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Almost periodic solutions of Volterra difference systems

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Abstract: We study the existence of an almost periodic solution of discrete Volterra systems by means of fixed point theory. Using discrete variant of exponential dichotomy, we provide sufficient conditions for the existence of an almost periodic solution. Hence, we provide an alternative solution for the open problem proposed in the literature.

Keywords: Almost periodic, discrete exponential dichotomy, Volterra, fixed point

MSC: Primary 39A10, 39A12; Secondary 34D09.

1 Introduction

Volterra type equations have tremendous potential for application in certain fields of applied mathematics. Qualitative properties of Volterra equations on continuous, discrete and hybrid domains became topic of many studies in the literature. By a quick literature review, we refer to [1–9] and references therein. Investigation of periodic solutions of dynamic equations and systems with periodic structures is of special importance for studies on population dynamics and control theory. However, it should be pointed out that periodicity is a very strong restriction for a class of functions. As an alternative relaxation of strict periodicity condition, the theory of almost periodic functions was first introduced by H. Bohr [10] and generalized by A. S. Besicovitch, W. Stepanoff, S. Bochner, and J. von Neumann at the beginning of 20th century (see [11–14]).

A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be almost periodic if the following characteristic property holds:

A. *For any* $\varepsilon > 0$, *the set*

$$E\left(\varepsilon, f(x)\right) := \left\{\tau : \left|f\left(x+\tau\right) - f(x)\right| < \varepsilon \text{ for all } x \in \mathbb{R}\right\}$$

is relatively dense in the real line $\mathbb{R}$. *That is, for any* $\varepsilon > 0$, there exists a number $l\left(\varepsilon\right) > 0$ such that any interval of length $l\left(\varepsilon\right)$ contains a number in $E\left(\varepsilon, f(x)\right)$.

Afterwards, S. Bochner showed that almost periodicity is equivalent to the following characteristic property which is also called *the normality condition*:

B. *From any sequence of the form* $\{f\left(x+h_n\right)\}$, *where* h_n *are real numbers, one can extract a subsequence converging uniformly on the real line (see [12], and [15, 16]).* For a comprehensive review on almost periodic functions on continuous and discrete domains, we refer the reader to [15–20].

There is a vast literature on the existence of almost periodic solutions of Volterra equations constructed on the real line, and researchers focused on the discrete analogues of Volterra equations in the last decade. For almost periodic solutions of Volterra integral equations, we may refer to [21]. In [22], Y. Song focused on

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the almost periodic and asymptotically almost periodic solutions of Volterra difference equations

$$x(n) = a(n) + \sum_{j=0}^{n} B\left(n, j, x(j)\right), \; n = 0, 1, 2, \ldots$$

and

$$x(n) = a(n) + \sum_{j=-\infty}^{n} B\left(n, j, x(j)\right), \; n = 0, 1, 2, \ldots,$$

where $a(n)$ and $B\left(n, j, x(j)\right)$ are vector sequences. In [5], S. Elaydi introduced an open problem for the existence of almost periodic solutions of nonconvolution type Volterra difference systems

$$y(t+1) = A(t)y(t) + \sum_{j=0}^{t} B(t,j)y(j) + g(t), \; t = 0, 1, 2, \ldots \tag{1}$$

In [23], Hamaya studied almost periodic solutions of the system

$$\Delta x(n) = f(n, x(n)) + \sum_{m=-\infty}^{n} F(n,m)\, x(m) + p(n), \; n = 0, 1, 2, \ldots, \tag{2}$$

by using fixed point theory. However, assumptions and the method used in [23] exclude a significantly large class of equations with almost periodic solutions. We have a discussion on this matter in Example 1.

In our study, we use the concept of exponential dichotomy to obtain necessary limit results leading to some sufficient conditions guaranteeing existence of almost periodic solutions of the system

$$x(t+1) = A(t)x(t) + \sum_{j=-\infty}^{t} B(t,j)x(j) + g(t), \; t \in \mathbb{Z}. \tag{3}$$

Establishing a linkage between the systems (1) and (3), we provide an alternative solution to the above mentioned open problem due to Elaydi (see [5]). Note that our solution provides a significant relaxation for the conditions proposed in [23].

