

Research Article

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Sasakian structures a foliated approach

DOI 10.1515/dema-2017-0008

Received March 21, 2016; accepted July 27, 2016.

Abstract: Recent renewed interest in Sasakian manifolds is due mainly to the fact that they can provide examples of generalized Einstein manifolds, manifolds which are of great interest in mathematical models of various aspects of physical phenomena. Sasakian manifolds are odd dimensional counterparts of Kählerian manifolds to which they are closely related. The paper presents a foliated approach to Sasakian manifolds on which the author gave several lectures. The paper concentrates on cohomological properties of Sasakian manifolds and of transversely holomorphic and Kählerian foliations. These properties permit to formulate obstructions to the existence of Sasakian structures on compact manifolds.

Keywords: Sasakian structure, Foliation, Basic cohomology, Contact metric structure, K-contact manifold

MSC: 53C25, 57R30

Recent renewed interest in Sasakian manifolds is due mainly to the fact that they can provide examples of generalized Einstein manifolds, manifolds which are of great interest in mathematical models of various aspects of physical phenomena. Sasakian manifolds are odd dimensional counterparts of Kählerian manifolds to which they are closely related. The book of Ch. Boyer and K. Galicki, *Sasakian Geometry*, is both the best introduction to the subject and at the same time it gathers state of the art information and results on these manifolds. However, although the authors are well aware that a Sasakian structure is a very special one-dimensional Riemannian foliation with Kählerian transverse structure, they use this fact only in a few very special cases.

The paper presents an approach to Sasakian manifolds on which the author gave several lectures, most recently at the *Workshop on almost hermitian and contact geometry* at the Banach Center in Będlewo in October 2015 and at University of the Basque Country in February 2016. The first lectures on the topic the author gave at Universidad de Sevilla in October 1988 and then presented the consequence for the geometry of Sasakian manifolds, in particular the relations between various curvatures and those of the transverse Kähler manifold. These results were published in several sections of [1] as well as in [2]. The most general theory of geometrical structures "adapted" to a foliation was presented in [3], see also [1]. The paper concentrates on cohomological properties of Sasakian manifolds and of transversely holomorphic and Kählerian foliations. These properties permit to formulate obstructions to the existence of Sasakian structures on compact manifolds. The presented results are due to the author as well as his former and present Ph.D. students.

1 Geometric structures on contact manifolds

In this paper we are going to work on odd dimensional smooth manifolds. Let M be a smooth connected manifold of dimension $m = 2n + 1$.

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Let η be a 1-form on M such that $\eta \wedge d\eta^n \neq 0$, i.e., it is a volume form of M . Then η is called a *contact form*. Two contact forms η and η' are said to be equivalent if there exists a smooth function f such that $\eta = f\eta'$. Such an equivalence class $[\eta]$ is called a *contact structure*. The pair (M, η) usually is called a *strict contact manifold*. A *contact manifold* is a smooth manifold M with a contact structure $[\eta]$.

On a strict contact manifold (M, η) there exists a unique vector field ξ , called the *Reeb vector field*, such that

$$\eta(\xi) = 1 \quad \wedge \quad i_\xi d\eta = 0.$$

This property ensures that the 1-dimensional foliation \mathcal{F}_ξ generated by the non-vanishing vector field ξ is transversely symplectic. This foliation is independent of the choice of the contact form η within the equivalence class, it is one of the objects we can associate to a contact manifold. On the contact manifold $(M, [\eta])$ we have the canonical splitting of the tangent bundle TM

$$TM = T\mathcal{F}_\xi \oplus D$$

where $D = \ker \eta$.

Geometers are more at home with a richer structure: *almost contact (manifold)*.

Definition 1.1. An *almost contact structure* on a smooth manifold M is a triple (ξ, η, ϕ) where

- i) ξ is a vector field on M ,
 - ii) η is a 1-form on M ,
 - iii) ϕ is an endomorphism of the tangent bundle TM
- such that

$$\eta(\xi) = 1, \quad \phi^2 = -id_{TM} + \xi \otimes \eta.$$

One can easily verify that $\phi(\xi) = 0$, $\eta\phi = 0$, and that the tangent bundle TM splits naturally into the direct sum $\mathcal{F}_\xi \oplus D$ where $D = \ker \eta = \text{im } \phi$.

The next step in the enrichment of the geometrical structure is a compatible Riemannian metric.

Definition 1.2. A Riemannian metric g is said to be *compatible* with an almost contact structure (ξ, η, ϕ) if for any vector fields X and Y

$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y).$$

Then the quadruple (g, ξ, η, ϕ) is called an *almost contact metric structure*.

Remark. The vector field ξ need not be a Killing vector field for the metric g even if the quadruple (g, ξ, η, ϕ) is an almost contact metric structure, cf. [4].

Combining the topological and geometrical structures we get the so-called *contact metric structure*.

Definition 1.3. An almost contact structure (ξ, η', ϕ) is said to be *compatible* with the strict contact structure (form) η if $\eta = \eta'$, ξ is its Reeb vector field, and for any vector fields X and Y

$$d\eta(\phi(X), \phi(Y)) = d\eta(X, Y) \quad \text{and} \quad d\eta(\phi(X), X) > 0, \quad X \in D, \quad X \neq 0.$$

Definition 1.4. A strict contact manifold (M, η) with a compatible almost contact metric structure (g, ξ, η', ϕ) such that for any two vector fields X and Y

$$g(X, \phi(Y)) = d\eta(X, Y)$$

is called a *contact metric structure*.

