{A}$. If \mathcal{A}_{12} is left-faithful to QM , then $QN = 0$. Thus $N = QN = 0$.

To see 2): Take any $T \in MQ$. Note that $\vee\{A^*Q^*T^*H : A \in \mathcal{A}\}$ is an invariant subspace of \mathcal{A}^* , it follows that $\vee\{A^*Q^*T^*H : A \in \mathcal{A}\} = N^\perp$ for some $N \in \mathcal{L}$. If $TQAP = 0$, for all $A \in \mathcal{A}$, then $P^*A^*Q^*T^* = 0$, thus $P^*N^\perp = 0$ and $N^\perp P = 0$. If P is right-faithful to \mathcal{L}^\perp , then $N^\perp = 0$, thus $Q^*T^* = 0$. Therefore $T = TQ = 0$.

For any $N \in \mathcal{L}$ such that $N^\perp P = 0$, then $N^\perp QAP = N^\perp QAN^\perp P = 0$ for all $A \in \mathcal{A}$. If \mathcal{A}_{21} is right-faithful to MQ , then $N^\perp Q = 0$, thus $N^\perp = 0$. ■

COROLLARY 2.11. *Let $\mathcal{A} = \text{alg}\mathcal{N}$ be a nest algebra on a Hilbert space H , $P \in \mathcal{A}$ be an idempotent, $Q = I - P$, $\mathcal{M} = B(H)$, and \mathcal{A}_{ij} be the Peirce decomposition of \mathcal{A} with respect to P . Then*

- 1) If $PH \supseteq N_0H$ for some $0 \neq N_0 \in \mathcal{N}$ then \mathcal{A}_{12} is left-faithful to QM .
- 2) If $PH \supseteq N_0^\perp H$ for some $I \neq N_0 \in \mathcal{N}$ then \mathcal{A}_{21} is right-faithful to MQ .

Proof. To see 1), for any $N \in \mathcal{N}$, if $PN = 0$ then $N_0N = PN_0N = PNN_0 = 0$, thus $N = 0$. The proof of 2) is similar. ■

Proof of Theorem 2.1. Let $\mathcal{M} = B(H)$. If $P = 0$, $\mathcal{R}_{\mathcal{A}}(0, 0) = \mathcal{A} \times \mathcal{A}$, the conclusion is clear. The case for $P = I$ follows from Lemma 2.5 and Theorem 2.8. Suppose $P \neq 0, I$, let \mathcal{A}_{ij} and \mathcal{M}_{ij} , $i, j \in \{1, 2\}$ be the Peirce decompositions of \mathcal{A} and \mathcal{M} with respect to P . Set $Q = I - P$, subtracting an inner derivation from δ if necessary, by Lemma 2.4, we can assume $\delta(Q) = 0$ and $\delta(\mathcal{A}_{ij}) \subseteq \mathcal{M}_{ij}$.

We will show that for any $A_{ij} \in \mathcal{A}_{ij}$ and $B_{kl} \in \mathcal{A}_{kl}$, $i, j, k, l \in \{1, 2\}$,

$$(*) \quad \delta(A_{ij}B_{kl}) = \delta(A_{ij})B_{kl} + A_{ij}\delta(B_{kl}) - A_{ij}\delta(P)B_{kl}.$$

By Eqs. (2.1) and (2.2), (*) holds if $j = 2$ or $k = 2$, so we assume $j = k = 1$. By Lemma 2.5, (*) holds if $i = j = 1$ or $k = l = 1$. It remains to show $\delta(A_{21}B_{12}) = \delta(A_{21})B_{12} + A_{21}\delta(B_{12}) - A_{21}\delta(P)B_{12}$.

For any $T_{12} \in \mathcal{A}_{12}$ and $A_{21} \in \mathcal{A}_{21}$,

$$\begin{aligned} \delta(T_{12}A_{21}B) &= \delta(T_{12}A_{21})B + T_{12}A_{21}\delta(B) - T_{12}A_{21}\delta(P)B \\ &= \delta(T_{12})A_{21}B + T_{12}\delta(A_{21})B + T_{12}A_{21}\delta(B) - T_{12}A_{21}\delta(P)B. \end{aligned}$$

On the other hand,

$$\delta(T_{12}A_{21}B) = \delta(T_{12})A_{21}B + T_{12}\delta(A_{21}B).$$

Combining the above two equations, we get

$$T_{12}\delta(A_{21}B) = T_{12}\delta(A_{21})B + T_{12}A_{21}\delta(B) - T_{12}A_{21}\delta(P)B.$$

If P is left-faithful to \mathcal{N} , then \mathcal{A}_{12} is left-faithful to $Q\mathcal{M}$ by Lemma 2.10, thus

$$\delta(A_{21}B) = \delta(A_{21})B + A_{21}\delta(B) - A_{21}\delta(P)B,$$

which implies (*).