In the next section, we introduce the discrete almost periodic functions, discrete variant of exponential dichotomy and its limiting property. In the latter part, we prove our existence theorem and discuss the efficacy of the existence result.

2 Preliminaries: Discrete almost periodicity and exponential dichotomy

Let $\mathcal{X}$ be a (real or complex) Banach space endowed with the norm $\|.\|_{\mathcal{X}}$ and $\mathcal{B}(\mathcal{X})$ a Banach space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{X}$ with the norm $\|.\|_{\mathcal{B}(\mathcal{X})}$ given by

$$\|L\|_{\mathcal{B}(\mathcal{X})} := \sup\left\{\|Lx\|_{\mathcal{X}} : x \in \mathcal{X} \text{ and } \|x\|_{\mathcal{X}} \leq 1\right\}.$$

Throughout the paper, we denote by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}_-$ the set of integers, the set of nonnegative integers and the set of negative integers, respectively.

Definition 1 ([22]). *Let Ω be a subset of the abstract Banach space $\mathcal{X}$. A function $f : \mathbb{Z}\times\Omega \to \mathcal{X}$ which is continuous on Ω for each $t \in \mathbb{Z}$ is said to be uniformly discrete almost periodic in $t \in \mathbb{Z}$ (uniformly for $x \in \Omega$) if for every $\varepsilon > 0$ and every compact $\Sigma \subset \Omega$ there corresponds an integer $N_\varepsilon(\Sigma) > 0$ such that among $N_\varepsilon(\Sigma)$ consecutive integers there is one denoted by p such that*

$$\|f(t+p, x) - f(t, x)\|_{\mathcal{X}} < \varepsilon \text{ for all } t \in \mathbb{Z}, \text{ uniformly for } x \in \Sigma.$$

Note that if $\Omega = \emptyset$, the function $f(t)$ is discrete almost periodic in t.

It is well known that the above given definition is equivalent to the normality condition for a function. Now, we give an alternative definition of discrete almost periodicity in the sense of S. Bochner:

Definition 2 ([17]). *A function $f : \mathbb{Z} \to \mathcal{X}$ is said to be discrete almost periodic if for every integer sequence $\{v'_n\}_{n\in\mathbb{Z}_+}$ there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}_+}$ such that*

$$\lim_{n\to\infty} f(t+v_n) =: \bar{f}(t) \tag{4}$$

uniformly for all $t \in \mathbb{Z}$.

Moreover, almost periodicity of functions with two variables is defined as follows:

Definition 3 ([23]). *The function $g : \mathbb{Z} \times \mathcal{X} \to \mathcal{X}$ is said to be discrete almost periodic in t uniformly for $x \in \mathcal{X}$ if for every integer sequence $\{v'_n\}_{n\in\mathbb{Z}_+}$ there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}_+}$ such that*

$$\lim_{n\to\infty} g(t+v_n, x) =: \bar{g}(t, x)$$

uniformly on $\mathbb{Z}\times K$, where K is a compact set in $\mathcal{X}$.

The following definition is useful for our further analysis.

Definition 4 ([19, 24]). *A sequence $\phi : \mathbb{Z}_+ \to \mathcal{X}$ is said to be almost periodic if for every nonnegative integer sequence $\{p'_n\}_{n\in\mathbb{Z}_+}$ there exists a subsequence $\{p_n\}_{n\in\mathbb{Z}_+}$ such that*

$$\lim_{n\to\infty} \phi(t+p_n) =: \bar{\phi}(t), \tag{5}$$

uniformly for all $t \in \mathbb{Z}_+$.