The Reeb vector field of a contact metric structure need not be Killing. If it is, the structure is called *K-contact*.

Definition 1.5. A contact metric structure (g, ξ, η', ϕ) on the manifold M is called *K-contact* if its Reeb vector field ξ is a Killing vector field of the Riemannian metric g . Then (M, g, ξ, η', ϕ) is called a *K-contact manifold*.

And finally, the most complex structure considered is the *Sasakian structure (manifold)*.

Definition 1.6. An almost contact structure (ξ, η', ϕ) on the manifold M is normal iff

$$N_\phi(X, Y) = [\phi(X), \phi(Y)] + \phi^2([X, Y]) - \phi([X, \phi(Y)]) - \phi([\phi(X), Y]) = -2\xi \otimes d\eta(X, Y)$$

for any vector field X and Y on M .

Definition 1.7. A K -contact manifold whose underlying almost contact structure is normal is called a *Sasakian manifold*.

2 Transverse properties of Sasakian manifolds

Let \mathcal{F} be a foliation on a Riemannian m -manifold (M, g) . Then \mathcal{F} is defined by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_{i,j \in I}$ modeled on a $2q$ -manifold N_0 such that

- (1) $\{U_i\}_{i \in I}$ is an open covering of M ,
- (2) $f_i : U_i \rightarrow N_0$ are submersions with connected fibres,
- (3) $g_{ij} : N_0 \rightarrow N_0$ are local diffeomorphisms of N_0 with $f_i = g_{ij} f_j$ on $U_i \cap U_j$.

The connected components of the trace of any leaf of \mathcal{F} on U_i consist of the fibres of f_i . The open subsets $N_i = f_i(U_i) \subset N_0$ form a q -manifold $N_{\mathcal{U}} = \sqcup N_i$, which can be considered as a transverse manifold of the foliation \mathcal{F} . The pseudogroup $\mathcal{H}_{\mathcal{U}}$ of local diffeomorphisms of N generated by g_{ij} is called the holonomy pseudogroup of the foliated manifold (M, \mathcal{F}) defined by the cocycle \mathcal{U} . Different cocycles can define the same foliation, then we have two different transverse manifolds and two holonomy pseudogroups. In fact, these two holonomy pseudogroups are equivalent in the sense of Haefliger, cf. [5].

According to Haefliger, cf. [6], a transverse property of a foliated manifold is a property of foliations which is shared by any two foliations with equivalent holonomy pseudogroup. For example, being Riemannian, transversely symplectic, transversely almost-complex, transversely Kähler, etc., is a transverse property. A Riemannian foliation, i.e., admitting a bundle-like metric, is defined by a cocycle \mathcal{U} modelled on a Riemannian manifold whose local submersions are Riemannian submersions. Then the associated transverse manifold $N_{\mathcal{U}}$ is Riemannian and the associated holonomy pseudogroup $\mathcal{H}_{\mathcal{U}}$ is a pseudogroup of local isometries. Any foliation defined by a cocycle \mathcal{V} whose holonomy pseudogroup $\mathcal{H}_{\mathcal{V}}$ is equivalent to $\mathcal{H}_{\mathcal{U}}$ is also Riemannian, as the equivalence of pseudogroups transports the Riemannian metric from $N_{\mathcal{U}}$ to $N_{\mathcal{V}}$ and ensures that the pseudogroup $\mathcal{H}_{\mathcal{V}}$ is a pseudogroup of local isometries of the transported metric. This metric can be lifted to a bundle-like metric (not unique) on the other foliated manifold making the second foliation Riemannian. The same procedure can be applied to any geometrical structure, for the discussion of this general procedure see [1, 3].

The space of $\mathcal{H}_{\mathcal{U}}$ -invariant k -forms on the manifold $N_{\mathcal{U}}$ can be identified with the space of foliated sections of the bundle $\bigwedge^k N(M, \mathcal{F})^*$ which in turn is isomorphic to the space of k basic forms

$$A^k(M, \mathcal{F}) = \{\alpha \in A^k(M) : i_X \alpha = i_X d\alpha = 0 \text{ for all vectors } X \in \mathcal{F}\}$$

The differential sends basic forms to basic forms and the cohomology of the complex $(A^*(M, \mathcal{F}), d)$ is called the *basic cohomology* of the foliated manifold (M, \mathcal{F}) . In the language of basic cohomology we can express a very important property of foliations.

Definition 2.1. A foliation \mathcal{F} on M is called *homologically orientable* if $H^{cod \mathcal{F}}(M, \mathcal{F}) = \mathbb{R}$.

For the discussion the meaning and importance of the condition see [7].

Let $\phi : U \rightarrow R^p \times R^q$, $\phi = (\phi^1, \phi^2) = (x_1, \dots, x_p, y_1, \dots, y_q)$ be an adapted chart on a foliated manifold (M, \mathcal{F}) . Then on U the vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p}$ span the bundle $T\mathcal{F}$ tangent to the leaves of the foliation \mathcal{F} , the equivalence classes denoted by $\frac{\bar{\partial}}{\partial y_1}, \dots, \frac{\bar{\partial}}{\partial y_q}$ of $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_q}$ span the normal bundle $N(M, \mathcal{F}) = TM/T\mathcal{F}$ which is isomorphic to the subbundle $T\mathcal{F}^\perp$.

