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MORE ON OSTROWSKI TYPE INEQUALITIES FOR SOME S-CONVEX FUNCTIONS IN THE SECOND SENSE

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Abstract. Some Ostrowski type inequalities for functions whose second derivatives in absolute value at certain powers are s-convex in the second sense are established. Two mistakes in a recently published paper are pointed out and corrected.

1. Introduction

We recall that Hudzik and Maligranda in [2] have defined a function $f : [0, \infty) \to \mathbf{R}$ to be s-convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in [0, \infty), \lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$. The class of s-convex functions in the second sense is usually denoted with K_s^2. It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$. It is proved in [2] that all functions from $K_s^2, s \in (0, 1)$ are nonnegative.

In a recent paper [5], Set et al. proved the following inequalities for functions whose second derivatives in absolute value at certain powers are s-convex in the second sense.

Theorem 1. ([5], Theorem 4) *Let $I \subset [0, \infty)$, $f : I \to \mathbf{R}$ be a twice differentiable function on I° such that $f'' \in L^1[a, b]$, where $a, b \in I$ with $a < b$. If $|f''|$ is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, then the following inequality holds:*

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$$
\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x) + \left(x-\frac{a+b}{2}\right)f'(x)\right|
\le \frac{1}{2(b-a)}\left\{\left[\frac{|f''(x)|}{s+3}+\frac{2|f''(a)|}{(s+1)(s+2)(s+3)}\right](x-a)^3
+\left[\frac{|f''(x)|}{s+3}+\frac{2|f''(b)|}{(s+1)(s+2)(s+3)}\right](b-x)^3\right\}, \tag{1}
$$

for each $x \in [a,b]$.

Theorem 2. ([5], Theorem 6) *Let* $I \subset [0,\infty)$, $f : I \to \mathbf{R}$ *be a twice differentiable function on* I° *such that* $f'' \in L^1[a,b]$, *where* $a,b \in I$ *with* $a<b$. *If* $|f''|^q$ *is s-convex in the second sense on* $[a,b]$ *for some fixed* $s \in (0,1]$ *and* $q \ge 1$, *then the following inequality holds:*

$$
\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x) + \left(x-\frac{a+b}{2}\right)f'(x)\right|
\le \frac{(x-a)^3}{2(b-a)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{|f''(x)|^q}{s+3}+\frac{2|f''(a)|^q}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}
+\frac{(b-x)^3}{2(b-a)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{|f''(x)|^q}{s+3}+\frac{2|f''(b)|^q}{(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}, \tag{2}
$$

for each $x \in [a,b]$.

However, it's a pity that Theorem 7 in [5] is not valid since a nonnegative $|f''|^q$ could not be an s-concave function for any fixed $s \in (0,1)$ which has been mentioned in [3], and, it is the Hölder inequality but not the power mean inequality, which has been used in proving the Theorem 6 of [5].

In this work, we will derive some Ostrowski type inequalities for functions whose second derivatives in absolute value at certain powers are s-convex in the second sense, which not only provide generalizations of Theorem 1 and Theorem 2, but also give some interesting special results.

2. Main results

In order to establish our main results, we need the following lemma.

Lemma 1. *Let* $I \subset \mathbf{R}$, $f : I \to \mathbf{R}$ *be a twice differentiable function on* I° *such that* $f'' \in L^1[a,b]$, *where* $a,b \in I$ *with* $a<b$. *Then for any* $\theta \in [0,1]$ *and* $x \in [a,b]$, *the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(t)\,dt-\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]+(1-\theta)\left(x-\frac{a+b}{2}\right)f'(x) \tag{3}$$
$$=\frac{(x-a)^3}{2(b-a)}\int_0^1\left(t^2-\theta\frac{b-a}{x-a}t\right)f''(tx+(1-t)a)\,dt$$
$$+\frac{(b-x)^3}{2(b-a)}\int_0^1\left(t^2-\theta\frac{b-a}{b-x}t\right)f''(tx+(1-t)b)\,dt.$$

Proof. From ([5], Lemma 1), we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt-f(x)+(x-\frac{a+b}{2})f'(x)$$
$$=\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2 f''(tx+(1-t)a)\,dt+\frac{(b-x)^3}{2(b-a)}\int_0^1 t^2 f''(tx+(1-t)b)\,dt,$$

for each $x\in[a,b]$.

On the other hand, we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}$$
$$=\frac{1}{2(b-a)}\int_a^b (t-a)(t-b)f''(t)\,dt$$
$$=\frac{1}{2(b-a)}\left[\int_a^x (t-a)(t-b)f''(t)\,dt+\int_x^b (t-a)(t-b)f''(t)\,dt\right]$$
$$=\frac{1}{2(b-a)}\left[(x-a)^3\int_0^1\left(t^2-\frac{b-a}{x-a}t\right)f''(tx+(1-t)a)\,dt\right.$$
$$\left.+(b-x)^3\int_0^1\left(t^2-\frac{b-a}{b-x}t\right)f''(tx+(1-t)b)\,dt\right],$$

by changing the variable t of the first integral to $tx+(1-t)a$ and the variable t of the second integral to $tx+(1-t)b$ on the third line.

