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MULTIPLICATION OPERATORS ON CESÀRO SEQUENCE SPACES

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Abstract. In this paper we characterize the compact, invertible and Fredholm multiplication operators on Cesàro sequence spaces.

1. Introduction and preliminaries

Let $\mathbb{N}$ be the set of positive integers. The Cesàro sequence space $Ces(l^2(\mathbb{N}))$ is defined as

$$Ces(l^2(\mathbb{N})) = \left\{ x : \mathbb{N} \to \mathbb{C} : \sum_{m=1}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |x_k| \right)^2 < \infty \right\}.$$

The space $Ces(l^2(\mathbb{N}))$ is a Banach space under the norm

$$\|x\| = \left(\sum_{m=1}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |x_k| \right)^2 \right)^{\frac{1}{2}}.$$

Cesàro sequence spaces Ces_p, $1 \leq p < \infty$, were introduced for the first time in 1968 in connection with the problem of finding their duals, which was posed by the Dutch Mathematical Society [6]. Shiue [16] and Leibowitz [12] studied the basic properties of these spaces. In 1974 Jagers [7] found the dual space of Ces_p (see also [11]). Later on, Mathematicians became interested in studying the geometric properties of these spaces. Cui and Płuciennik studied Local Uniform Nonsquareness [4], and Banach–Saks Property and Property β [5], Cui and Hudzik studied Fixed Point Property [3] and obtained the Packing Constant [2].

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Let $u : X \to \mathbb{C}$ be a function such that $u.f \in Ces(l^2(\mathbb{N}))$ for every $f \in Ces(l^2(\mathbb{N}))$. Then we can define a multiplication transformation $M_u : Ces(l^2(\mathbb{N})) \to Ces(l^2(\mathbb{N}))$ by

$$M_u f = u.f, \quad \forall\ f \in Ces(l^2(\mathbb{N})).$$

If M_u is continuous, we call it a multiplication operator induced by u. These operators received considerable attention of several mathematician over the past several decades ([1], [8], [9], [10], [13], [14], [15]).

A bounded linear operator $A : E \to E$ (where E is a Banach space) is called compact if $A(B_1)$ has compact closure, where B_1 denotes the closed unit ball of E.

A bounded linear operator $A : E \to E$ is called Fredholm if A has closed range, $\dim(\ker A)$ and co-$\dim(\mathrm{ran} A)$ are finite.

The sequence e^n is defined as $e^n(k) = \delta_{n_k}$, the Kronecker delta. By $B(E)$ we denote the Banach algebra of bounded linear operators from E into itself.

The main purpose of this paper is to characterize the bounded, compact, invertible and Fredholm multiplication operators on Cesàro sequence spaces.

2. Bounded multiplication operators on Cesàro sequence spaces

Theorem 1. *Let $\theta : \mathbb{N} \to \mathbb{C}$ be a mapping. Then $M_\theta : Ces(l^2(\mathbb{N})) \to Ces(l^2(\mathbb{N}))$ is a bounded operator if and only if θ is a bounded function.*

Proof. Suppose θ is a bounded mapping. Then there exists $M > 0$ such that $|\theta_n| \leq M\ \forall\ n \in \mathbb{N}$. Let $x \in Ces(l^2(\mathbb{N}))$. Then

$$\begin{aligned}
\|M_\theta x\|^2 &= \sum_{m=1}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |(\theta x)_k| \right)^2 \\
&= \sum_{m=1}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |\theta_k||x_k| \right)^2 \\
&\leq M^2 \sum_{m=1}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |x_k| \right)^2 \\
&= M^2 \|x\|^2.
\end{aligned}$$

Thus, $\|M_\theta x\| \leq M\|x\|\ \forall\ x \in Ces(l^2(\mathbb{N}))$ which implies that M_θ is a bounded operator.

Conversely, suppose that M_θ is a bounded operator. We will show that θ is a bounded function. For, if θ is not a bounded function, then for every $n \in \mathbb{N}$, there exists some $p_n \in \mathbb{N}$ such that $|\theta_{p_n}| > n$. Now $\|e^{p_n}\| = \left(\sum_{m=p_n}^{\infty} \frac{1}{m^2} \right)^{\frac{1}{2}}$.