All the definitions of Section 1 have been formulated in a purely geometrical way without any reference to the characteristic foliation. Let us look at the transverse structure of the characteristic foliation.

The characteristic foliations of a contact manifold $(M, [\eta])$ is transversely symplectic as the 2-form $d\eta$ is basic and defines a transverse symplectic form.

The basic cohomology class $[d\eta] \in H^2(M, \mathcal{F})$ is in the kernel of the natural mapping $H^2(M, \mathcal{F}) \rightarrow H^2(M)$, and $[d\eta]^n \in H^{2n}(M, \mathcal{F})$ is in the kernel of the natural mapping $H^{2n}(M, \mathcal{F}) \rightarrow H^{2n}(M)$. Therefore if transverse volume form $[d\eta]^n$ defines a non-zero basic cohomology class, then this $2n$ -form is in the kernel the natural mapping $H^{2n}(M, \mathcal{F}) \rightarrow H^{2n}(M)$, thus this mapping cannot be injective providing an obstruction to a transversely symplectic 1-dimensional foliation being the characteristic foliation of a contact structure.

For example, such a 1-dimensional foliation cannot admit a transverse foliation with a compact leaf, as then according to the result of Molino-Sergiescu this mapping should be injective, cf. Theorem 2 of [8].

In the case of an almost contact structure (ξ, η, ϕ) on a smooth manifold M , the following conditions are equivalent, cf. [9] or Lemma 6.3.3 of [4]:

- (1) there exists a Riemannian metric for which the orbits of ξ are geodesics,
- (2) $L_\xi d\eta = 0$,
- (3) $i_\xi d\eta = 0$.

The conditions (2) and (3) are evidently equivalent, and they just say that the 2-form $d\eta$ is basic.

In the case of a contact metric structure (g, ξ, η', ϕ) we have the following equivalent conditions (Proposition 6.4.8 of [4])

- i) the characteristic foliation is Riemannian for g ,
- ii) the metric g is bundle-like,
- iii) the vector field ξ is Killing,
- iv) the vector field ξ preserves the $(1,1)$ -tensor field ϕ , i.e., $L_\xi \phi = 0$,
- v) the contact metric structure (g, ξ, η', ϕ) is K-contact.

Therefore the characteristic foliation of a K-contact manifold is transversely almost-Kähler, and the characteristic foliation of a Sasakian manifold is transversely Kähler, [4] Theorem 7.1.3. However, even if the characteristic foliation of a K-contact manifold is transversely Kähler, it does not imply that the structure is Sasakian.

3 Transversely Kähler foliations

We have noticed that the characteristic foliation of a Sasakian manifold is transversely Kähler. In this section we will gather the results which are particular to transversely Kähler foliations and therefore are also true for the characteristic foliations of a Sasakian manifold. It will facilitate the search for characterizations of Sasakian manifolds and properties which can distinguish between K-contact and Sasakian manifolds.

Let \mathcal{F} be a foliation of dimension p and codimension $2q$ on a smooth manifold M of dimension $m = p + 2q$. It is a transversely Kähler foliation if there is a cocycle $\mathcal{U} = \{(U_i, f_i, g_{ij})\}_{i,j \in I}$ defining the foliation \mathcal{F} modelled on a Kähler manifold (N, g_N, J_N) such that the local diffeomorphisms g_{ij} of N are Kähler isometries, or equivalently that the associated holonomy pseudogroup $\mathcal{H}_{\mathcal{U}}$ is a pseudogroup of Kähler isometries of a Kähler structure on the transverse manifold $N_{\mathcal{U}}$.

We assume that the foliation \mathcal{F} is transversely holomorphic, of complex codimension q and that the manifold M is compact. Therefore on the normal bundle $N(M, \mathcal{F})$ of the foliation \mathcal{F} we have a foliated Kähler structure, i.e. a foliated Riemannian metric \bar{g} and an endomorphism \bar{J} of the normal bundle such that $\bar{J}^2 = -Id$ (an almost complex structure in the normal bundle) compatible with \bar{g} , i.e. for any $X, Y \in N(M, \mathcal{F})$

$$\bar{g}(\bar{J}(X), \bar{J}(Y)) = \bar{g}(X, Y)$$

and satisfying the "integrability" condition: for any X and Y

$$N_{\bar{J}}(X, Y) = [\bar{J}(X), \bar{J}(Y)] + \bar{J}^2([X, Y]) - \bar{J}([\bar{J}(X), Y]) - \bar{J}([X, \bar{J}(Y)]) = 0$$