Consequently,

$$\frac{1}{b-a}\int_a^b f(t)\,dt-\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]+(1-\theta)\left(x-\frac{a+b}{2}\right)f'(x)$$
$$=(1-\theta)\left[\frac{1}{b-a}\int_a^b f(t)\,dt-f(x)+\left(x-\frac{a+b}{2}\right)f'(x)\right]$$
$$+\theta\left[\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right]$$

$$= (1-\theta)\bigg[\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2 f''(tx+(1-t)a)\,dt$$

$$+ \frac{(b-x)^3}{2(b-a)}\int_0^1 t^2 f''(tx+(1-t)b)\,dt\bigg]$$

$$+ \theta\bigg[\frac{(x-a)^3}{2(b-a)}\int_0^1 (t^2 - \frac{b-a}{x-a}t) f''(tx+(1-t)a)\,dt$$

$$+ \frac{(b-x)^3}{2(b-a)}\int_0^1 \left(t^2 - \frac{b-a}{b-x}t\right) f''(tx+(1-t)b)\,dt\bigg]$$

$$= \frac{(x-a)^3}{2(b-a)}\int_0^1 \left(t^2 - \theta\frac{b-a}{x-a}t\right) f''(tx+(1-t)a)\,dt$$

$$+ \frac{(b-x)^3}{2(b-a)}\int_0^1 \left(t^2 - \theta\frac{b-a}{b-x}t\right) f''(tx+(1-t)b)\,dt.$$

The proof is completed. ■

Theorem 3. *Let $I \subset [0,\infty)$, $f : I \to \mathbf{R}$ be a twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a,b \in I$ with $a<b$. If $|f''|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequalities hold:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right] + (1-\theta)\left(x-\frac{a+b}{2}\right)f'(x)\right| \tag{4}$$

$$\leq \frac{(x-a)^3}{2(b-a)}\left[\frac{1}{3} - \frac{\theta}{2}\frac{b-a}{x-a} + \frac{1}{3}\left(\theta\frac{b-a}{x-a}\right)^3\right]^{1-\frac{1}{q}}$$

$$\times\Bigg\{\left[\frac{1}{s+3} - \frac{\theta}{s+2}\frac{b-a}{x-a} + \frac{2}{(s+2)(s+3)}\left(\theta\frac{b-a}{x-a}\right)^{s+3}\right]|f''(x)|^q$$

$$+ \Bigg[2\left(1-\theta\frac{b-a}{x-a}\right)^{s+2}\left(\frac{\theta}{(s+2)(s+3)}\frac{b-a}{x-a} + \frac{2}{(s+1)(s+2)(s+3)}\right)$$

$$+ \frac{\theta}{(s+1)(s+2)}\frac{b-a}{x-a} - \frac{2}{(s+1)(s+2)(s+3)}\Bigg]|f''(a)|^q\Bigg\}^{\frac{1}{q}}$$

$$+ \frac{(b-x)^3}{2(b-a)}\left[\frac{1}{3} - \frac{\theta}{2}\frac{b-a}{b-x} + \frac{1}{3}\left(\theta\frac{b-a}{b-x}\right)^3\right]^{1-\frac{1}{q}}$$

$$\times\Bigg\{\left[\frac{1}{s+3} - \frac{\theta}{s+2}\frac{b-a}{b-x} + \frac{2}{(s+2)(s+3)}\left(\theta\frac{b-a}{b-x}\right)^{s+3}\right]|f''(x)|^q$$

$$+\left[2\left(1-\theta\frac{b-a}{b-x}\right)^{s+2}\left(\frac{\theta}{(s+2)(s+3)}\frac{b-a}{b-x}+\frac{2}{(s+1)(s+2)(s+3)}\right)\right.$$
$$\left.+\frac{\theta}{(s+1)(s+2)}\frac{b-a}{b-x}-\frac{2}{(s+1)(s+2)(s+3)}\right]|f''(b)|^q\Bigg\}^{\frac{1}{q}}$$

for each $x \in [a, \frac{a+b}{2}]$ *with* $0 \le \theta \le \frac{x-a}{b-a} \le \frac{b-x}{b-a} \le 1$, *or for each* $x \in [\frac{a+b}{2}, b]$ *with* $0 \le \theta \le \frac{b-x}{b-a} \le \frac{x-a}{b-a} \le 1$,