Let $\widehat{e^{p_n}} = \frac{e^{p_n}}{\|e^{p_n}\|}$. Then $\|\widehat{e^{p_n}}\| = 1$. But

$$\begin{aligned}\|M_\theta \widehat{e^{p_n}}\| &= \frac{\|M_\theta e^{p_n}\|}{\|e^{p_n}\|} \\ &= \frac{\left(\sum\limits_{m=p_n}^{\infty} \frac{|\theta_{p_n}|^2}{m^2}\right)^{\frac{1}{2}}}{\|e^{p_n}\|} \\ &= |\theta_{p_n}| > n.\end{aligned}$$

This contradicts the boundedness of M_θ. Hence, θ must be a bounded function. ∎

Example 1. Define $\theta : \mathbb{N} \to \mathbb{C}$ by $\theta(n) = \frac{e^{in}}{n}$ for all $n \in \mathbb{N}$. Then for every $x \in Ces(l^2(\mathbb{N}))$,

$$\begin{aligned}\|M_\theta x\| &= \left(\sum_{m=1}^{\infty}\left(\frac{1}{m}\sum_{k=1}^{m}\frac{1}{k^2}|x_k|\right)^2\right)^{\frac{1}{2}} \\ &\leq \|x\|.\end{aligned}$$

Therefore, M_θ is a bounded operator.

Theorem 2. *M_θ is an isometry if and only if $|\theta_n| = 1 \ \forall\ n \in \mathbb{N}$.*

Proof. The proof of the necessary part is trivial.

Conversely, assume that M_θ is an isometry. A straightforward computation shows that if $|\theta_{n_0}| > 1$, for some $n_0 \in \mathbb{N}$, then $\|M_\theta e^{n_0}\| > \|e^{n_0}\|$ and if $|\theta_{n_0}| < 1$, then $\|M_\theta e^{n_0}\| < \|e^{n_0}\|$. This contradicts our assumption that M_θ is an isometry. Hence, $|\theta_n| = 1 \ \forall\ n \in \mathbb{N}$. ∎

3. Compact multiplication operators on Cesàro sequence spaces

Theorem 3. *Let $M_\theta \in B(Ces(l^2(\mathbb{N})))$. Then M_θ is a compact operator if and only if $\theta(n) \to 0$ as $n \to \infty$.*

Proof. We first assume that M_θ is a compact operator. We will show that $\theta(n) \to 0$ as $n \to \infty$. For, if this is not true, then there exists $\epsilon > 0$ such that the set $N_\epsilon = \{k \in \mathbb{N} : |\theta_k| \geq \epsilon\}$ is an infinite set. Let $p_1, p_2, ..., p_n, ..$ be in N_ϵ. Write $\widehat{e^{p_n}} = \frac{e^{p_n}}{\|e^{p_n}\|}$. Then $\{\widehat{e^{p_n}} : p_n \in N_\epsilon\}$ is an infinite bounded set in $Ces(l^2(\mathbb{N}))$. Now

$$\begin{aligned}\|M_\theta \widehat{e^{p_n}} - M_\theta \widehat{e^{p_s}}\| &= \|\theta \widehat{e^{p_n}} - \theta \widehat{e^{p_s}}\| \\ &\geq \epsilon \|\widehat{e^{p_n}} - \widehat{e^{p_s}}\|.\end{aligned}$$

This proves that $\{M_\theta \widehat{e^{p_n}} : p_n \in N_\epsilon\}$ cannot have a convergent subsequence. This contradicts the compactness of M_θ. Hence, $\theta_n \to 0$ as $n \to \infty$.

Conversely, suppose $\theta(n) \to 0$ as $n \to \infty$. Then for every $\epsilon > 0$, the set $N_\epsilon = \{n \in \mathbb{N} : |\theta(n)| \geq \epsilon\}$ is a finite set. Then $Ces(l^2(N_\epsilon))$ is a finite dimensional space for each $\epsilon > 0$. Therefore, $M_\theta | Ces(l^2(N_\epsilon))$ is a compact operator. For each $n \in \mathbb{N}$, define $\theta_n : \mathbb{N} \to \mathbb{C}$ by

$$\theta_n(m) = \begin{cases} \theta(m), & \forall m \in N_{\frac{1}{n}}, \\ 0, & \forall m \notin N_{\frac{1}{n}}. \end{cases}$$