as a section of $N(M, \mathcal{F})$. Then the fundamental Kähler 2-form $\bar{\Omega}(X, Y) = \bar{g}(X, \bar{J}Y)$ is basic and correspond to the Kähler form of the transverse Kähler structure of the foliation. For any point x of M there exists an adapted chart $(x_1, \dots, x_{m-2q}, z_1, \dots, z_q)$ modelled on $R^{m-2q} \times C^q$ defined on an open neighbourhood of x . Basic forms on (M, \mathcal{F}) are in one-to-one correspondence with holonomy invariant (\mathcal{H} -invariant) forms on the transverse manifold. Basic k -forms are just foliated sections of the k th exterior product of the conormal bundle $N(M, \mathcal{F})^*$ foliated by the natural lift of the foliation \mathcal{F} , cf. [3]. If the foliation \mathcal{F} is transversely holomorphic its normal bundle has a (almost) complex structure corresponding to the complex structure of the transverse manifold. Therefore any complex valued basic k -form can be represented as a sum of k -forms of pure type (r, s) corresponding to the decomposition of k -forms on the complex manifold N . It is possible as elements of the holonomy pseudogroup are bi-holomorphic local diffeomorphisms. This decomposition can be also obtained by considering the decomposition of sections of the complex bundle $\Lambda_C^k N(M, \mathcal{F})^*$. Then a basic k -form α is of pure type (r, s) if for any point of M there exists an adapted chart $(x_1, \dots, x_{m-2q}, z_1, \dots, z_q)$ such that

$$\alpha = \sum f_{IJ} dz_{i_1} \wedge \dots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_s}$$

where $1 \leq i_1 < \dots < i_r \leq q, 1 \leq j_1 < \dots < j_s \leq q, I = (i_1, \dots, i_r), J = (j_1, \dots, j_s)$.

Let us denote by $A_C^k(M, \mathcal{F})$ the space of complex valued basic k -forms on the foliated manifold (M, \mathcal{F}) , and by $A_C^{r,s}(M, \mathcal{F})$ the space of complex valued basic forms of pure type (r, s) . Then

$$A_C^k(M, \mathcal{F}) = \sum_{r+s=k} A_C^{r,s}(M, \mathcal{F})$$

for short

$$A^k = \sum_{r+s=k} A^{r,s}.$$

The exterior differential $d: A_C^k(M, \mathcal{F}) \rightarrow A_C^{k+1}(M, \mathcal{F})$ is decomposed into two components ∂ and $\bar{\partial}$ of bidegree $(1, 0)$ and $(0, 1)$, correspondingly,

$$\partial: A^{r,s} \rightarrow A^{r+1,s} \quad \text{and} \quad \bar{\partial}: A^{r,s} \rightarrow A^{r,s+1}.$$

The basic cohomology of transversely holomorphic and transversely Kähler foliations was studied in depth by A. El Kacimi-Alaoui, cf. [10]. We recall some basic results from this paper.

We assume that the foliation is transversely Hermitian. The operator

$$*: A^k(M, \mathcal{F}) \rightarrow A^{2q-k}(M, \mathcal{F})$$

defined in [11] using the transverse part of the bundle-like metric, and the corresponding standard $*$ operator on the level of the transverse manifold, can be extended to an operator

$$\bar{*}: A_C^k(M, \mathcal{F}) \rightarrow A_C^{2q-k}(M, \mathcal{F}).$$

The normal part of the bundle-like metric g , or the corresponding transverse metric, defines a Riemannian (Hermitian) metric g^k on the bundle $\Lambda_C^k N(M, \mathcal{F})^*$, and therefore we can define a scalar product on

$$A_C^*(M, \mathcal{F}) = \sum A_C^k(M, \mathcal{F})$$

as follows, [7]:

if $\alpha \in A_C^k(M, \mathcal{F}), \beta \in A_C^l(M, \mathcal{F})$, and $k \neq l$

$$\langle \alpha, \beta \rangle = 0$$

if $\alpha \in A_C^k(M, \mathcal{F})$, and $\beta \in A_C^k(M, \mathcal{F})$,

$$\langle \alpha, \beta \rangle = \int_M g^k(\alpha, \beta)$$

The operator $\delta: A_C^k(M, \mathcal{F}) \rightarrow A_C^{k-1}(M, \mathcal{F})$ defined as $\delta = \bar{*}d\bar{*}$ is the adjoint operator of d with respect to the scalar product \langle, \rangle .

Following the classical (manifold) case we define the "foliated" Laplacian

$$\Delta = d\delta + \delta d.$$

The foliated Laplacian sends basic forms into basic forms, it is a self-adjoint foliated (transversely) elliptic operator, cf. [10].

We can also define basic Dolbeault cohomology of the foliated manifold (M, \mathcal{F}) . For a fixed r , $0 \leq r \leq q$, consider the differential complex:

$$0 \rightarrow A^{r,0} \xrightarrow{\bar{\partial}} A^{r,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} A^{r,q} \xrightarrow{\bar{\partial}} 0.$$

Its cohomology is called the basic Dolbeault cohomology of the foliated manifold (M, \mathcal{F}) , and denoted

$$H^{r,*}(M, \mathcal{F}) = \Sigma_{s=0}^q H^{r,s}(M, \mathcal{F}).$$

The operator $\bar{*}$ induces an isomorphism $\bar{*}: A^{r,s} \rightarrow A^{q-r,q-s}$. Using the same procedure as for the operator δ we define an operator $\bar{\delta}$ by the formula

$$\bar{\delta} = -\bar{*}\bar{\partial}\bar{*}.$$

The operator $\bar{\delta}$ is the adjoint of $\bar{\partial}$ with respect to the just defined scalar product. Moreover, the operator

$$\Delta'' = \bar{\partial}\bar{\delta} + \bar{\delta}\bar{\partial}$$

is a self-adjoint foliated (transversely) elliptic operator.

In the case of transversely Kähler foliations we can say much more about the basic cohomology and operators just defined.