$$\text{(5)}\quad \left|\frac{1}{b-a}\int_a^b f(t)\,dt-\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]\right.$$
$$\left.+(1-\theta)\left(x-\frac{a+b}{2}\right)f'(x)\right|$$
$$\le \frac{(x-a)^3}{2(b-a)}\left(\frac{\theta}{2}\frac{b-a}{x-a}-\frac{1}{3}\right)^{1-\frac{1}{q}}\Bigg\{\left(\frac{\theta}{s+2}\frac{b-a}{x-a}-\frac{1}{s+3}\right)|f''(x)|^q$$
$$+\left[\frac{\theta}{(s+1)(s+2)}\frac{b-a}{x-a}-\frac{2}{(s+1)(s+2)(s+3)}\right]|f''(a)|^q\Bigg\}^{\frac{1}{q}}$$
$$+\frac{(b-x)^3}{2(b-a)}\left[\frac{1}{3}-\frac{\theta}{2}\frac{b-a}{b-x}+\frac{1}{3}\left(\theta\frac{b-a}{b-x}\right)^3\right]^{1-\frac{1}{q}}$$
$$\times\Bigg\{\left[\frac{1}{s+3}-\frac{\theta}{s+2}\frac{b-a}{b-x}+\frac{2}{(s+2)(s+3)}\left(\theta\frac{b-a}{b-x}\right)^{s+3}\right]|f''(x)|^q$$
$$+\left[2\left(1-\theta\frac{b-a}{b-x}\right)^{s+2}\left(\frac{\theta}{(s+2)(s+3)}\frac{b-a}{b-x}+\frac{2}{(s+1)(s+2)(s+3)}\right)\right.$$
$$\left.+\frac{\theta}{(s+1)(s+2)}\frac{b-a}{b-x}-\frac{2}{(s+1)(s+2)(s+3)}\right]|f''(b)|^q\Bigg\}^{\frac{1}{q}}$$

for each $x \in [a, \frac{a+b}{2}]$ *with* $0 \le \frac{x-a}{b-a} \le \theta \le \frac{b-x}{b-a} \le 1$,

$$\text{(6)}\quad \left|\frac{1}{b-a}\int_a^b f(t)\,dt-\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]+(1-\theta)\left(x-\frac{a+b}{2}\right)f'(x)\right|$$
$$\le \frac{(x-a)^3}{2(b-a)}\left(\frac{\theta}{2}\frac{b-a}{x-a}-\frac{1}{3}\right)^{1-\frac{1}{q}}\Bigg\{\left(\frac{\theta}{s+2}\frac{b-a}{x-a}-\frac{1}{s+3}\right)|f''(x)|^q$$
$$+\left[\frac{\theta}{(s+1)(s+2)}\frac{b-a}{x-a}-\frac{2}{(s+1)(s+2)(s+3)}\right]|f''(a)|^q\Bigg\}^{\frac{1}{q}}$$
$$+\frac{(b-x)^3}{2(b-a)}\left(\frac{\theta}{2}\frac{b-a}{b-x}-\frac{1}{3}\right)^{1-\frac{1}{q}}\Bigg\{\left(\frac{\theta}{s+2}\frac{b-a}{b-x}-\frac{1}{s+3}\right)|f''(x)|^q$$
$$+\left[\frac{\theta}{(s+1)(s+2)}\frac{b-a}{b-x}-\frac{2}{(s+1)(s+2)(s+3)}\right]|f''(b)|^q\Bigg\}^{\frac{1}{q}}$$

for each $x \in [a, \frac{a+b}{2}]$ *with* $0 \le \frac{x-a}{b-a} \le \frac{b-x}{b-a} \le \theta \le 1$, *or for each* $x \in [\frac{a+b}{2}, b]$ *with* $0 \le \frac{b-x}{b-a} \le \frac{x-a}{b-a} \le \theta \le 1$,

$$
\begin{aligned}
(7)\quad & \left|\frac{1}{b-a}\int_a^b f(t)\,dt - \left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right] \right.\\
&\qquad\left. + (1-\theta)\left(x - \frac{a+b}{2}\right)f'(x)\right|\\
&\le \frac{(x-a)^3}{2(b-a)}\left[\frac{1}{3} - \frac{\theta}{2}\frac{b-a}{x-a} + \frac{1}{3}\left(\theta\frac{b-a}{x-a}\right)^3\right]^{1-\frac{1}{q}}\\
&\times\left\{\left[\frac{1}{s+3} - \frac{\theta}{s+2}\frac{b-a}{x-a} + \frac{2}{(s+2)(s+3)}\left(\theta\frac{b-a}{x-a}\right)^{s+3}\right]|f''(x)|^q\right.\\
&+\left[2\left(1-\theta\frac{b-a}{x-a}\right)^{s+2}\left(\frac{\theta}{(s+2)(s+3)}\frac{b-a}{x-a} + \frac{2}{(s+1)(s+2)(s+3)}\right)\right.\\
&\left.\left.+\frac{\theta}{(s+1)(s+2)}\frac{b-a}{x-a} - \frac{2}{(s+1)(s+2)(s+3)}\right]|f''(a)|^q\right\}^{\frac{1}{q}}\\
&+\frac{(b-x)^3}{2(b-a)}\left(\frac{\theta}{2}\frac{b-a}{b-x} - \frac{1}{3}\right)^{1-\frac{1}{q}}\left\{\left(\frac{\theta}{s+2}\frac{b-a}{b-x} - \frac{1}{s+3}\right)|f''(x)|^q\right.\\
&\left.+\left[\frac{\theta}{(s+1)(s+2)}\frac{b-a}{b-x} - \frac{2}{(s+1)(s+2)(s+3)}\right]|f''(b)|^q\right\}^{\frac{1}{q}}
\end{aligned}
$$

for each $x \in [\frac{a+b}{2}, b]$ *with* $0 \le \frac{b-x}{b-a} \le \theta \le \frac{x-a}{b-a} \le 1$.

