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On the approximation of integrable functions by some special matrix means of Fourier series

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Abstract: The results concerninig pointwise approximation and product of summability methods corresponding to the theorems of Xh. Z. Krasniqi [Poincare J. Anal. Appl., 2014, 1, 1-8] and W. Łenski and B. Szal [Math. Slovaca, 2016, 66(6), 1-12] are generalized.
Some special cases are also formulated as corollaries.

Keywords: Degree of approximation, Fourier series, matrix means

MSC: 42A24

1 Introduction

Let L^p $(1 \leq p < \infty)$ be the class of all 2π–periodic real–valued functions, integrable in the Lebesgue sense, with p–th power over $Q = [-\pi, \pi]$ with the norm

$$\|f(\cdot)\|_{L^p} := \left(\int_Q | f(t) |^p \, dt \right)^{1/p} \quad \text{when } 1 \leq p < \infty.$$

Consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

and denote by

$$S_k f(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{k} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

the k-th partial sums of Sf.

Let $A := (a_{n,k})$ and $B := (b_{n,k})$ be infinite lower-triangular matrices with real entries, such that

$$\begin{aligned} a_{n,-1} &= 0, \ a_{n,k} \geq 0 \ \text{ and } \ b_{n,k} \geq 0 \quad \text{when } k = 0, 1, 2, \ldots n, \\ a_{n,k} &= 0 \ \text{ and } \ b_{n,k} = 0 \quad \text{when } k > n\,, \end{aligned} \tag{1}$$

$$\sum_{k=0}^{n} a_{n,k} = 1 \quad \text{and} \quad \sum_{k=0}^{n} b_{n,k} = 1, \quad \text{where } n = 0, 1, 2, \ldots \tag{2}$$

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and let, for $m = 0, 1, 2, .., n$,

$$A_{n,m} = \sum_{k=0}^{m} a_{n,k} \quad \text{and} \quad \overline{A}_{n,m} = \sum_{k=m}^{n} a_{n,k}.$$

Let the AB–transformation of $(S_k f)$ be given by

$$T_{n,A,B} f(x) := \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} S_k f(x) \qquad (n = 0, 1, 2, \ldots).$$

Following L. Leindler [1] a sequence $c := (c_r)$ of nonnegative numbers tending to zero is called the *Mean Rest Variation Sequence*, or briefly $c \in MRBVS$, if it has the property

$$\sum_{r=m}^{\infty} |c_r - c_{r+1}| \le K(c) \frac{1}{m+1} \sum_{m \ge r \ge m/2} c_r, \tag{3}$$

for every positive integer m.

Similarly, following W. Łenski and B. Szal [2], a sequence $c := (c_r)$ of nonnegative numbers will be called the *Mean Head Bounded Variation Sequence*, or briefly $c \in MHBVS$, if it has the property

$$\sum_{r=0}^{n-m-1} |c_r - c_{r+1}| \le K(c) \frac{1}{m+1} \sum_{r=n-m}^{n} c_r, \tag{4}$$

for all positive integers $m < n$, where the sequence c has finitely many nonzero terms and the last nonzero term is c_n.

Moreover, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K, such that

$$0 \le K(\alpha_n) \le K$$

holds for every n, where $K(\alpha_n)$ denote the constants appearing in the inequalities (3) or (4) for the sequences $\alpha_n = (a_{n,r})_{r=0}^{n}$, $n = 0, 1, 2, \ldots$.

Next, we assume that for every n and $0 \le m < n$

$$\sum_{r=m}^{n-1} |a_{n,r} - a_{n,r+1}| \le K \frac{1}{m+1} \sum_{m \ge r \ge m/2}^{m} a_{n,r}$$

or

$$\sum_{r=0}^{n-m-1} |a_{n,r} - a_{n,r+1}| \le K \frac{1}{m+1} \sum_{r=n-m}^{n} a_{n,r}$$

hold if $(a_{n,r})_{r=0}^{n}$ belongs to $MRBVS$ or $MHBVS$, for $n = 1, 2, \ldots$, respectively.