For any $T_{21} \in \mathcal{A}_{21}$ and $A_{12} \in \mathcal{A}_{12}$,

$$\begin{aligned} \delta(BA_{12}T_{21}) &= \delta(B)A_{12}T_{21} + B\delta(A_{12}T_{21}) - B\delta(P)A_{12}T_{21} \\ &= \delta(B)A_{12}T_{21} + B\delta(A_{12})T_{21} + BA_{12}\delta(T_{21}) - B\delta(P)A_{12}T_{21}. \end{aligned}$$

On the other hand,

$$\delta(BA_{12}T_{21}) = \delta(BA_{12})T_{21} + BA_{12}\delta(T_{21}).$$

Combining the above two equations, we get

$$\delta(BA_{12})T_{21} = \delta(B)A_{12}T_{21} + B\delta(A_{12})T_{21} - B\delta(P)A_{12}T_{21}.$$

If P is right-faithful to \mathcal{N}^\perp , then \mathcal{A}_{21} is right-faithful to $\mathcal{M}Q$ by Lemma 2.10, thus

$$\delta(BA_{12}) = \delta(B)A_{12} + B\delta(A_{12}) - B\delta(P)A_{12},$$

which again implies (*).

Since $\delta(I) = \delta(P)$, $(*)$ implies δ is a generalized derivation.

By Theorem 2.8, Lemma 2.5, and Lemma 2.9, $\delta(P) = P\delta(P)P \in \{\mathcal{A}_{11}\}' = \mathbb{C}P + \mathcal{M}_{22}$. Thus $\delta(I) = \delta(P) \in \mathbb{C}P$. ■

Our next example shows that the assumption “ P is either left-faithful to \mathcal{N} or right-faithful to \mathcal{N}^\perp ” cannot be dropped in Theorem 2.1.

EXAMPLE. Let $H = \mathbb{C}^3$ with orthonormal basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. Take $N_1 = \mathbb{C}e_1$ and $N_2 = \mathbb{C}e_1 + \mathbb{C}e_2$, then $\mathcal{N} = \{0, N_1, N_2, H\}$ forms a nest of H . Let $\mathcal{A} = \text{alg}\mathcal{N}$, then \mathcal{A} can be identified with \mathcal{T}_3 , the set of all 3×3 upper triangular matrices with respect to the orthonormal basis. Let E_{ij} be the matrix units and $P = E_{22}$, then $P \in \mathcal{A}$ is an idempotent. In this case P is neither left-faithful to \mathcal{N} nor right-faithful to \mathcal{N}^\perp . For any $A = (a_{ij}) \in \text{alg}\mathcal{N} = \mathcal{T}_3$, define $\delta \in L(\mathcal{A}, \mathcal{A})$ by $\delta(A) = E_{11}AE_{22} = a_{12}E_{12}$, then $\delta(I) = 0$. We will see that δ is derivable on $\mathcal{R}_{\mathcal{A}}(P, 0)$, but it is not a generalized derivation. First note that $\forall j > i$, $E_{jj}AE_{ii} = 0$, $\forall A \in \mathcal{A}$. Let $A = (a_{ij}) \in \mathcal{A}$ and $B = (b_{ij}) \in \mathcal{A}$. If $APB = AE_{22}B = 0$, then $\delta(AB) = \delta(A)B + A\delta(B)$; indeed

$$\begin{aligned}\delta(AB) &= E_{11}ABE_{22} = E_{11}AIBE_{22} = E_{11}A(E_{11} + E_{22} + E_{33})BE_{22} \\ &= E_{11}AE_{11}BE_{22} = a_{11}b_{12}E_{12}\end{aligned}$$

and

$$\delta(A)B + A\delta(B) = E_{11}AE_{22}B + AE_{11}BE_{22} = 0 + a_{11}b_{12}E_{12}.$$

Thus, δ is derivable on $\mathcal{R}_{\mathcal{A}}(P, 0)$. However, setting $A = E_{12}$ and $B = E_{23}$ we have $\delta(AB) = \delta(E_{12}E_{23}) = \delta(E_{13}) = 0$, but $\delta(A)B + A\delta(B) - A\delta(I)B = \delta(E_{12})E_{23} + E_{12}\delta(E_{23}) - 0 = E_{13}$, so δ is not a generalized derivation.