For details on the almost periodic sequences, we refer the reader to the pioneering work of Diagana et al. (see [24]). The basic properties of discrete almost periodic functions are given in the following theorem:

Theorem 1 ([18]). *Let $f_1, f_2 : \mathbb{Z} \to \mathcal{X}$ and $g_1, g_2 : \mathbb{Z} \times \mathcal{X} \to \mathcal{X}$ be discrete almost periodic functions in $t \in \mathbb{Z}$, then*
i. *$f_1 + f_2$ and $g_1 + g_2$ are discrete almost periodic in $t \in \mathbb{Z}$*
ii. *cf_1 and cg_1 are discrete almost periodic in $t \in \mathbb{Z}$ for every scalar c*
iii. $\sup_{t\in\mathbb{Z}} \|f_{1,2}(t)\|_{\mathcal{X}} < \infty$ *for each $t \in \mathbb{Z}$ and* $\sup_{t\in\mathbb{Z}} \|g_{1,2}(t,x)\|_{\mathcal{X}} < \infty$ *for each $t \in \mathbb{Z}$ and $x \in K$.*

Definition 5 (Discrete exponential dichotomy). *Let $X(t)$ be the principal fundamental matrix solution for the linear homogeneous system*

$$x(t+1) = A(t)x(t),\ x(t_0) = x_0, \tag{6}$$

where A is a matrix function which is invertible for all $t \in \mathbb{Z}$. Then (6) is said to admit an exponential dichotomy if there exist a projection P and positive constants $\alpha_1, \alpha_2, \beta_1$ and β_2 such that

$$\left\|X(t)PX^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \le \beta_1 (1+\alpha_1)^{s-t},\ t \ge s, \tag{7}$$

$$\left\|X(t)(I-P)X^{-1}(s)\right\|_{\mathcal{B}(\mathcal{X})} \le \beta_2 (1+\alpha_2)^{t-s},\ s \ge t. \tag{8}$$

Remark 1. *Notice that in [17] and [25], the discrete exponential dichotomy is defined by using the exponential function* $\exp(\alpha(s-t))$ *instead of the discrete exponential function $e_\alpha(t,s) = (1+\alpha)^{s-t}$ (satisfying $\Delta_t e_\alpha(t,s) = \alpha e_\alpha(t,s)$) in (7) and (8), respectively. For convenience we prefer using Definition 5 which is evidently equivalent to [25, Definition 2.11].*

Theorem 2 ([25]). *If the system (6) admits an exponential dichotomy and the function f is bounded, then the nonhomogeneous system*

$$x(t+1) = A(t)x(t) + f(t, x(t)),\ x(t_0) = x_0 \tag{9}$$

has a bounded solution of the form

$$x(t) = \sum_{j=-\infty}^{t-1} X(t)PX^{-1}(j+1)f(j, x(j)) - \sum_{j=t}^{\infty} X(t)PX^{-1}(j+1)f(j, x(j)).$$

Since all almost periodic solutions are almost automorphic, the following result is also valid for an almost periodic coefficient matrix A.

Theorem 3 ([26]). *Suppose that the system (6) admits an exponential dichotomy with the projection P and the positive constants α_1, α_2, β_1, and β_2. Let the matrix valued function $A(t)$ in (6) be almost automorphic. That is, for any sequence $\{\widetilde{\theta}_k\}_{k\in\mathbb{Z}_+}$ of integers there exists a subsequence $\{\theta_k\}_{k\in\mathbb{Z}_+}$ such that*

$$\lim_{k\to\infty} A(t+\theta_k) := \bar{A}(t)$$

is well defined and

$$\lim_{k\to\infty} \bar{A}(t-\theta_k) = A(t)$$

for each $t \in \mathbb{Z}$. Then

$$\lim_{k\to\infty} X(t+\theta_k)PX^{-1}(s+\theta_k) := \bar{X}(t)\bar{P}\bar{X}^{-1}(s) \text{ for } s \in (-\infty, t] \cap \mathbb{Z} \tag{10}$$

and

$$\lim_{k\to\infty} X(t+\theta_k)(I-P)X^{-1}(s+\theta_k) := \bar{X}(t)\left(I-\bar{P}\right)\bar{X}^{-1}(s) \text{ for } s \in [t, \infty) \cap \mathbb{Z} \tag{11}$$

are well defined for each $t \in \mathbb{Z}$ and the limiting system

$$x(t+1) = \bar{A}(t)x(t), \;\; x(t_0) = x_0 \tag{12}$$

admits an exponential dichotomy with the projection $\bar{P}$ and the same constants. Furthermore, for each $t \in \mathbb{Z}$ we have