Clearly, M_{θ_n} is a compact operator as the space $Ces(l^2(N_{\frac{1}{n}}))$ is finite dimensional for each $n \in \mathbb{N}$. Now

$$\begin{aligned} \|(M_{\theta_n} - M_\theta)x\|^2 &= \sum_{m=1}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |\theta_n(k)x_k - \theta(k)x_k| \right)^2 \\ &= \sum_{m \in N_{\frac{1}{n}}}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |\theta_n(k)x_k - \theta(k)x_k| \right)^2 \\ &\quad + \sum_{m \in N'_{\frac{1}{n}}}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |\theta_n(k)x_k - \theta(k)x_k| \right)^2 \\ &= \sum_{m \in N'_{\frac{1}{n}}}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |\theta(k)x_k| \right)^2 \\ &< \frac{1}{n^2} \sum_{m \in N'_{\frac{1}{n}}}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |x_k| \right)^2 \\ &\leq \frac{1}{n^2} \|x\|^2, \end{aligned}$$

equivalently, $\|(M_{\theta_n} - M_\theta)(x)\| \leq \frac{1}{n}\|x\|$. This proves that $\|(M_{\theta_n} - M_\theta)\| \leq \frac{1}{n}$ and that M_θ is a limit of compact operators and hence M_θ is a compact operator. ∎

Theorem 4. *Let $M_\theta \in B(Ces(l^2(\mathbb{N})))$. Then M_θ has closed range if and only if θ is bounded away from zero on $\mathbb{N} \backslash ker\theta =: S$.*

Proof. Suppose θ is bounded away from zero on S. Then there exists $\epsilon > 0$ such that $|\theta_n| \geq \epsilon \ \forall \ n \in \mathbb{N} \backslash ker\theta$. We have to prove that $ranM_\theta$ is closed. Let y be a limit point of $ranM_\theta$. Then there exists a sequence $\{y^{(n)}\}$ in $ranM_\theta$ such that $y^{(n)} \to y$. We can write $y^{(n)} = M_\theta x^{(n)}$, where $x^{(n)} = \{x_k^{(n)}\}_{k=1}^{\infty}$ in $Ces(l^2(\mathbb{N}))$. Clearly, the sequence $\{M_\theta x^{(n)}\}$ is a Cauchy sequence. Now,

$$\begin{aligned}
\|M_\theta x^{(n)} - M_\theta x^{(m)}\|^2 &= \sum_{m=1}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |\theta_k x_k^{(n)} - \theta_k x_k^{(m)}| \right)^2 \\
&= \sum_{m=1}^{\infty} \left(\frac{1}{m} \sum_{\substack{k=1 \\ k \in S}}^{m} |\theta_k| |x_k^{(n)} - x_k^{(m)}| \right)^2 \\
&\geq \epsilon^2 \sum_{m=1}^{\infty} \left(\sum_{\substack{k=1 \\ k \in S}}^{m} \frac{|x_k^{(n)} - x_k^{(m)}|}{m} \right)^2 \\
&= \epsilon^2 \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} \frac{|\tilde{x_k^{(n)}} - \tilde{x_k^{(m)}}|}{m} \right)^2 \\
&= \epsilon^2 \, ||\tilde{x^{(n)}} - \tilde{x^{(m)}}||^2,
\end{aligned}$$

where

$$\tilde{x_k^{(n)}} = \begin{cases} x_k^{(n)}, & if k \in S, \\ 0, & if k \notin S. \end{cases}$$

This proves that $\{\tilde{x^{(n)}}\}$ is a Cauchy sequence in $Ces(l^2(\mathbb{N}))$. But $Ces(l^2(\mathbb{N}))$ is complete. Therefore, there exists $x \in Ces(l^2(\mathbb{N}))$ such that $\tilde{x^{(n)}} \to x$ as $n \to \infty$. In view of continuity of M_θ, $M_\theta \tilde{x^{(n)}} \to M_\theta x$. But $M_\theta x^{(n)} = M_\theta \tilde{x^{(n)}} \to y$. Therefore, $M_\theta x = y$. Hence $y \in ran M_\theta$. This proves that M_θ has closed range.