Let

$$|A|_{n,m} = \begin{cases} A_{n,m}, & \text{when } (a_{n,r})_{r=0}^{n} \in MRBVS, \\ \overline{A}_{n,n-m}, & \text{when } (a_{n,r})_{r=0}^{n} \in MHBVS \end{cases}$$

and

$$\Phi_x(t) = \int_0^t |\varphi_x(u)| \, du,$$

where

$$\varphi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

As a measure of pointwise approximation of a function f by $T_{n,A,B} f$ we will use the generalized pointwise modulus of continuity of f defined by the formula

$$\omega_x[f, \beta](t) := \frac{1}{t} \beta\left(\frac{\pi}{t}\right) \Phi_x(t) = \frac{1}{t} \beta\left(\frac{\pi}{t}\right) \int_0^t |\varphi_x(u)| \, du, \tag{5}$$

where β is a positive and non-decreasing function of t.

The deviation $T_{n,A,B}f(\cdot) - f(\cdot)$ with lower-triangular infinite matrices A and B was investigated among others by W. Łenski and B. Szal [3], Xh. Z. Krasniqi [4], R. Kranz and A. Rzepka [5]. In this paper the results corresponding to the following theorems of Xh. Z. Krasniqi [4] and W. Łenski and B. Szal [2] are shown.

The series $Sf(x)$ with partial sums $S_jf(x)$ is said to be *summable to s by A(E,r)-means* when the transformation

$$T_{n,A,(E,r)}f(x) := \sum_{\nu=0}^{n} \frac{a_{n,\nu}}{(1+r)^{\nu}} \sum_{j=0}^{\nu} \binom{\nu}{j} r^{\nu-j} S_jf(x), \qquad (r>0,\ n=0,1,2,\ldots),$$

tends to s, as $n \longrightarrow \infty$, where s is a finite number.

Theorem A [4] *Let $A := (a_{n,k})$ be a positive lower-triangular matrix, such that $\sum_{k=0}^{n} a_{n,k} = 1$ and $(a_{n,k})_{k=0}^{n}$ is a non-increasing sequence, ($n = 0,1,2,\ldots$). Let $\beta(t)$ be a positive and non-decreasing function of t. If*

$$\Phi_x(t) = o_x\left(\frac{t}{\beta\left(\frac{1}{t}\right)}\right), \text{ as } t \to 0^+,$$

and $\beta(t) \to \infty$, as $t \to \infty$, then a sufficient condition for the Fourier series $Sf(x)$ to be summable to $f(x)$ by $A(E,r)$ means is

$$\int_1^n \frac{A_{n,[u]}}{u\beta(u)} du = O(1), \text{ as } n \to \infty. \tag{6}$$

A lower-triangular matrix $C = (c_{n,k})$ is called a *maximal hump matrix* if , for each n, there exists an integer $k_0 = k_0(n)$, such that $(c_{n,k})_{k=0}^{k_0-1}$ is non-decreasing for $0 \le k < k_0$, and $(c_{n,k})_{k=k_0}^{n}$ is non-increasing for $k_0 \le k \le n$. Denote by

$$w_x^p f(\delta)_\beta = \begin{cases} \left(\frac{1}{\delta}\int_0^\delta |\,\varphi_x(u)\sin^\beta \frac{u}{2}\,|^p\, du\right)^{1/p} & \text{when } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{0<u\le\delta} |\,\varphi_x(u)\sin^\beta \frac{u}{2}\,| & \text{when } p = \infty. \end{cases}$$