Proof. By Lemma 1 and using the Hölder inequality, we have

$$
\begin{aligned}
(8)\quad & \left|\frac{1}{b-a}\int_a^b f(t)\,dt - \left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right] \right.\\
&\qquad\left. + (1-\theta)\left(x - \frac{a+b}{2}\right)f'(x)\right|\\
&\le \frac{(x-a)^3}{2(b-a)}\int_0^1\left|t^2 - \theta\frac{b-a}{x-a}t\right||f''(tx+(1-t)a)|\,dt\\
&\quad+\frac{(b-x)^3}{2(b-a)}\int_0^1\left|t^2 - \theta\frac{b-a}{b-x}t\right||f''(tx+(1-t)b)|\,dt\\
&\le \frac{(x-a)^3}{2(b-a)}\left(\int_0^1\left|t^2 - \theta\frac{b-a}{x-a}t\right|dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1\left|t^2 - \theta\frac{b-a}{x-a}t\right||f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(b-x)^3}{2(b-a)}\left(\int_0^1 |t^2-\theta\frac{b-a}{b-x}t|\,dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 \left|t^2-\theta\frac{b-a}{b-x}t\right||f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\\
\le\ &\frac{(x-a)^3}{2(b-a)}\left(\int_0^1 \left|t^2-\theta\frac{b-a}{x-a}t\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 \left|t^2-\theta\frac{b-a}{x-a}t\right|[t^s|f''(x)|^q+(1-t)^s|f''(a)|^q]\,dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^3}{2(b-a)}\left(\int_0^1 \left|t^2-\theta\frac{b-a}{b-x}t\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 \left|t^2-\theta\frac{b-a}{b-x}t\right|[t^s|f''(x)|^q+(1-t)^s|f''(b)|^q]\,dt\right)^{\frac{1}{q}}\\
=\ &\frac{(x-a)^3}{2(b-a)}\left(\int_0^1 \left|t^2-\theta\frac{b-a}{x-a}t\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(|f''(x)|^q\int_0^1 \left|t^2-\theta\frac{b-a}{x-a}t\right|t^s\,dt+|f''(a)|^q\int_0^1 \left|t^2-\theta\frac{b-a}{x-a}t\right|(1-t)^s\,dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^3}{2(b-a)}\left(\int_0^1 \left|t^2-\theta\frac{b-a}{b-x}t\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(|f''(x)|^q\int_0^1 \left|t^2-\theta\frac{b-a}{b-x}t\right|t^s\,dt+|f''(b)|^q\int_0^1 \left|t^2-\theta\frac{b-a}{b-x}t\right|(1-t)^s\,dt\right)^{\frac{1}{q}},
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^1 \left|t^2-\theta\frac{b-a}{x-a}t\right|dt &= \int_0^{\theta\frac{b-a}{x-a}} t\left(\theta\frac{b-a}{x-a}-t\right)dt\\
&\quad+\int_{\theta\frac{b-a}{x-a}}^1 t\left(t-\theta\frac{b-a}{x-a}\right)dt\\
&=\frac{1}{3}-\frac{\theta}{2}\frac{b-a}{x-a}+\frac{1}{3}\left(\theta\frac{b-a}{x-a}\right)^3,
\end{aligned}\tag{9}
$$

$$
\begin{aligned}
\int_0^1 \left|t^2-\theta\frac{b-a}{x-a}t\right|t^s\,dt &= \int_0^{\theta\frac{b-a}{x-a}} t\left(\theta\frac{b-a}{x-a}-t\right)t^s\,dt\\
&\quad+\int_{\theta\frac{b-a}{x-a}}^1 t\left(t-\theta\frac{b-a}{x-a}\right)t^s\,dt\\
&=\frac{1}{s+3}-\frac{\theta}{s+2}\frac{b-a}{x-a}+\frac{2}{(s+2)(s+3)}\left(\theta\frac{b-a}{x-a}\right)^{s+3},
\end{aligned}\tag{10}
$$

$$\begin{aligned}(11)\qquad &\int_0^1 \left|t^2-\theta\frac{b-a}{x-a}t\right|(1-t)^s\,dt\\ &=\int_0^{\theta\frac{b-a}{x-a}} t\left(\theta\frac{b-a}{x-a}-t\right)(1-t)^s\,dt+\int_{\theta\frac{b-a}{x-a}}^1 t\left(t-\theta\frac{b-a}{x-a}\right)(1-t)^s\,dt\\ &=2\left(1-\theta\frac{b-a}{x-a}\right)^{s+2}\left(\frac{\theta}{(s+2)(s+3)}\frac{b-a}{x-a}+\frac{2}{(s+1)(s+2)(s+3)}\right)\\ &\quad+\frac{\theta}{(s+1)(s+2)}\frac{b-a}{x-a}-\frac{2}{(s+1)(s+2)(s+3)},\end{aligned}$$

in case $0\le\theta\frac{b-a}{x-a}\le 1$, or equivalently, $0\le\theta\le\frac{x-a}{b-a}\le 1$, and

$$(12)\qquad \int_0^1\left|t^2-\theta\frac{b-a}{x-a}t\right|dt=\int_0^1 t\left(\theta\frac{b-a}{x-a}-t\right)dt=\frac{\theta}{2}\frac{b-a}{x-a}-\frac{1}{3},$$