The Kähler form of the transverse manifold N corresponds to a basic $(1, 1)$ -form on (M, \mathcal{F}) which we call the (transverse) Kähler form of the foliated manifold. Using this form we define the L operator

$$L: A_C^k(M, \mathcal{F}) \rightarrow A_C^{k+2}(M, \mathcal{F})$$

$$L\alpha = \alpha \wedge \omega.$$

Its adjoint with respect to \langle, \rangle is $\Lambda = -\bar{*}L\bar{*}$.

For transversely Kähler foliations on compact manifolds we have the following relations:

$$\Lambda\bar{\partial} - \bar{\partial}\Lambda = -\sqrt{-1}\bar{\delta},$$

$$\Lambda\bar{\delta} - \bar{\delta}\Lambda = -\sqrt{-1}\bar{\partial},$$

$$\bar{\partial}\bar{\delta} + \bar{\delta}\bar{\partial} = \bar{\delta}\bar{\partial} + \bar{\partial}\bar{\delta} = 0,$$

$$\Delta = 2\Delta'',$$

$$\Delta L = L\Delta, \quad \Delta\Lambda = \Lambda\Delta.$$

These identities permitted A. ElKacimi Alaoui to prove the following theorem, cf. [10].

Theorem 3.1. *Let \mathcal{F} be a transversely Kähler foliation on a compact manifold M . If \mathcal{F} is homologically oriented, then*

i) *a basic k -form $\alpha = \Sigma_{r+s=k} \alpha_{r,s}$, $\alpha_{r,s} \in A^{r,s}$, is harmonic if and only if the forms $\alpha_{r,s}$ are harmonic, thus*

$$H_C^k(M, \mathcal{F}) \cong \Sigma_{r+s=k} H^{r,s}(M, \mathcal{F}).$$

ii) *the conjugation induces isomorphisms*

$$H^{r,s}(M, \mathcal{F}) \cong H^{s,r}(M, \mathcal{F}).$$

iii) *for any $0 \leq r \leq q$, the form ω^r is harmonic, thus $H^{r,r}(M, \mathcal{F}) \neq 0$.*

The complex $A^* = \sum_{r,s} A^{r,s}$ of complex valued basic forms can be filtered by

$$F^k A = \sum_{r \geq k} A^{r,*}.$$

The filtration is compatible with the bigradation of the complex. Therefore we can define the associated spectral sequence which is called the basic Frölicher spectral sequence of the transversely holomorphic foliation \mathcal{F} , cf. [12]. It converges to the complex basic cohomology of the foliated manifold (M, \mathcal{F}) .

The terms $E_1^{r,s}$ are just the basic (r, s) -Dolbeault cohomology groups. If the foliation \mathcal{F} is homologically oriented and transversely Kähler, then it is a simple consequence of Theorem 1 that the Dolbeault spectral sequence collapses at the first term, cf. Theorem 2 of [12]. Indeed, the Hodge theorem combined with the just mentioned theorem ensures that

$$E_1^{r,s} \cong \mathcal{H}^{r,s}(M, \mathcal{F})$$

where $\mathcal{H}^{r,s}(M, \mathcal{F})$ is the space of (r, s) -pure basic harmonic forms. As harmonic forms are closed the operator d_1 is trivial (vanishes).

In [12] the authors noticed that the so called dd^c -lemma for Kähler manifolds is an algebraic consequence of several identities. These identities have their counterparts for the basic cohomology of the transversely Kähler foliation on a compact manifold so the dd^c -lemma is also true for the basic cohomology of a transversely Kähler foliation on a compact manifold. On the other hand this lemma is the key element of the proof of the formality of the cohomology of a compact Kähler manifold, [13]. Therefore retracing the steps of the original proof we obtain

Theorem 3.2. *Let \mathcal{F} be a transversely Kähler foliation on a compact manifold M . If \mathcal{F} is homologically oriented then the minimal model of the complex basic cohomology of \mathcal{F} is formal and thus Massey products of complex valued basic forms vanish.*

4 Obstructions to existence of Sasakian structures

We have remarked that the characteristic foliation of the Sasakian manifold is transversely Kähler. Therefore we have a 1-dimensional (tangentially) orientable foliation with a very sophisticated transverse structure. Moreover, the normality condition is not a *transverse property* as its formulation involves vectors tangent to leaves of the foliation, in particular the characteristic vector field ξ . The corresponding transverse property can be formulated as follows:

Let $\bar{J}: N(M, \mathcal{F}) \rightarrow N(M, \mathcal{F})$ be the endomorphism of the normal bundle defined for any tangent vector X as

$$\bar{J}(\bar{X}) = \overline{\phi(X)}$$

where \bar{X} is the vector in the normal bundle corresponding to a tangent vector X . The endomorphism \bar{J} is well defined and $\bar{J}^2 = -id$. Therefore it is an almost complex structure in the normal bundle.