Conversely, suppose that M_θ has closed range. Then M_θ is bounded away from zero on $(ker M_\theta)^\perp = Cesl^2(\mathbb{N} \backslash ker\theta)$. That is, there exists $\epsilon > 0$ such that

$$\|M_\theta x\| \geq \epsilon \|x\| \; \forall \; x \in Cesl^2(\mathbb{N}|ker\theta). \tag{1}$$

Let $E = \{k \in \mathbb{N}|ker\theta : |\theta_k| < \frac{\epsilon}{2}\}$. If $E \neq \phi$, then for $n_0 \in E$, we have

$$\begin{aligned}
\|M_\theta e^{n_0}\|^2 &= \sum_{m=1}^{\infty} \left(\frac{1}{m} \sum_{k=1}^{m} |\theta_k e^{n_0}(k)| \right)^2 \\
&= \sum_{m=n_0}^{\infty} \left(\frac{1}{m} |\theta_{n_0}| \right)^2 \\
&< \epsilon^2 \sum_{m=n_0}^{\infty} \frac{1}{m^2} \\
&= \epsilon^2 \|e^{n_0}\|^2.
\end{aligned}$$

That is, $\|M_\theta e^{n_0}\| < \epsilon \|e^{n_0}\|$ which contradicts (3.1). Hence, $E = \phi$ so that $|\theta_k| \geq \epsilon \; \forall \; k \in \mathbb{N} \backslash ker\theta$. This proves the theorem. ∎

4. Invertible and Fredholm multiplication operators on Cesàro sequence spaces

Theorem 5. *Let $\theta : \mathbb{N} \to \mathbb{C}$ be a mapping. Then $M_\theta : Ces(l^2(\mathbb{N})) \to Ces(l^2(\mathbb{N}))$ is invertible if and only if there exist $m > 0$ and $M > 0$ such that $m < \theta_n < M,\ \forall\ n \in \mathbb{N}$.*

Proof. Suppose that the condition is true. Define $\beta : \mathbb{N} \to \mathbb{C}$ by $\beta_n = \frac{1}{\theta_n}$. Then M_θ and M_β are bounded linear operators in view of Theorem 2.1. Also $M_\theta.M_\beta = M_\beta.M_\theta = I$. Hence, M_β is the inverse of M_θ.

Conversely, suppose that M_θ is invertible. Then $ranM_\theta = Ces(l^2(\mathbb{N}))$. Therefore, $ranM_\theta$ is closed. Hence by Theorem 3.2, there exists $\epsilon > 0$ such that $|\theta_n| \geq \epsilon\ \forall\ n \in \mathbb{N}\backslash ker\theta$. Now $ker\theta = \phi$, otherwise $\theta_{n_0} = 0$, for some $n_0 \in \mathbb{N}$, in which case $e^{n_0} \in kerM_\theta$ which is a contradiction, since $kerM_\theta$ is trivial. Hence, $|\theta_n| \geq \epsilon\ \forall\ n \in \mathbb{N}$. Since M_θ is bounded, so by Theorem 2.1, there exists $M > 0$ such that $|\theta_n| \leq M\ \forall\ n \in \mathbb{N}$. Thus, we have proved that $\epsilon \leq |\theta_n| \leq M,\ \forall\ n \in \mathbb{N}$. ∎

Theorem 6. *Let $M_\theta : Ces(l^2(\mathbb{N})) \to Ces(l^2(\mathbb{N}))$ be a bounded operator. Then M_θ is Fredholm operator if and only if*

(i) *$ker\theta$ is a finite subset of $\mathbb{N}$.*
(ii) *$|\theta_n| \geq \epsilon\ \forall\ n \in \mathbb{N}\backslash ker\theta$.*

Proof. Suppose M_θ is Fredholm. If $ker\theta$ is an infinite subset of $\mathbb{N}$, then $e^n \in kerM_\theta\ \forall\ n \in ker\theta$. But e^n's are linearly independent, which shows that $kerM_\theta$ is an infinite dimensional which is a contradiction. Hence $ker\theta$ must be a finite subset of $\mathbb{N}$. Condition (ii) follows from Theorem 3.2.

Conversely, if conditions (i) and (ii) are true, then we prove that M_θ is Fredholm. In view of Theorem 3.2, condition (ii) implies that M_θ has closed range. Condition (i) implies that $kerM_\theta$ and $kerM_\theta^\star$ are finite dimensional. This proves that M_θ is Fredholm. ∎

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