$$(13)\qquad \int_0^1\left|t^2-\theta\frac{b-a}{x-a}t\right|t^s\,dt=\int_0^1\left(\theta\frac{b-a}{x-a}-t\right)t^{s+1}\,dt=\frac{\theta}{s+2}\frac{b-a}{x-a}-\frac{1}{s+3},$$

$$\begin{aligned}(14)\qquad \int_0^1\left|t^2-\theta\frac{b-a}{x-a}t\right|(1-t)^s\,dt&=\int_0^1 t\left(\theta\frac{b-a}{x-a}-t\right)(1-t)^s\,dt\\ &=\frac{\theta}{(s+1)(s+2)}\frac{b-a}{x-a}-\frac{2}{(s+1)(s+2)(s+3)},\end{aligned}$$

in case $\theta\frac{b-a}{x-a}\ge 1$, or equivalently, $0\le\frac{x-a}{b-a}\le\theta\le 1$, and similarly,

$$\begin{aligned}(15)\qquad &\int_0^1\left|t^2-\theta\frac{b-a}{b-x}t\right|dt\\ &=\int_0^{\theta\frac{b-a}{b-x}} t\left(\theta\frac{b-a}{b-x}-t\right)dt+\int_{\theta\frac{b-a}{b-x}}^1 t\left(t-\theta\frac{b-a}{b-x}\right)dt\\ &=\frac{1}{3}-\frac{\theta}{2}\frac{b-a}{b-x}+\frac{1}{3}\left(\theta\frac{b-a}{b-x}\right)^3,\end{aligned}$$

$$\begin{aligned}(16)\qquad &\int_0^1\left|t^2-\theta\frac{b-a}{b-x}t\right|t^s\,dt\\ &=\int_0^{\theta\frac{b-a}{b-x}} t\left(\theta\frac{b-a}{b-x}-t\right)t^s\,dt+\int_{\theta\frac{b-a}{b-x}}^1 t\left(t-\theta\frac{b-a}{b-x}\right)t^s\,dt\\ &=\frac{1}{s+3}-\frac{\theta}{s+2}\frac{b-a}{b-x}+\frac{2}{(s+2)(s+3)}\left(\theta\frac{b-a}{b-x}\right)^{s+3},\end{aligned}$$

$$\begin{aligned}(17)\quad &\int_0^1 \left|t^2-\theta\frac{b-a}{b-x}t\right|(1-t)^s\,dt\\ &=\int_0^{\theta\frac{b-a}{b-x}} t\left(\theta\frac{b-a}{b-x}-t\right)(1-t)^s\,dt+\int_{\theta\frac{b-a}{b-x}}^1 t\left(t-\theta\frac{b-a}{b-x}\right)(1-t)^s\,dt\\ &=2\left(1-\theta\frac{b-a}{b-x}\right)^{s+2}\left(\frac{\theta}{(s+2)(s+3)}\frac{b-a}{b-x}+\frac{2}{(s+1)(s+2)(s+3)}\right)\\ &\quad+\frac{\theta}{(s+1)(s+2)}\frac{b-a}{b-x}-\frac{2}{(s+1)(s+2)(s+3)},\end{aligned}$$

in case $0 \le \theta\frac{b-a}{b-x} \le 1$, or equivalently, $0 \le \theta \le \frac{b-x}{b-a} \le 1$, and

$$(18)\quad \int_0^1 \left|t^2-\theta\frac{b-a}{b-x}t\right|dt=\int_0^1 t\left(\theta\frac{b-a}{b-x}-t\right)dt=\frac{\theta}{2}\frac{b-a}{b-x}-\frac{1}{3},$$

$$(19)\quad \int_0^1 \left|t^2-\theta\frac{b-a}{b-x}t\right|t^s\,dt=\int_0^1 \left(\theta\frac{b-a}{b-x}-t\right)t^{s+1}\,dt=\frac{\theta}{s+2}\frac{b-a}{b-x}-\frac{1}{s+3},$$

$$\begin{aligned}(20)\quad \int_0^1 \left|t^2-\theta\frac{b-a}{b-x}t\right|(1-t)^s\,dt&=\int_0^1 t\left(\theta\frac{b-a}{b-x}-t\right)(1-t)^s\,dt\\ &=\frac{\theta}{(s+1)(s+2)}\frac{b-a}{b-x}-\frac{2}{(s+1)(s+2)(s+3)},\end{aligned}$$

in case $\theta\frac{b-a}{b-x} \ge 1$, or equivalently, $0 \le \frac{b-x}{b-a} \le \theta \le 1$.

Moreover, it should be noticed that for each $x \in [a, \frac{a+b}{2}]$, we have $0 \le x-a \le b-x \le b-a$ and it follows $0 \le \frac{x-a}{b-a} \le \frac{b-x}{b-a} \le 1$, which indicates that three cases have to be considered as (i) $0 \le \theta \le \frac{x-a}{b-a} \le \frac{b-x}{b-a} \le 1$, (ii) $0 \le \frac{x-a}{b-a} \le \theta \le \frac{b-x}{b-a} \le 1$, (iii) $0 \le \frac{x-a}{b-a} \le \frac{b-x}{b-a} \le \theta \le 1$, and for each $x \in [\frac{a+b}{2}, b]$, we have $0 \le b-x \le x-a \le b-a$ and it follows $0 \le \frac{b-x}{b-a} \le \frac{x-a}{b-a} \le 1$, which indicates that another three cases have to be considered as (iv) $0 \le \theta \le \frac{b-x}{b-a} \le \frac{x-a}{b-a} \le 1$, (v) $0 \le \frac{b-x}{b-a} \le \theta \le \frac{x-a}{b-a} \le 1$, (vi) $0 \le \frac{b-x}{b-a} \le \frac{x-a}{b-a} \le \theta \le 1$.