The vector field ξ acts on the normal bundle, and therefore on the endomorphism \bar{J} . It is a foliated endomorphism iff

$$L_\xi \bar{J} = 0,$$

i.e. iff $L_\xi \bar{J}(\bar{X}) = 0 = \overline{[\xi, \phi(X)]} - \overline{\phi([\xi, X])}$ for any foliated section of the bundle $N(M, \mathcal{F})$. Then \bar{J} corresponds to an almost complex structure J on the transverse manifold of the characteristic foliation. The normality condition insures that J is integrable (i.e., the Nijenhuis tensor $N_J = 0$). However, the normality condition is stronger, the equality $N_J = 0$ equivalent to $N_{\bar{J}} = 0$, and thus to the fact that for any sections X, Y of D , $N_\phi(X, Y)$ is a vector field tangent to the characteristic foliation \mathcal{F}_ξ , i.e., of the form $h\xi$ for some smooth function h on M , but not necessarily $2d\eta(X, Y)$ as requires the normality condition.

Therefore having given a 1-dimensional foliation we can ask many questions like:

Is this foliation Riemannian, (transversely) Hermitian, transversely symplectic, transversely holomorphic, transversely Kähler?

These questions are about the transverse structure of the foliation and can be answered in the language of transverse properties, so the basic cohomology can provide some obstructions to the existence of such structures.

It is not difficult to see that a 1-dimensional transversely Kähler foliation admits a contact metric structure in the sense that it is the characteristic foliation of this structure. If the manifold M is compact, the non-triviality of the top dimensional basic cohomology ensures that one can modify the Riemannian metric to ensure that the foliation is Riemannian and minimal, i.e. generated by a Killing vector field.

Let ξ be a non-vanishing vector field on the manifold M . Assume that the foliation \mathcal{F}_ξ generated by ξ is transversely Kähler. Therefore on the transverse manifold N of the foliated manifold (M, \mathcal{F}_ξ) there exists a holonomy invariant Kähler structure (\hat{g}, \hat{J}) , i.e. \hat{g} is a Riemannian metric, \hat{J} a complex structure, and for any tangent vectors X, Y of N

$$\hat{g}(\hat{J}(X), \hat{J}(Y)) = g(X, Y) \quad \text{and} \quad \hat{\Omega}(X, Y) = \hat{g}(X, \hat{J}(Y)) \quad \text{is a closed 2-form.}$$

We can lift the Riemannian metric \hat{g} to a Riemannian metric \bar{g} in the normal bundle by the formula

$$\bar{g}_y(\bar{X}, \bar{Y}) = \hat{g}_{f_i(y)}(df_i(\bar{X}), df_i(\bar{Y}))$$

where $f_i: U_i \rightarrow N$ is a submersion from a cocycle defining the foliation \mathcal{F}_ξ , $\bar{X}, \bar{Y} \in N(M, \mathcal{F}_\xi)_y$, and $y \in U_i$. The complex structure is lifted to an almost complex structure \bar{J} in the normal bundle in a similar way:

$$df_i(\bar{J}_y(\bar{X})) = \hat{J}_{f_i(y)}(df_i(\bar{X})).$$

Then the associated 2-form $\bar{\Omega}(\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \bar{J}\bar{Y})$ is (locally) the pull-back of the Kähler 2-form $\hat{\Omega}$, i.e.

$$f_i^* \hat{\Omega} = \bar{\Omega}$$

for any submersion f_i from the cocycle defining the foliation \mathcal{F}_ξ . Next choose a supplementary subbundle D to the foliation, which as a vector bundle is isomorphic to the normal bundle, and define the Riemannian metric g on M as follows: the subbundles $T\mathcal{F}_\xi$ and D are orthogonal, $g(\xi, \xi) = 1$, and transport \bar{g} via the isomorphism to D .

The tensor field ϕ is defined in a similar fashion: for vectors from the subbundle D we define ϕ as the pull-back of \bar{J} via the isomorphism from the normal bundle, and $\phi(\xi) = 0$.

Let us define the 1-form η as

$$\eta(X) = g(\xi, X).$$

Then, obviously, the triple (ξ, η, ϕ) is an almost contact structure on the manifold M . Let X, Y be any vectors on M . Taking into account the splitting $T\mathcal{F}_\xi \oplus D$ we can write $X = a_X \xi + \bar{X}$ and $Y = a_Y \xi + \bar{Y}$. Thus

$$\begin{aligned} g(X, Y) &= g(\bar{X}, \bar{Y}) + g(a_X \xi, a_Y \xi) = g(\bar{X}, \bar{Y}) + a_X a_Y = g(\bar{X}, \bar{Y}) + \eta(X)\eta(Y) \\ &= g(\phi(\bar{X}), \phi(\bar{Y})) + \eta(X)\eta(Y) = g(\phi(X), \phi(Y)) + \eta(X)\eta(Y) \end{aligned}$$

as the metric \bar{g} is \bar{J} -invariant. Therefore the quadruple (g, ξ, η, ϕ) is an almost contact metric structure.

Assume that the vector field ξ is the characteristic vector field of a (strict) contact structure η on the manifold M . The 2-form $d\eta$ is basic and defines a foliated symplectic form, so it projects to a symplectic form $\hat{\Omega}$ on the transverse manifold N . Assume that the 2-form $\hat{\Omega}$ is the Kähler form of the transverse Kähler structure (\hat{g}, \hat{J}) . Take $D = \ker \eta$. Then it is not difficult to verify that (g, ξ, η, ϕ) is a contact metric structure, as $\hat{\Omega}(X, Y) = \hat{\Omega}(\hat{J}(X), \hat{J}(Y))$ and $\hat{\Omega}(X, Y) = \hat{g}(X, \hat{J}(Y))$.