Theorem B [2] *Let $f \in L^p$ with $1 \le p \le \infty$. If matrix A is a maximal hump matrix with $k_0^{-1} = O(n^{-1})$, such that $(a_{n,r})_{r=0}^{n} \in MRBVS$ and*

$$\left|\sum_{r=\mu}^{\nu}\sum_{k=0}^{r} b_{r,k}\sin\frac{(2k+1)t}{2}\right| \ll \tau,$$

for $0 \le \mu \le \nu$ and $\tau = \left[\frac{\pi}{t}\right]$, with $t \in \left[\frac{\pi}{n+1}, \pi\right]$ *, when $n = 1,2,\ldots$, then*

$$\left|T_{n,A,B}f(x) - f(x)\right| = O_x\left((n+1)^\beta\left[w_x^p f\left(\frac{\pi}{n+1}\right)_\beta + \frac{1}{n+1}\sum_{k=0}^{n} w_x^1 f\left(\frac{\pi}{k+1}\right)_\beta\right]\right)$$

for almost all considered x and $0 \le \beta < 1 - \frac{1}{p}$, when $p > 1$, and $\beta = 0$, when $p = 1$.

We shall write $J_1 \ll J_2$, if there exists a positive constant C, sometimes depending on some parameters, such that $J_1 \le CJ_2$.

2 Statement of the results

Let w_x be a positive function of modulus continuity type, i.e., $w_x(0) = 0$, $w_x(t_1) \le w_x(t_2) \le w_x(t_1+t_2) \le w_x(t_1) + w_x(t_2)$, for any $0 \le t_1 \le t_2 \le t_1 + t_2 \le 2\pi$.

Now we can formulate our main results:

Theorem 1. *Let $f \in L^p$ $(1 \le p < \infty)$ and let the entries of the matrices $A = (a_{n,r})$ and $B = (b_{r,k})$ satisfy the conditions (1) and (2). Additionally, let $(b_{r,k})$ be such that*

$$\left|\sum_{r=\mu}^{\nu}\sum_{k=0}^{r} b_{r,k}\sin\frac{(2k+1)t}{2}\right| \ll \tau \tag{7}$$

holds, for $0 \le \mu \le \nu \le n$, *and* $\tau = \left[\frac{\pi}{t}\right]$, *with* $t \in \left[\frac{\pi}{n+1}, \pi\right]$, *when* $n = 1, 2, \ldots$

If a sequence $(a_{n,r})_{r=0}^{n} \in MRBVS \cup MHBVS$,

$$|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right), \tag{8}$$

and for some x

$$\frac{1}{t}\beta\left(\frac{\pi}{t}\right)\int\limits_{0}^{t}|\varphi_x(u)|\,du = O(w_x(t)), \tag{9}$$

where w_x *has a continuous derivative,* $\beta(t)$ *and* $t\beta(\frac{\pi}{t})$ *are positive and non-decreasing functions of* t, *then*

$$\left|T_{n,A,B}f(x) - f(x)\right| = O_x(1)\left\{w_x\left(\frac{\pi}{n+1}\right) + \frac{1}{\beta(n+1)}\int\limits_{\frac{\pi}{n+1}}^{\pi}\frac{w_x(t) + tw_x'(t)}{t}dt\right\}$$

holds, for all considered x.

More generally:

Theorem 2. *Let* $f \in L^p$ $(1 \le p < \infty)$ *and let the entries of the matrices* $A = (a_{n,r})$ *and* $B = (b_{r,k})$ *satisfy the conditions (1) and (2). Additionally, let* $(b_{r,k})$ *satisfy (7), for* $0 \le \mu \le \nu \le n$, *and* $\tau = \left[\frac{\pi}{t}\right]$, *with* $t \in \left[\frac{\pi}{n+1}, \pi\right]$, *when* $n = 1, 2, \ldots$

If a sequence $(a_{n,r})_{r=0}^{n} \in MRBVS \cup MHBVS$ *satisfies (8),* $t\beta(\frac{\pi}{t})$, $\beta(t)$ *are positive and non-decreasing functions of* t, *then*

$$\left|T_{n,A,B}f(x) - f(x)\right| = O_x\left(\frac{1}{\beta(n+1)}\omega_x[f,\beta]\left(\frac{\pi}{n+1}\right) + \right.$$

$$\left. + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{\beta(k+1)}\omega_x[f,\beta]\left(\frac{\pi}{k}\right) + \sum_{k=1}^{n}\frac{a_n(k)}{\beta(k+1)}\omega_x[f,\beta]\left(\frac{\pi}{k}\right)\right)$$

holds, for all considered x, *where*

$$a_n(k) = \begin{cases} a_{n,k+1}, & \text{when } (a_{n,r})_{r=0}^{n} \in MRBVS, \\ a_{n,n-k-1}, & \text{when } (a_{n,r})_{r=0}^{n} \in MHBVS. \end{cases}$$

3 Corollaries

Finally, we give some corollaries as an application of our results.