REMARKS. The following question is raised in [12]: For what Banach algebra \mathcal{A} does it hold that $\delta \in L(\mathcal{A}, \mathcal{A})$ is a generalized derivation iff δ is derivable on $\mathcal{R}_{\mathcal{A}}(m, 0)$, for some $m \in \mathcal{A}$? In particular, does it hold for nest algebras in general and $B(H)$? The previous example gives a negative answer for general nest algebras. Our main result in the following section gives a positive answer for $\mathcal{A} = B(H)$, in fact for any unital standard algebra on a Banach space X .

3. Derivable maps and generalized derivations on standard algebras

THEOREM 3.1. *If \mathcal{A} is a unital standard algebra on a Banach space X then $\Delta \in L(\mathcal{A}, B(X))$ is a generalized derivation iff Δ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$ for some $M \in B(X)$; in this case $\Delta(I) \in \mathbb{C}M$. In particular, the above holds if $\mathcal{A} = B(X)$.*

REMARKS. The main result of [12] is a special case of Theorem 3.1 with X being finite-dimensional, the proofs there make extensive use of Jordan canonical forms of matrices, thus cannot be adapted to the infinite-dimensional settings.

LEMMA 3.2. *Let \mathcal{A} be a unital standard algebra on a Banach space X and $M \in B(X)$. If $\Delta \in L(\mathcal{A}, B(X))$ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$ then $\Delta(I) \in \mathbb{C}M$.*

Proof. By [5], we only need to show that $\forall x \in X, \Delta(I)x \in \mathbb{C}Mx$. Let $y = Mx$.

If $y = 0$ then for any $0 \neq f \in X^*$, $IMx \otimes f = 0$. Since Δ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$, $\Delta(Ix \otimes f) = \Delta(I)x \otimes f + I\Delta(x \otimes f)$. Thus $\Delta(I)x = 0 \in \mathbb{C}Mx$.

If $y \neq 0$, take $f \in X^*$ such that $f(y) = 1$ and $f(x) \neq 0$. It follows $(I - y \otimes f)Mx \otimes f = 0$. Since Δ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$, $\Delta[(I - y \otimes f)x \otimes f] = \Delta(I - y \otimes f)x \otimes f + (I - y \otimes f)\Delta(x \otimes f)$. Thus $-f(x)\Delta(y \otimes f) = \Delta(I)x \otimes f - \Delta(y \otimes f)x \otimes f - y \otimes f\Delta(x \otimes f)$. Applying both sides to x , we get $-f(x)\Delta(y \otimes f)x = f(x)\Delta(I)x - f(x)\Delta(y \otimes f)x - y \otimes f\Delta(x \otimes f)x$. It follows $\Delta(I)x \in \mathbb{C}y = \mathbb{C}Mx$. ■

LEMMA 3.3. *Let \mathcal{A} be a unital standard algebra on a Banach space X and $M \in B(X)$. If $\delta \in L(\mathcal{A}, B(X))$ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$ and $\delta(I) = 0$ then $\forall x \in X, y = Mx$, and $f, g \in X^*$, we have $\delta(y \otimes g \cdot x \otimes f) = \delta(y \otimes g)x \otimes f + y \otimes g\delta(x \otimes f)$.*

Proof. If $g(y) = 0$ then $y \otimes g \cdot Mx \otimes f = 0$. Since δ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$, the conclusion follows.