Combining (8) with (9)–(20), and taking into account the above explanation, the inequalities (4)–(7) follow.

The proof is thus completed. ∎

Corollary 1. *Let $I \subset [0,\infty)$, $f : I \to \mathbf{R}$ be a twice differentiable function on I° such that $f'' \in L^1[a,b]$, where $a, b \in I$ with $a < b$. If $|f''|$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequalities hold:*

$$\begin{aligned}
(21)\quad & \left|\frac{1}{b-a}\int_a^b f(t)\,dt - \left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]\right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad \left. +(1-\theta)\left(x-\frac{a+b}{2}\right)f'(x)\right| \\
& \leq \frac{(x-a)^3}{2(b-a)}\left\{\left[\frac{1}{s+3}-\frac{\theta}{s+2}\frac{b-a}{x-a}+\frac{2}{(s+2)(s+3)}\left(\theta\frac{b-a}{x-a}\right)^{s+3}\right]|f''(x)|\right. \\
& \quad +\left[2\left(1-\theta\frac{b-a}{x-a}\right)^{s+2}\left(\frac{\theta}{(s+2)(s+3)}\frac{b-a}{x-a}+\frac{2}{(s+1)(s+2)(s+3)}\right)\right. \\
& \quad \left.\left. +\frac{\theta}{(s+1)(s+2)}\frac{b-a}{x-a}-\frac{2}{(s+1)(s+2)(s+3)}\right]|f''(a)|\right\} \\
& \quad +\frac{(b-x)^3}{2(b-a)}\left\{\left[\frac{1}{s+3}-\frac{\theta}{s+2}\frac{b-a}{b-x}+\frac{2}{(s+2)(s+3)}(\theta\frac{b-a}{b-x})^{s+3}\right]|f''(x)|\right. \\
& \quad +\left[2\left(1-\theta\frac{b-a}{b-x}\right)^{s+2}\left(\frac{\theta}{(s+2)(s+3)}\frac{b-a}{b-x}+\frac{2}{(s+1)(s+2)(s+3)}\right)\right. \\
& \quad \left.\left. +\frac{\theta}{(s+1)(s+2)}\frac{b-a}{b-x}-\frac{2}{(s+1)(s+2)(s+3)}\right]|f''(b)|\right\}
\end{aligned}$$

for each $x \in [a, \frac{a+b}{2}]$ *with* $0 \leq \theta \leq \frac{x-a}{b-a} \leq \frac{b-x}{b-a} \leq 1$, *or for each* $x \in [\frac{a+b}{2}, b]$ *with* $0 \leq \theta \leq \frac{b-x}{b-a} \leq \frac{x-a}{b-a} \leq 1$,

$$\begin{aligned}
(22)\quad & \left|\frac{1}{b-a}\int_a^b f(t)\,dt - \left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]+(1-\theta)\left(x-\frac{a+b}{2}\right)f'(x)\right| \\
& \leq \frac{(x-a)^3}{2(b-a)}\left\{\left(\frac{\theta}{s+2}\frac{b-a}{x-a}-\frac{1}{s+3}\right)|f''(x)|\right. \\
& \quad \left. +\left[\frac{\theta}{(s+1)(s+2)}\frac{b-a}{x-a}-\frac{2}{(s+1)(s+2)(s+3)}\right]|f''(a)|\right\} \\
& \quad +\frac{(b-x)^3}{2(b-a)}\left\{\left[\frac{1}{s+3}-\frac{\theta}{s+2}\frac{b-a}{b-x}+\frac{2}{(s+2)(s+3)}\left(\theta\frac{b-a}{b-x}\right)^{s+3}\right]|f''(x)|\right. \\
& \quad +\left[2\left(1-\theta\frac{b-a}{b-x}\right)^{s+2}\left(\frac{\theta}{(s+2)(s+3)}\frac{b-a}{b-x}+\frac{2}{(s+1)(s+2)(s+3)}\right)\right. \\
& \quad \left.\left. +\frac{\theta}{(s+1)(s+2)}\frac{b-a}{b-x}-\frac{2}{(s+1)(s+2)(s+3)}\right]|f''(b)|\right\}
\end{aligned}$$

for each $x \in [a, \frac{a+b}{2}]$ *with* $0 \leq \frac{x-a}{b-a} \leq \theta \leq \frac{b-x}{b-a} \leq 1$,

$$\begin{aligned}
(23)\quad & \left|\frac{1}{b-a}\int_a^b f(t)\,dt - \left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]\right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad \left. +(1-\theta)\left(x-\frac{a+b}{2}\right)f'(x)\right|
\end{aligned}$$