Corollary 1. *Theorem A, from Xh. Z. Krasniqi's paper [4, Theorem 3.1], and Theorem 1 are comparable. Condition (6) is more general than (8), but condition (7) is satisfied by a more general class of summability methods.*

Corollary 2. *Taking* $\beta(t) = t^{\eta}$, $(t > 0, 0 < \eta \le 1)$ *and assuming*

$$\frac{1}{t}\int\limits_{0}^{t}|\varphi_x(u)|\,du = O_x\left(t^{\alpha}\right), \quad (0 < \alpha \le 1)$$

we get

$$\omega_x[f,\beta](t) = O_x\left(t^{\alpha-\eta}\right), \qquad \text{for } \alpha > \eta.$$

Hence, under the assumptions for the matrices $A = (a_{n,r})$ *and* $B = (b_{r,k})$ *from Theorem 2, we have*

$$\left|T_{n,A,B}f(x) - f(x)\right| = o_x(1)$$

for almost all considered x.

Example 3. *Let* $a_{n,r} = \frac{1}{n+1}$, *when* $r = 0, 1, 2, \ldots, n$, $a_{n,r} = 0$, *when* $r > n$, $b_{r,k} = \binom{r}{k}\frac{\gamma^k}{(1+\gamma)^r}$, *when* $k = 0, 1, 2, \ldots, r$, *and* $b_{n,r} = 0$, *when* $k > r$ *with* $\gamma > 0$. *Clearly, the sequence* $(a_{n,r})_{r=0}^{n} \in MRBVS \cup MHBVS$ *and* $|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right)$, *for* $\tau = \left[\frac{\pi}{t}\right]$, *with* $t \in \left[\frac{\pi}{n+1}, \pi\right]$, *when* $n = 1, 2, \ldots$. *W. Łenski and B. Szal proved in [2, proof of Corollary 2.4.1] that the sequence* $\left(b_{r,k}\right)_{k=0}^{r}$ *satisfies the condition*

$$\left|\sum_{r=\mu}^{\nu}\sum_{k=0}^{r} b_{r,k}\sin\frac{(2k+1)t}{2}\right| \ll \tau,$$

for $0 \le \mu \le \nu$. *So conditions (7) and (8) are satisfied. If* $f \in L^p$ *and (9) holds , where* $\beta(t)$ *and* $t\beta(\frac{\pi}{t})$ *are positive and non-decreasing functions of* t, *then*

$$\left|\frac{1}{n+1}\sum_{r=0}^{n}\frac{1}{(1+\gamma)^r}\sum_{k=0}^{r}\binom{r}{k}\gamma^k S_k f(x) - f(x)\right| = O_x(1)\left\{w_x\left(\frac{\pi}{n+1}\right) + \frac{1}{\beta(n+1)}\int_{\frac{\pi}{n+1}}^{\pi}\frac{w_x(t) + tw_x'(t)}{t}dt\right\}.$$

Remark 1. *Analogously, for sequences* $(a_{n,r})_{r=0}^{n}$ *and* $\left(b_{r,k}\right)_{k=0}^{r}$ *defined as above, we can derive the following estimation from Theorem 2*

$$\begin{aligned}&\left|\frac{1}{n+1}\sum_{r=0}^{n}\frac{1}{(1+\gamma)^r}\sum_{k=0}^{r}\binom{r}{k}\gamma^k S_k f(x) - f(x)\right| \\ = \; & O_x\left(\frac{1}{\beta(n+1)}\omega_x[f,\beta]\left(\frac{\pi}{n+1}\right) + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{\beta(k+1)}\omega_x[f,\beta]\left(\frac{\pi}{k}\right)\right),\end{aligned}$$

where $\beta(t)$ *and* $t\beta(\frac{\pi}{t})$ *are positive and non-decreasing functions of* t.