$$\lim_{k\to\infty} \bar{X}(t-\theta_k)\bar{P}\bar{X}^{-1}(s-\theta_k) = X(t)PX^{-1}(s), \;\; s \in (-\infty, t] \cap \mathbb{Z} \tag{13}$$

and

$$\lim_{k\to\infty} \bar{X}(t-\theta_k)\left(I-\bar{P}\right)\bar{X}^{-1}(s-\theta_k) = X(t)(I-P)X^{-1}(s), s \in [t, \infty) \cap \mathbb{Z}. \tag{14}$$

3 Existence results

In this section, we provide sufficient conditions for existence of an almost periodic solution of the following Volterra difference system with infinite delay

$$x(t+1) = A(t)x(t) + \sum_{k=-\infty}^{t} B(t,k)x(k) + g(t), \;\; t \in \mathbb{Z}, \tag{15}$$

where $A(t) = \left[a_{ij}(t)\right]$, $B(t,k) = \left[b_{ij}(t,k)\right]$ are $n\times n$ matrix functions and $g(t)$ is a vector function. By a solution of system (15), we refer to a vector valued function x defined on $\mathbb{Z}$ satisfying (15) for all $t \in \mathbb{Z}_+$. By defining the initial vector function $\theta : \mathbb{Z}_- \to \mathbb{R}^n$ with $\sup_{t\in\mathbb{Z}_-} |\theta(t)| < M_\theta < \infty$, we denote an almost periodic solution of (15) by x^θ so that $x^\theta(t)$ is almost periodic sequence satisfying (15) for $t \in \mathbb{Z}_+$ and $x^\theta(t) = \theta(t)$ for all $t \in \mathbb{Z}_-$.

Let $\mathcal{AP}(\mathcal{X})$ be the set of functions on $\mathbb{Z}$ satisfying the condition (5) for all $t \in \mathbb{Z}_+$ given in Definition 4. Then $\mathcal{AP}(\mathcal{X})$ is a Banach space endowed by the norm

$$\|f\|_{\mathcal{AP}(\mathcal{X})} := \sup_{t\in\mathbb{Z}_+} \left\|f(t)\right\|_{\mathcal{X}}. \tag{16}$$

Theorem 4 ([24, Theorem 2.12]). *Let $g : \mathbb{Z}\times\mathcal{X}\to\mathcal{X}$ be discrete almost periodic in $t \in \mathbb{Z}$, for each $x, y \in \mathcal{X}$ satisfying Lipschitz condition in x uniformly in t, that is*

$$\left\|g(t,x) - g(t,y)\right\|_{\mathcal{AP}(\mathcal{X})} \leq L\,\|x-y\|_{\mathcal{X}},\ \forall x, y \in \mathcal{X}.$$

Suppose $\varphi : \mathbb{Z} \to \mathcal{X}$ is discrete almost periodic function, then the function $g(t, \varphi(t))$ is discrete almost periodic.

We employ the following fixed point theorem to prove the main result of this section.

Theorem 5 (Schauder). *Let $\mathbb{B}$ be a Banach Space. Assume that K is a closed, bounded and convex subset of $\mathbb{B}$. If $T : K \to K$ is a compact operator, then it has a fixed point in K.*

Henceforth, we suppose that the following conditions hold:

A1 Functions $A(t)$ and $g(t)$ are discrete almost periodic on $\mathbb{Z}$

A2 The matrix function $B(t, k)$ is discrete almost periodic in t and k (bi-almost periodic, see [27]) i.e., for every integer sequence $\{v'_n\}_{n\in\mathbb{Z}_+}$ there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}_+}$ such that

$$\lim_{n\to\infty} B(t+v_n, k+v_n) = \bar{B}(t,k)$$

uniformly for all $t, k \in \mathbb{Z}$

A3 There exists a positive constant U_B such that $0 < \sup_{t\in\mathbb{Z}_+}\sum_{k=-\infty}^{t}\left\|B(t,k)\right\| \leq U_B < \infty$

A4 The homogeneous system (6) admits an exponential dichotomy with positive constants $\alpha_{1,2}$ and $\beta_{1,2}$.