This equality when lifted to the foliated manifold (M, \mathcal{F}_ξ) reads

$$g(X, \phi(Y)) = d\eta(X, Y)$$

The fact that the Kähler form $\hat{\Omega}$ is \hat{J} invariant on the foliated manifold (M, \mathcal{F}_ξ) reads as

$$d\eta(\phi(X), \phi(Y)) = d\eta(X, Y).$$

The condition $d\eta(\phi(X), X) > 0$ for $0 \neq X \in D$, translates itself on the level of the transverse manifold to $\hat{\Omega}(\hat{J}(X), X) > 0$ which follows immediately from the definition of the form $\Omega: \Omega(\hat{J}(X), X) = \hat{g}(\hat{J}(X), \hat{J}(X)) > 0$ provided that $X \neq 0$. Therefore strict contact structure η whose characteristic foliation is transversely Kähler admits a contact metric structure whose 1-form is the contact form η .

If the manifold M is compact and its characteristic foliation homologically oriented, the foliated symplectic form $\omega = d\eta$ is basic and the $2n$ -basic form ω^n defines a non-zero $2n$ -basic cohomology class, so the characteristic foliation is taut, cf. [14]. Therefore we can modify the bundle-like metric g along the tangent bundle to the characteristic foliation to a Riemannian metric g' making the characteristic foliation minimal, i.e. the tangent vector field of unit length in the metric g' is Killing. The modification of the metric did preserve the splitting of the tangent bundle. Therefore on the contact manifold M we have a K-contact structure (g', ξ', η', ϕ) whose characteristic foliation is the same \mathcal{F}_ξ .

These considerations can be summed up by the following statement

No transverse property can distinguish K-contact manifolds from Sasakian manifolds.

Transverse properties can only say that a given foliation is not transversely Kähler. However, the characteristic foliation of a K-contact manifold can be transversely Kähler without the structure itself being Sasakian.

Thus if we want to prove that a given K-contact structure on a compact manifold is not Sasakian (i.e. it is not normal) we have to look for some properties which are not transverse, e.g., it is useless to study properties of the basic cohomology of the characteristic foliation.

The Sasakian version of the Hard Lefschetz Theorem proved by B. Cappelletti Montana et al. provides precisely a true obstruction to "being Sasakian," cf. [15]

Theorem 4.1. *Let (M, g, ξ, η, ϕ) be a compact connected Sasaki manifold of dimension $m=2n+1$. Then for any $0 \leq p \leq n$ the multiplication by the form $\eta \wedge d\eta^p$ induces an isomorphism between $H^{n-p}(M)$ and $H^{n+p+1}(M)$.*

To complement this result the authors constructed two nilmanifolds of dimension 5 and 7, respectively, which are K-contact but do not admit any Sasakian structure. To prove that they use the properties of the cohomology ring which can be derived from the Hard Lefschetz Theorem, cf. [16].

The theorem coupled with these examples demonstrates that the Hard Lefschetz property is an obstruction to being Sasakian for compact manifolds.

5 Other cohomology theories

In search for more cohomological obstructions one can turn to other cohomology theories which have been developed for complex manifolds, for the most recent and up-to-date information see [17]. The foliated versions of several of these cohomology theories have been defined and studied by P. Rażny, cf. [18], a Ph.D. student at the Jagiellonian University.

5.1 Basic Bott-Chern cohomology of foliations

Let M be a manifold of dimension $m = p + 2q$, endowed with a transversely Hermitian (i.e. transversely holomorphic, possessing a transverse Hermitian metric) foliation \mathcal{F} of complex codimension q . We can define the basic de Rham complex (denoted $A_C^*(M, \mathcal{F})$) as the subcomplex of the standard de Rham complex of M consisting of basic forms. As in the manifold case the transversely holomorphic structure induces a decomposition of the cotangent spaces into forms of type $(0,1)$ and $(1,0)$, cf. Section 3. The basic Bott-Chern cohomology of \mathcal{F} is defined as

$$H_{BC}^{*,*}(M, \mathcal{F}) := \frac{\text{Ker } \partial \cap \text{Ker } \bar{\partial}}{\text{Im } \partial \bar{\partial}}$$

Remark. *Complex conjugation induces an antilinear isomorphism:*

$$H_{BC}^{p,q}(M, \mathcal{F}) \rightarrow H_{BC}^{q,p}(M, \mathcal{F})$$

in particular:

$$\dim_{\mathbb{C}} H_{BC}^{p,q}(M, \mathcal{F}) = \dim_{\mathbb{C}} H_{BC}^{q,p}(M, \mathcal{F})$$

P. Ražny proves a decomposition theorem for basic Bott-Chern cohomology using the operator:

$$\Delta_{BC} := \partial\bar{\partial}(\partial\bar{\partial})^* + (\partial\bar{\partial})^*\partial\bar{\partial} + \bar{\partial}^*\partial(\bar{\partial}^*\partial)^* + (\bar{\partial}^*\partial)^*\bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} + \partial^*\partial$$

where ∂^* and $\bar{\partial}^*$ are the adjoint operators to ∂ and $\bar{\partial}$, respectively, with respect to the Hermitian product, defined by the transverse Hermitian structure, as defined in [10]. He notices that the operator Δ_{BC} is transversely elliptic and self-adjoint. To prove ellipticity he uses the fact that the operator projects, on the local quotient manifold, to the manifold version of the Δ_{BC} operator, which is elliptic, cf. [19].