4 Auxiliary results

We begin this section with some notation following A. Zygmund [6, Section 5 of Chapter II].

It is clear that

$$S_k f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, D_k(t)\, dt$$

and

$$T_{n,A,B} f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\sum_{r=0}^{n}\sum_{k=0}^{r} a_{n,r} b_{r,k} D_k(t)\, dt,$$

where

$$D_k(t) = \frac{1}{2} + \sum_{\nu=1}^{k}\cos\nu t = \frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}.$$

Hence,

$$T_{n,A,B} f(x) - f(x) = \frac{1}{\pi}\int_{0}^{\pi}\varphi_x(t)\sum_{r=0}^{n}\sum_{k=0}^{r} a_{n,r} b_{r,k} D_k(t)\, dt.$$

Now, we formulate some estimates of the considered kernel.

Lemma 1. *(see [6]) If* $t \in \mathbb{R}$, *then*

$$|D_k(t)| \le k + \frac{1}{2}.$$

Lemma 2. *(see [2]) Let $B = \left(b_{r,k}\right)_{k=0}^{r}$ be such that condition (7) holds for $0 \leq \mu \leq \nu$. If $\left(a_{n,k}\right)_{k=0}^{n} \in MRBVS$, then*

$$\left|\sum_{r=0}^{n} a_{n,r} \sum_{k=0}^{r} b_{r,k} D_k(t)\right| \ll \tau A_{n,\tau}$$

and if $\left(a_{n,k}\right)_{k=0}^{n} \in MHBVS$, then

$$\left|\sum_{r=0}^{n} a_{n,r} \sum_{k=0}^{r} b_{r,k} D_k(t)\right| \ll \tau \overline{A}_{n,n-\tau},$$

where $\tau = \left[\pi/t\right]$ and $t \in \left[\frac{\pi}{n+1}, \pi\right]$, for $n = 0, 1, 2, \ldots$

5 Proofs of the results

5.1 Proof of Theorem 1

Let

$$T_{n,A,B}f(x) - f(x) = \frac{1}{\pi}\left(\int_{0}^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}\right) G_n(t)\,\varphi_x(t)\,dt,$$

where

$$G_n(t) = \sum_{r=0}^{n}\sum_{k=0}^{r} a_{n,r} b_{r,k} D_k(t)$$

and

$$\left|T_{n,A,B}f(x) - f(x)\right| \leq \left|\frac{1}{\pi}\int_{0}^{\frac{\pi}{n+1}} G_n(t)\,\varphi_x(t)\,dt\right| + \left|\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi} G_n(t)\,\varphi_x(t)\,dt\right| = |I_1| + |I_2|.$$

For the first term, from Lemma 1 and (9) we obtain

$$|I_1| \leq \frac{1}{\pi}\sum_{r=0}^{n}\sum_{k=0}^{r} a_{n,r} b_{r,k} \left|D_k(t)\right| \int_{0}^{\frac{\pi}{n+1}} \left|\varphi_x(t)\right| dt$$

$$\ll (n+1)\int_{0}^{\frac{\pi}{n+1}} \left|\varphi_x(t)\right| dt \ll \frac{1}{\beta(n+1)} w_x\left(\frac{\pi}{n+1}\right).$$