Remark 2. *If the matrix function $B(t, s)$ has a component $f(t, s)$ which is a function of the convolution type i.e. $f(t, s) = f(t - s)$, then we do not require f to satisfy the condition in (A2). However, the almost periodicity condition in (A2) is a compulsory condition for all nonconvolution type components of the matrix valued function B.*

Now, define the mapping

$$\left(Hx^{\theta}\right)(t) := \begin{cases} \theta(t),\ t \in \mathbb{Z}_- \\ \displaystyle\sum_{k=-\infty}^{t-1} X(t)PX^{-1}(k+1)\,\Lambda(k, x(k)) - \sum_{k=t}^{\infty} X(t)(I-P)X^{-1}(k+1)\,\Lambda(k, x(k)),\ t \in \mathbb{Z}_+, \end{cases}$$

where

$$\Lambda(k, x(k)) := \sum_{s=-\infty}^{k} B(k,s)x(s) + g(k). \tag{17}$$

Lemma 1. *If $\omega \in \mathcal{AP}(\mathcal{X})$, then $\Lambda(., \omega(.)) \in \mathcal{AP}(\mathcal{X})$.*

Proof. For any $t \in \mathbb{Z}$ and $\xi, \psi \in \mathcal{X}$, consider

$$\begin{aligned}\|\Lambda(t,\xi) - \Lambda(t,\psi)\|_{\mathcal{AP}(\mathcal{X})} &= \left\|\sum_{k=-\infty}^{t} B(t,k)\xi - \sum_{k=-\infty}^{t} B(t,k)\psi\right\|_{\mathcal{AP}(\mathcal{X})} \\ &= \sup_{t\in\mathbb{Z}_+}\left\|\sum_{k=-\infty}^{t} B(t,k)\xi - \sum_{k=-\infty}^{t} B(t,k)\psi\right\|_{\mathcal{X}} \\ &\leq \sup_{t\in\mathbb{Z}_+}\sum_{k=-\infty}^{t}\left\|B(t,k)\right\|\,\|\xi-\psi\|_{\mathcal{X}} \\ &\leq U_B\,\|\xi-\psi\|_{\mathcal{X}},\end{aligned}$$

where we use assumption A3. Then by employing Theorem 4, we conclude that $\Lambda(., x(.)) \in \mathcal{AP}(\mathcal{X})$ for any x in a compact subset of $\mathcal{AP}(\mathcal{X})$. □

In preparation for the next result, we define the subset Ω_M of $\mathcal{AP}(\mathcal{X})$ by

$$\Omega_M := \left\{ x^\theta \in \mathcal{AP}(\mathcal{X}) : \left\| x^\theta \right\|_{\mathcal{AP}(\mathcal{X})} \leq M \right\}$$

for a fixed M. Then, Ω_M is a bounded, closed and convex subset of $\mathcal{AP}(\mathcal{X})$.

Theorem 6. *Assume (A1-A4). Then (15) has an almost periodic solution.*

Proof. At first, we need to show that H maps Ω_M into Ω_M. Suppose that $x^\theta \in \mathcal{AP}(\mathcal{X})$ and by Lemma 1 we know that the function $\Lambda(t, x(t))$, which is defined in (17), is an almost periodic sequence in t for any $t \in \mathbb{Z}_+$ uniformly for x. That is for every nonnegative integer sequence $\{p'_n\}_{n\in\mathbb{Z}_+}$ there exists a subsequence $\{p_n\}_{n\in\mathbb{Z}_+}$ such that

$$\lim_{n\to\infty} \Lambda(t+p_n, x(t+p_n)) =: \bar{\Lambda}(t, \bar{x}(t))$$

is uniformly for each $t \in \mathbb{Z}_+$ on any compact subset of $\mathbb{Z}\times\mathcal{AP}(\mathcal{X})$. Additionally, we have