Theorem 5.1 (Decomposition of the basic Bott-Chern cohomology). *If M is a compact manifold, endowed with a transversely Hermitian foliation \mathcal{F} , then we have the following decomposition*

$$A^{*,*}(M, \mathcal{F}) = \text{Ker}\Delta_{BC} \oplus \text{Im}\partial\bar{\partial} \oplus (\text{Im}\partial^* + \text{Im}\bar{\partial}^*)$$

In particular,

$$H_{BC}^{*,*}(M, \mathcal{F}) \cong \text{Ker}\Delta_{BC}$$

and the dimension of $H_{BC}^{*,*}(M, \mathcal{F})$ is finite.

5.2 Basic Aeppli cohomology of foliations

We define the basic Aeppli cohomology of \mathcal{F} as

$$H_A^{*,*}(M, \mathcal{F}) := \frac{\text{Ker}\partial\bar{\partial}}{\text{Im}\partial + \text{Im}\bar{\partial}}$$

Remark. As in the Bott-Chern case complex conjugation induces an antilinear isomorphism

$$H_A^{p,q}(M, \mathcal{F}) \rightarrow H_A^{q,p}(M, \mathcal{F})$$

To obtain a decomposition theorem for the basic Aeppli cohomology of \mathcal{F} we define a basic self-adjoint, transversely elliptic differential operator Δ_A

$$\Delta_A := \partial\partial^* + \bar{\partial}\bar{\partial}^* + (\partial\bar{\partial})^*\partial\bar{\partial} + \partial\bar{\partial}(\partial\bar{\partial})^* + (\bar{\partial}\partial^*)^*\bar{\partial}\partial^* + \bar{\partial}\partial^*(\bar{\partial}\partial^*)^*$$

and thus we have

Theorem 5.2 (Decomposition of basic Aeppli cohomology). *Let M be a compact manifold, endowed with a Hermitian foliation \mathcal{F} . Then we have the following decomposition*

$$A^{*,*}(M, \mathcal{F}) = \text{Ker}\Delta_A \oplus (\text{Im}\partial + \text{Im}\bar{\partial}) \oplus \text{Im}(\partial\bar{\partial})^*$$

In particular, there is an isomorphism

$$H_A^{*,*}(M, \mathcal{F}) \cong \text{Ker}(\Delta_A)$$

and the dimension of $H_A^{*,*}(M, \mathcal{F})$ is finite.

A duality theorem for basic Bott-Chern and Aeppli cohomology is also true, but we have to assume that our foliation is homologically orientable.

Remark. The above condition guaranties, that the following equalities hold for basic r -forms:

$$\partial^* = (-1)^r * \partial *, \quad \bar{\partial}^* = (-1)^r * \bar{\partial} *$$

Corollary 5.3. *If M is a compact manifold endowed with a Hermitian, homologically orientable foliation \mathcal{F} , then the transverse star operator induces an isomorphism*

$$H_{BC}^{p,q}(M, \mathcal{F}) \rightarrow H_A^{n-p, n-q}(M, \mathcal{F})$$

The theorem below is the main result concerning the basic Bott-Chern and Aeppli cohomologies proved in [18].

Theorem 5.4 (Basic Frölicher-type inequality). *Let M be a manifold of dimension n , endowed with a transversely holomorphic foliation \mathcal{F} of complex codimension q . Let us assume that the basic Dolbeault cohomology of \mathcal{F} are finitely dimensional. Then, for every $k \in \mathbb{N}$, the following inequality holds*

$$\sum_{p+q=k} \dim_{\mathbb{C}} H_{BC}^{p,q}(M, \mathcal{F}) + \dim_{\mathbb{C}} H_A^{p,q}(M, \mathcal{F}) \geq 2 \dim_{\mathbb{C}} H^k(M, \mathcal{F}, \mathbb{C})$$

Furthermore, the equality holds for every $k \in \mathbb{N}$, iff \mathcal{F} satisfies the $\partial\bar{\partial}$ -lemma (i.e. it's basic Dolbeault double complex satisfies the $\partial\bar{\partial}$ -lemma).

In the case when \mathcal{F} is a transversely Hermitian foliation on a closed manifold M , we get the following corollary:

Corollary 5.5. *Let F be a transversely Hermitian, homologically orientable foliation on a closed manifold M . Then for all $k \in \mathbb{N}$ the following inequality holds*

$$\sum_{p+q=k} \dim_{\mathbb{C}} H_{BC}^{p,q}(M, \mathcal{F}) + \dim_{\mathbb{C}} H_A^{p,q}(M, \mathcal{F}) \geq 2 \dim_{\mathbb{C}} H^k(M, \mathcal{F}, \mathbb{C})$$

Furthermore, the equality holds for every $k \in \mathbb{N}$, iff \mathcal{F} satisfies the $\partial\bar{\partial}$ -lemma (i.e. it's basic Dolbeault double complex satisfies the $\partial\bar{\partial}$ -lemma).

These properties of new basic cohomology theories provide new tools to distinguish various transverse structures of Riemannian and holomorphic foliations. They can be used to check whether a given Riemannian foliation admits a rich transverse holomorphic structure, in particular whether it is transversely Kähler.

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