For the second one, from (7) and Lemma 2 we get

$$|I_2| \leq \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi} \left|G_n(t)\right|\left|\varphi_x(t)\right| dt \ll \int_{\frac{\pi}{n+1}}^{\pi} \frac{|A|_{n,\tau}}{t}\frac{d}{dt}\left(\Phi_x(t)\right) dt$$

$$= \left[\frac{|A|_{n,\tau}}{t}\Phi_x(t)\right]_{t=\frac{\pi}{n+1}}^{t=\pi} - \int_{\frac{\pi}{n+1}}^{\pi} \Phi_x(t)\frac{d}{dt}\left(\frac{|A|_{n,\tau}}{t}\right) dt$$

$$\ll \Bigg\{ |A|_{n,1}\,\Phi_x(\pi) - (n+1)\,|A|_{n,n+1}\,\Phi_x\left(\frac{\pi}{n+1}\right) + \int_{\frac{\pi}{n+1}}^{\pi} \Phi_x(t)\left[-\frac{d}{dt}\left(\frac{|A|_{n,\tau}}{t\beta(\frac{\pi}{t})}\right)\beta(\tfrac{\pi}{t}) - \frac{d}{dt}\left(\beta(\tfrac{\pi}{t})\right)\frac{|A|_{n,\tau}}{t\beta(\frac{\pi}{t})}\right] dt \Bigg\}$$

$$\ll \left[|A|_{n,1}\,\Phi_x(\pi)-(n+1)\Phi_x\left(\frac{\pi}{n+1}\right)\right]+\int_{\frac{\pi}{n+1}}^{\pi}\beta(\frac{\pi}{t})\Phi_x(t)\left[-\frac{d}{dt}\left(\frac{|A|_{n,\tau}}{t\beta(\frac{\pi}{t})}\right)\right]dt$$

$$+\int_{\frac{\pi}{n+1}}^{\pi}\frac{\beta(\frac{\pi}{t})\Phi_x(t)}{t}\frac{|A|_{n,\tau}}{\left[\beta(\frac{\pi}{t})\right]^2}\left[-\frac{d}{dt}\left(\beta(\frac{\pi}{t})\right)\right]dt \tag{10}$$

$$=\Sigma_1+\Sigma_2+\Sigma_3.$$

Now we estimate Σ_1, Σ_2 and Σ_3 separately. Namely, from (8), (9) and by $\frac{w_x(t)}{t}\le 2\frac{w_x(u)}{u}$, for $0<u<t$,

$$\Sigma_1\le|A|_{n,1}\,\Phi_x(\pi)\le\frac{1}{(n+1)\beta(1)}w_x(\pi)\ll\frac{1}{\beta(1)}w_x\left(\frac{\pi}{n+1}\right).$$

For the second term, integrating by parts and from (8) and (9), we obtain

$$\Sigma_2\le\int_{\frac{\pi}{n+1}}^{\pi}tw_x(t)\left[-\frac{d}{dt}\left(\frac{|A|_{n,\tau}}{t\beta(\frac{\pi}{t})}\right)\right]dt=\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}tw_x(t)\left[-\frac{d}{dt}\left(\frac{|A|_{n,\tau}}{t\beta(\frac{\pi}{t})}\right)\right]dt$$

$$=\sum_{k=1}^{n}\left[\frac{-|A|_{n,\tau}}{\beta(\frac{\pi}{t})}w_x(t)\right]_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}+\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}(w_x(t)+tw_x'(t))\frac{|A|_{n,\tau}}{t\beta(\frac{\pi}{t})}dt$$

$$\le\frac{|A|_{n,n+1}}{\beta(n+1)}w_x\left(\frac{\pi}{n+1}\right)+\int_{\frac{\pi}{n+1}}^{\pi}(w_x(t)+tw_x'(t))\frac{|A|_{n,\tau}}{t\beta(\frac{\pi}{t})}dt$$

$$\ll\frac{1}{\beta(n+1)}w_x\left(\frac{\pi}{n+1}\right)+\frac{1}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{w_x(t)+tw_x'(t)}{t}\frac{1}{t\beta(\frac{\pi}{t})}dt.$$

By the monotonicity of the function $t\beta(\frac{\pi}{t})$ we get

$$\Sigma_2\ll\frac{1}{\beta(n+1)}w_x\left(\frac{\pi}{n+1}\right)+\frac{1}{\beta(n+1)}\int_{\frac{\pi}{n+1}}^{\pi}\frac{w_x(t)+tw_x'(t)}{t}dt.$$