If $g(y) \neq 0$, rescaling if necessary, we can assume $g(y) = 1$. Thus $(I - y \otimes g)Mx \otimes f = 0$. Since δ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$, $\delta[(I - y \otimes g)x \otimes f] = \delta(I - y \otimes g)x \otimes f + (I - y \otimes g)\delta(x \otimes f)$. Since $\delta(I) = 0$, again the conclusion follows. ■

LEMMA 3.4. *Let \mathcal{A} be a unital standard algebra on a Banach space X and $M \in B(X)$. If $\delta \in L(\mathcal{A}, B(X))$ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$ and $\delta(I) = 0$ then $\forall x \otimes f, x \otimes h \in \mathcal{A}$, $\delta(x \otimes h \cdot x \otimes f) = \delta(x \otimes h)x \otimes f + x \otimes h\delta(x \otimes f)$.*

Proof. If $Mx = 0$ then $x \otimes h \cdot Mx \otimes f = 0$. Since δ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$, the conclusion follows.

Let $y = Mx \neq 0$. For any $g \in X^*$, by Lemma 3.3,

$$\begin{aligned} \delta(y \otimes g \cdot x \otimes h \cdot x \otimes f) &= \delta(y \otimes g(x \otimes h \cdot x \otimes f)) \\ &= \delta(y \otimes g)x \otimes h \cdot x \otimes f + y \otimes g\delta(x \otimes h \cdot x \otimes f). \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \delta(y \otimes g \cdot x \otimes h \cdot x \otimes f) \\
 &= \delta[(y \otimes g \cdot x \otimes h)x \otimes f] = \delta(y \otimes g \cdot x \otimes h)x \otimes f + y \otimes g \cdot x \otimes h\delta(x \otimes f) \\
 &= [\delta(y \otimes g)x \otimes h + y \otimes g\delta(x \otimes h)]x \otimes f + y \otimes g \cdot x \otimes h\delta(x \otimes f) \\
 &= \delta(y \otimes g)x \otimes h \cdot x \otimes f + y \otimes g\delta(x \otimes h)x \otimes f + y \otimes g \cdot x \otimes h\delta(x \otimes f).
 \end{aligned}$$

It follows $y \otimes g\delta(x \otimes h \cdot x \otimes f) = y \otimes g\delta(x \otimes h)x \otimes f + y \otimes g \cdot x \otimes h\delta(x \otimes f)$. Thus $\delta(x \otimes h \cdot x \otimes f) = \delta(x \otimes h)x \otimes f + x \otimes h\delta(x \otimes f)$. ■

LEMMA 3.5. *Let \mathcal{A} be a unital standard algebra on a Banach space X and $M \in B(X)$. If $\delta \in L(\mathcal{A}, B(X))$ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$ and $\delta(I) = 0$ then there exists a $B \in L(X, X)$ such that for any $f \in X^*$, there is a $b_f \in X^*$ satisfying*

- i) $\delta(x \otimes f) = x \otimes b_f + Bx \otimes f, \forall x \in X$.
- ii) $b_f(x) + f(Bx) = 0, \forall x \in X$.

Proof. For any $0 \neq x \otimes f \in \mathcal{A}$, take $h \in X^*$ such that $h(x) = 1$. By Lemma 3.4, $\delta(x \otimes f) = \delta(x \otimes h \cdot x \otimes f) = \delta(x \otimes h)x \otimes f + x \otimes h\delta(x \otimes f)$. It follows that $\forall u \in \ker(f), \delta(x \otimes f)u \in \mathbb{C}x$, i.e. $\delta(x \otimes f) \ker(f) \subseteq \mathbb{C}x$, which yields Part i) using an argument similar to [11, Lemma 3.5(iii)].

To see Part ii), first by Part i),

$$\delta(x \otimes f)x \otimes f = [x \otimes b_f + Bx \otimes f]x \otimes f = b_f(x)x \otimes f + f(x)Bx \otimes f.$$

On the other hand, by Lemma 3.4 and Part i),

$$\begin{aligned}
 & \delta(x \otimes f)x \otimes f \\
 &= \delta(x \otimes f \cdot x \otimes f) - x \otimes f\delta(x \otimes f) = f(x)\delta(x \otimes f) - x \otimes f\delta(x \otimes f) \\
 &= f(x)[x \otimes b_f + Bx \otimes f] - x \otimes f[x \otimes b_f + Bx \otimes f] \\
 &= f(x)Bx \otimes f - f(Bx)x \otimes f.
 \end{aligned}$$

Thus $b_f(x)x \otimes f + f(x)Bx \otimes f = f(x)Bx \otimes f - f(Bx)x \otimes f$, which gives Part ii). ■