$$
\leq \frac{(x-a)^3}{2(b-a)}\bigg\{\bigg(\frac{\theta}{s+2}\frac{b-a}{x-a}-\frac{1}{s+3}\bigg)|f''(x)| + \bigg[\frac{\theta}{(s+1)(s+2)}\frac{b-a}{x-a}-\frac{2}{(s+1)(s+2)(s+3)}\bigg]|f''(a)|\bigg\}
+ \frac{(b-x)^3}{2(b-a)}\bigg\{\bigg(\frac{\theta}{s+2}\frac{b-a}{b-x}-\frac{1}{s+3}\bigg)|f''(x)| + \bigg[\frac{\theta}{(s+1)(s+2)}\frac{b-a}{b-x}-\frac{2}{(s+1)(s+2)(s+3)}\bigg]|f''(b)|\bigg\}
$$

for each $x \in [a, \frac{a+b}{2}]$ *with* $0 \leq \frac{x-a}{b-a} \leq \frac{b-x}{b-a} \leq \theta \leq 1$, *or for each* $x \in [\frac{a+b}{2}, b]$ *with* $0 \leq \frac{b-x}{b-a} \leq \frac{x-a}{b-a} \leq \theta \leq 1$,

$$
\tag{24} \bigg|\frac{1}{b-a}\int_a^b f(t)\,dt - \bigg[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\bigg] + (1-\theta)\bigg(x-\frac{a+b}{2}\bigg)f'(x)\bigg|
$$

$$
\leq \frac{(x-a)^3}{2(b-a)}\bigg\{\bigg[\frac{1}{s+3}-\frac{\theta}{s+2}\frac{b-a}{x-a}+\frac{2}{(s+2)(s+3)}\bigg(\theta\frac{b-a}{x-a}\bigg)^{s+3}\bigg]|f''(x)| + \bigg[2\bigg(1-\theta\frac{b-a}{x-a}\bigg)^{s+2}\bigg(\frac{\theta}{(s+2)(s+3)}\frac{b-a}{x-a}+\frac{2}{(s+1)(s+2)(s+3)}\bigg) + \frac{\theta}{(s+1)(s+2)}\frac{b-a}{x-a}-\frac{2}{(s+1)(s+2)(s+3)}\bigg]|f''(a)|\bigg\}
+ \frac{(b-x)^3}{2(b-a)}\bigg\{\bigg(\frac{\theta}{s+2}\frac{b-a}{b-x}-\frac{1}{s+3}\bigg)|f''(x)| + \bigg[\frac{\theta}{(s+1)(s+2)}\frac{b-a}{b-x}-\frac{2}{(s+1)(s+2)(s+3)}\bigg]|f''(b)|\bigg\}
$$

for each $x \in [\frac{a+b}{2}, b]$ *with* $0 \leq \frac{b-x}{b-a} \leq \theta \leq \frac{x-a}{b-a} \leq 1$.

Proof. The inequalities (21)–(24) are immediate by setting $q = 1$ in (4)–(7) of the Theorem 3. ∎

Remark 1. If we take $\theta = 0$ in Theorem 3 and Corollary 1, respectively, then we recapture Theorem 2 and Theorem 1.

Corollary 2. *Let* $I \subset [0, \infty)$, $f : I \to \mathbf{R}$ *be a twice differentiable function on* I° *such that* $f'' \in L^1[a, b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f''|^q$ *is s-convex in the second sense on* $[a, b]$ *for some fixed* $s \in (0, 1]$ *and* $q \geq 1$, *then the following inequalities hold:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\left[(1-\theta)f\left(\frac{a+b}{2}\right)+\theta\frac{f(a)+f(b)}{2}\right]\right| \tag{25}$$
$$\le\frac{(b-a)^2}{16}\left(\frac{1}{3}-\theta+\frac{8}{3}\theta^3\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\left(\frac{1}{s+3}-\frac{2\theta}{s+2}+\frac{2(2\theta)^{s+3}}{(s+2)(s+3)}\right)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.\right.$$
$$+\left(2(1-2\theta)^{s+2}\left(\frac{2\theta}{(s+2)(s+3)}+\frac{2}{(s+1)(s+2)(s+3)}\right)\right.$$
$$\left.\left.+\frac{2\theta}{(s+1)(s+2)}-\frac{2}{(s+1)(s+2)(s+3)}\right)|f''(a)|^q\right]^{\frac{1}{q}}$$
$$+\left[\left(\frac{1}{s+3}-\frac{2\theta}{s+2}+\frac{2(2\theta)^{s+3}}{(s+2)(s+3)}\right)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.$$
$$+\left(2(1-2\theta)^{s+2}\left(\frac{2\theta}{(s+2)(s+3)}+\frac{2}{(s+1)(s+2)(s+3)}\right)\right.$$
$$\left.\left.\left.+\frac{2\theta}{(s+1)(s+2)}-\frac{2}{(s+1)(s+2)(s+3)}\right)|f''(b)|^q\right]^{\frac{1}{q}}\right\}$$