Further, from (8) and by the monotonicity of the function w_x

$$\begin{aligned}\Sigma_3 &\le \int_{\frac{\pi}{n+1}}^{\pi}\frac{w_x(t)}{t}\frac{t\,|A|_{n,\tau}}{\left[\beta(\frac{\pi}{t})\right]^2}\left[-\frac{d}{dt}\left(\beta(\frac{\pi}{t})\right)\right]dt\le\\ &\le \frac{1}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{w_x(t)}{t}\frac{1}{\left[\beta(\frac{\pi}{t})\right]^2}\left[-\frac{d}{dt}\left(\beta(\frac{\pi}{t})\right)\right]dt\le\\ &\le w_x\left(\frac{\pi}{n+1}\right)\int_{\frac{\pi}{n+1}}^{\pi}\frac{1}{\left[\beta(\frac{\pi}{t})\right]^2}\left[-\frac{d}{dt}\left(\beta(\frac{\pi}{t})\right)\right]dt\ll\frac{1}{\beta(1)}w_x\left(\frac{\pi}{n+1}\right).\end{aligned}$$

Combining these estimates we obtain the desired result. □

5.2 Proof of Theorem 2

As usual let

$$\left|T_{n,A,B}f(x)-f(x)\right|\le|I_1|+|I_2|.$$

From Lemma 1 and (5), we get

$$|I_1| \leq \frac{1}{\beta(n+1)}\omega_x[f,\beta]\left(\frac{\pi}{n+1}\right).$$

For the second term, from Lemma 2 and by (10) in the previous proof we have

$$|I_2| \leq \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi} |G_n(t)|\,|\varphi_x(t)|\,dt \ll \int_{\frac{\pi}{n+1}}^{\pi} \frac{|A|_{n,\tau}}{t}\frac{d}{dt}\left(\Phi_x(t)\right)dt$$

$$\leq \left[|A|_{n,1}\Phi_x(\pi) - (n+1)\Phi_x\left(\frac{\pi}{n+1}\right)\right] + \int_{\frac{\pi}{n+1}}^{\pi} \beta\left(\frac{\pi}{t}\right)\Phi_x(t)\left[-\frac{d}{dt}\left(\frac{|A|_{n,\tau}}{t\beta(\frac{\pi}{t})}\right)\right]dt$$

$$+\int_{\frac{\pi}{n+1}}^{\pi} \frac{|A|_{n,\tau}}{\beta\left(\frac{\pi}{t}\right)}\frac{\Phi_x(t)}{t}\left[-\frac{d}{dt}\left(\beta\left(\frac{\pi}{t}\right)\right)\right]dt = \Lambda_1 + \Lambda_2 + \Lambda_3.$$

Now we estimate Λ_1, Λ_2 and Λ_3. For Λ_1 from (5) and (8) we obtain

$$\begin{aligned}\Lambda_1 &\ll \frac{1}{n+1}\Phi_x(\pi) - (n+1)\Phi_x\left(\frac{\pi}{n+1}\right) \ll \frac{1}{n+1}\frac{\pi}{\beta(1)}\omega_x[f,\beta](\pi)\\ &\leq \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{\beta(k+1)}\omega_x[f,\beta]\left(\frac{\pi}{k}\right).\end{aligned}$$

For the second term, by the monotonicity of the function $\beta\left(\frac{\pi}{t}\right)$ and $\Phi_x(t)$ we get

$$\Lambda_2 \leq \sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} \frac{1}{t}\beta\left(\frac{\pi}{t}\right)\Phi_x(t)\,t\left[-\frac{d}{dt}\left(\frac{|A|_{n,\tau}}{t\beta(\frac{\pi}{t})}\right)\right]dt$$

$$\leq \sum_{k=1}^{n}\frac{k+1}{\pi}\beta(k+1)\Phi_x\left(\frac{\pi}{k}\right)\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} t\left[-\frac{d}{dt}\left(\frac{|A|_{n,\tau}}{t\beta\left(\frac{\pi}{t}\right)}\right)\right]dt.$$