LEMMA 3.6. *Let \mathcal{A} be a unital standard algebra on a Banach space X and $M \in B(X)$. If $\delta \in L(\mathcal{A}, B(X))$ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$ and $\delta(I) = 0$ then $\forall x, z \in X$ and $f, h \in X^*$, we have $\delta(z \otimes h \cdot x \otimes f) = \delta(z \otimes h)x \otimes f + z \otimes h\delta(x \otimes f)$.*

Proof. By Lemma 3.5, there exists a $B \in L(X, X)$, $b_f, b_h \in X^*$ such that

$$\begin{aligned}
 & \delta(z \otimes h)x \otimes f + z \otimes h\delta(x \otimes f) \\
 &= [z \otimes b_h + Bz \otimes h]x \otimes f + z \otimes h[x \otimes b_f + Bx \otimes f] \\
 &= b_h(x)z \otimes f + h(x)Bz \otimes f + h(x)z \otimes b_f + h(Bx)z \otimes f \\
 &= [b_h(x) + h(Bx)]z \otimes f + h(x)[z \otimes b_f + Bz \otimes f] \\
 &= [0]z \otimes f + h(x)[\delta(z \otimes f)] = \delta(z \otimes h \cdot x \otimes f). \quad \blacksquare
 \end{aligned}$$

LEMMA 3.7. *Let \mathcal{A} be a unital standard algebra on a Banach space X and $M \in B(X)$. If $\delta \in L(\mathcal{A}, B(X))$ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$ and $\delta(I) = 0$ then $\forall x \in X, f \in X^*$ and $T \in \mathcal{A}$, we have $\delta(Tx \otimes f) = \delta(T)x \otimes f + T\delta(x \otimes f)$.*

Proof. For any $x \in X$ and $T \in \mathcal{A}$, let $y = Mx$ and $z = Ty$. If $y = 0$ then $TMx \otimes f = 0$, the conclusion follows. If $y \neq 0$, take $h \in X^*$ such that $h(y) = 1$. Then $(T - z \otimes h)Mx \otimes f = 0$. Since δ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$,

$$\delta[(T - z \otimes h)x \otimes f] = \delta(T - z \otimes h)x \otimes f + (T - z \otimes h)\delta(x \otimes f).$$

By Lemma 3.6,

$$\delta(z \otimes h \cdot x \otimes f) = \delta(z \otimes h)x \otimes f + z \otimes h\delta(x \otimes f).$$

By the above two equations we get $\delta(Tx \otimes f) = \delta(T)x \otimes f + T\delta(x \otimes f)$. ■

Proof of Theorem 3.1. For any $M \in B(X)$ and $\Delta \in L(\mathcal{A}, B(X))$, if Δ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$, by Lemma 3.2, there exists a $k \in \mathbb{C}$ such that $\Delta(I) = kM$. Define a left multiplier $L_{\Delta} \in L(\mathcal{A}, B(X))$ by $L_{\Delta}(A) = kMA$, $\forall A \in \mathcal{A}$. Let $\delta = \Delta - L_{\Delta}$. Then δ is derivable on $\mathcal{R}_{\mathcal{A}}(M, 0)$ and $\delta(I) = 0$.

For any $A, B \in \mathcal{A}$ and $0 \neq x \otimes f \in \mathcal{A}$, by Lemma 3.7,

$$\begin{aligned} \delta(ABx \otimes f) &= \delta(A)Bx \otimes f + A\delta(Bx \otimes f) \\ &= \delta(A)Bx \otimes f + A[\delta(B)x \otimes f + B\delta(x \otimes f)] \\ &= \delta(A)Bx \otimes f + A\delta(B)x \otimes f + AB\delta(x \otimes f). \end{aligned}$$

On the other hand, also by Lemma 3.7,

$$\delta(ABx \otimes f) = \delta(AB)x \otimes f + AB\delta(x \otimes f).$$

It follows $\delta(AB)x \otimes f = \delta(A)Bx \otimes f + A\delta(B)x \otimes f$. Since $x \otimes f$ is arbitrary, $\delta(AB) = \delta(A)B + A\delta(B)$, or equivalently

$$\Delta(AB) = \Delta(A)B + A\Delta(B) - A\Delta(I)B.$$

For the other direction, if $\Delta \in L(\mathcal{A}, B(X))$ is a generalized derivation then clearly it is derivable on $\mathcal{R}_{\mathcal{A}}(M_1, 0)$, with $M_1 = \Delta(I)$. ■

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