for $0\le\theta\le\frac{1}{2}$, *and*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\left[(1-\theta)f\left(\frac{a+b}{2}\right)+\theta\frac{f(a)+f(b)}{2}\right]\right| \tag{26}$$
$$\le\frac{(b-a)^2}{16}\left(\theta-\frac{1}{3}\right)^{1-\frac{1}{q}}\left\{\left[\left(\frac{2\theta}{s+2}-\frac{1}{s+3}\right)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.\right.$$
$$\left.+\left(\frac{2\theta}{(s+1)(s+2)}-\frac{2}{(s+1)(s+2)(s+3)}\right)|f''(a)|^q\right]^{\frac{1}{q}}$$
$$+\left[\left(\frac{2\theta}{s+2}-\frac{1}{s+3}\right)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right.$$
$$\left.\left.+\left(\frac{2\theta}{(s+1)(s+2)}-\frac{2}{(s+1)(s+2)(s+3)}\right)|f''(b)|^q\right]^{\frac{1}{q}}\right\}$$

for $\frac{1}{2}\le\theta\le 1$.

Proof. Set $x=\frac{a+b}{2}$ in Theorem 3. ∎

Remark 2. If we take $q=1$ in (25) and (26), then we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\left[(1-\theta)f\left(\frac{a+b}{2}\right)+\theta\frac{f(a)+f(b)}{2}\right]\right| \tag{27}$$

$$\leq \frac{(b-a)^2}{8}\left\{\left[\frac{1}{s+3} - \frac{2\theta}{s+2} + \frac{2(2\theta)^{s+3}}{(s+2)(s+3)}\right]\left|f''\left(\frac{a+b}{2}\right)\right| + \left[(1-2\theta)^{s+2}\left(\frac{2\theta}{(s+2)(s+3)} + \frac{2}{(s+1)(s+2)(s+3)}\right) + \frac{\theta}{(s+1)(s+2)} - \frac{1}{(s+1)(s+2)(s+3)}\right](|f''(a)| + |f''(b)|)\right\}$$

for $0 \leq \theta \leq \frac{1}{2}$, and

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \left[(1-\theta)f\left(\frac{a+b}{2}\right) + \theta\frac{f(a)+f(b)}{2}\right]\right| \leq \frac{(b-a)^2}{8}\left\{\left[\frac{2\theta}{s+2} - \frac{1}{s+3}\right]\left|f''\left(\frac{a+b}{2}\right)\right| + \left[\frac{\theta}{(s+1)(s+2)} - \frac{1}{(s+1)(s+2)(s+3)}\right](|f''(a)| + |f''(b)|)\right\} \tag{28}$$

for $\frac{1}{2} \leq \theta \leq 1$.

Remark 3. If we take $\theta = 0$ in (27), then we get a midpoint type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{8}\left[\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right| + \frac{1}{(s+1)(s+2)(s+3)}(|f''(a)| + |f''(b)|)\right]. \tag{29}$$

If we take $\theta = 1$ in (28), then we get a trapezoid type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)^2}{8}\left[\frac{s+4}{(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right| + \frac{1}{(s+1)(s+3)}(|f''(a)| + |f''(b)|)\right]. \tag{30}$$

If we take $\theta = \frac{1}{3}$ in (27), then we get a Simpson type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{(b-a)^2}{8}\left[\frac{1}{s+3} - \frac{2}{3(s+2)} + \frac{2}{(s+2)(s+3)}\left(\frac{2}{3}\right)^{s+3}\right]\left|f''\left(\frac{a+b}{2}\right)\right| + \left[\frac{2s+8}{3(s+1)(s+2)(s+3)}\left(\frac{1}{3}\right)^{3+2} + \frac{s}{3(s+1)(s+2)(s+3)}(|f''(a)| + |f''(b)|)\right]. \tag{31}$$

If we take $\theta = \frac{1}{2}$ in (27) or (28), then we get an averaged midpoint-trapezoid type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{(b-a)^2}{16(s+2)(s+3)}\left[|f''(a)| + 2\left|f''\left(\frac{a+b}{2}\right)\right| + |f''(b)|\right]. \tag{32}$$

Remark 4. If we put $M = \sup_{x\in[a,b]} |f''|$ in (29)–(32), then we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \leq \frac{M(s^2+3s+4)(b-a)^2}{8(s+1)(s+2)(s+3)}, \tag{33}$$

in which a misprint in Corollary 3 of [5] has been corrected,

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \leq \frac{M(s^2+7s+8)(b-a)^2}{8(s+1)(s+2)(s+3)}, \tag{34}$$

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| &\leq \frac{M(b-a)^2}{24(s+1)(s+2)(s+3)} \\ &\times \left[s^2 + 3s + (4s+16)\left(\frac{1}{3}\right)^{s+2} + 6(s+1)\left(\frac{2}{3}\right)^{s+3}\right],\end{aligned} \tag{35}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{M(b-a)^2}{4(s+2)(s+3)}. \tag{36}$$

Remark 5. If we further take $s = 1$ in (32)–(36), i.e., for functions f with convex $|f''|$, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \leq \frac{M(b-a)^2}{24}, \tag{37}$$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\right| \leq \frac{M(b-a)^2}{12}, \tag{38}$$

which recaptures a result in [4].

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{M(b-a)^2}{81}, \tag{39}$$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{M(b-a)^2}{48}. \tag{40}$$

Obviously, (37)–(40) indicate that the Simpson type inequality has the best error estimation for functions f with convex $|f''|$.

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