Further, integrating by parts and using (5) and (8) we obtain

$$\Lambda_2 \leq \sum_{k=1}^{n}\omega_x[f,\beta]\left(\frac{\pi}{k}\right)\left[\frac{|A|_{n,k+1}}{\beta(k+1)} - \frac{|A|_{n,k}}{\beta(k)} + \frac{\pi}{n+1}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\frac{1}{t^2\beta\left(\frac{\pi}{t}\right)}dt\right].$$

Since

$$\frac{|A|_{n,k+1}}{\beta(k+1)} - \frac{|A|_{n,k}}{\beta(k)} \leq \frac{|A|_{n,k+1} - |A|_{n,k}}{\beta(k+1)} =$$

$$= \begin{cases} \frac{a_{n,k+1}}{\beta(k+1)}, & \text{when } (a_{n,r})_{r=0}^{n} \in MRBVS \\ \frac{a_{n,n-k-1}}{\beta(k+1)}, & \text{when } (a_{n,r})_{r=0}^{n} \in MHBVS \end{cases} := \frac{a_n(k)}{\beta(k+1)},$$

therefore

$$\Lambda_2 \ll \sum_{k=1}^{n}\omega_x[f,\beta]\left(\frac{\pi}{k}\right)\left[\frac{a_n(k)}{\beta(k+1)} + \frac{1}{n+1}\frac{1}{\beta(k+1)}\right].$$

For the third term by (5) and (8) we obtain

$$\begin{aligned}\Lambda_3 &\leq \sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\frac{\Phi_x(t)}{t}\frac{|A|_{n,\tau}}{\beta\left(\frac{\pi}{t}\right)}\left[-\frac{d}{dt}\left(\beta\left(\frac{\pi}{t}\right)\right)\right]dt\\ &\ll \frac{\pi}{n+1}\sum_{k=1}^{n}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\frac{\Phi_x(t)}{t}\frac{1}{t\beta\left(\frac{\pi}{t}\right)}\left[-\frac{d}{dt}\left(\beta\left(\frac{\pi}{t}\right)\right)\right]dt\end{aligned}$$

$$\ll \frac{\pi}{n+1}\sum_{k=1}^{n}\frac{k+1}{\pi}\Phi_x\left(\frac{\pi}{k}\right)\frac{k+1}{\pi\beta(k+1)}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\left[-\frac{d}{dt}\left(\beta\left(\frac{\pi}{t}\right)\right)\right]dt$$

$$\leq \frac{\pi}{n+1}\sum_{k=1}^{n}\omega_x[f,\beta]\left(\frac{\pi}{k}\right)\frac{1}{k\beta(k)}\frac{(k+1)^2}{\beta(k+1)}\int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}}\left[-\frac{d}{dt}\left(\beta\left(\frac{\pi}{t}\right)\right)\right]dt$$

$$\ll \frac{1}{n+1}\sum_{k=1}^{n}\omega_x[f,\beta]\left(\frac{\pi}{k}\right)\frac{1}{\beta(k)}\frac{k+1}{\beta(k+1)}\left[\beta(k+1)-\beta(k)\right].$$

By the monotonicity of $t\beta(\frac{\pi}{t})$, we can notice that $\frac{\pi}{k}\beta(k)\leq\frac{\pi}{k+1}\beta(k+1)$, for $k=1,2,3,\ldots$, and thus

$$\begin{aligned}\beta(k+1)-\beta(k) &= \frac{1}{k}(k\beta(k+1)-k\beta(k))\leq\frac{1}{k}((k+1)\beta(k)-k\beta(k))\\ &= \frac{1}{k}\beta(k).\end{aligned}$$

So

$$\Lambda_3\ll\frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{\beta(k+1)}\omega_x[f,\beta]\left(\frac{\pi}{k}\right).$$

The desired result follows by combining these estimates. □

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