$$\begin{aligned}(Hx)(t+p_n) = & \sum_{k=-\infty}^{t-1} X(t+p_n)PX^{-1}(k+p_n+1)\Lambda(k+p_n, x(k+p_n)) \\ & - \sum_{k=t}^{\infty} X(t+p_n)(I-P)X^{-1}(k+p_n+1)\Lambda(k+p_n, x(k+p_n))\end{aligned}$$

for $t \in \mathbb{Z}_+$. If we let the limit as $n \to \infty$ and employ the Lebesgue Convergence Theorem and Theorem 3, we get the uniform convergence

$$\begin{aligned}\overline{(Hx)}(t) = & \sum_{k=-\infty}^{t-1} \bar{X}(t)\overline{PX}^{-1}(k+1)\bar{\Lambda}(k, \bar{x}(k)) \\ & - \sum_{k=t}^{\infty} \bar{X}(t)\left(I-\bar{P}\right)\overline{X}^{-1}(k+1)\bar{\Lambda}(k, \bar{x}(k)), \text{ for each } t \in \mathbb{Z}_+.\end{aligned}$$

Moreover

$$\begin{aligned}\left\|(Hx)(t)\right\|_{\mathcal{AP}(\mathcal{X})} & \leq \left\| \sum_{k=-\infty}^{t-1} X(t)PX^{-1}(k+1)\Lambda(k, x(k)) \right\|_{\mathcal{AP}(\mathcal{X})} \\ & + \left\| \sum_{k=t}^{\infty} X(t)(I-P)X^{-1}(k+1)\Lambda(k, x(k)) \right\|_{\mathcal{AP}(\mathcal{X})} \\ & \leq \sum_{k=-\infty}^{t-1} \left\| X(t)PX^{-1}(k+1) \right\| \left\|\Lambda(k, x(k))\right\|_{\mathcal{AP}(\mathcal{X})} \\ & + \sum_{k=t}^{\infty} \left\| X(t)(I-P)X^{-1}(k+1) \right\| \left\|\Lambda(k, x(k))\right\|_{\mathcal{AP}(\mathcal{X})} \\ & \leq U_\Lambda \sum_{k=-\infty}^{t-1} \beta_1(1+\alpha_1)^{k+1-t} + U_\Lambda \sum_{k=t}^{\infty} \beta_2(1+\alpha_2)^{t-k-1} \\ & = U_\Lambda \left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right),\end{aligned}$$

where U_Λ is the upper bound for $\Lambda(t, x(t))$ for all $t \in \mathbb{Z}_+$. By setting $M := \max\left\{ M_\theta, U_\Lambda \left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right) \right\}$ and using the boundedness of the initial function θ, we prove that $H : \Omega_M \to \Omega_M$ for all $t \in \mathbb{Z}$. Now, we have to show that H is continuous. Let $\xi, \psi \in \Omega_M$ and define the number $\delta(\varepsilon) > 0$ by

$$\delta := \frac{\varepsilon}{L\left\|\xi - \psi\right\|_{\mathcal{AP}(\mathcal{X})} \left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right)}.$$

If $\|\xi - \psi\|_{\mathcal{AP}(\mathcal{X})} < \delta$, then we have

$$\begin{aligned}
\|(H\xi)(t) - (H\psi)(t)\|_{\mathcal{X}} &\leq \sum_{k=-\infty}^{t-1} \left\|X(t)PX^{-1}(k+1)\right\| \|\Lambda(k,\xi(k)) - \Lambda(k,\psi(k))\|_{\mathcal{AP}(\mathcal{X})} \\
&+ \sum_{k=t}^{\infty} \left\|X(t)(I-P)X^{-1}(k+1)\right\| \|\Lambda(k,\xi(k)) - \Lambda(k,\psi(k))\|_{\mathcal{AP}(\mathcal{X})} \\
&\leq L\|\xi-\psi\|_{\mathcal{AP}(\mathcal{X})} \left[\sum_{k=-\infty}^{t-1} \beta_1 (1+\alpha_1)^{k+1-t} + \sum_{k=t}^{\infty} \beta_2 (1+\alpha_2)^{t-k-1}\right] \\
&= L\|\xi-\psi\|_{\mathcal{AP}(\mathcal{X})} \left(\beta_1 \frac{1+\alpha_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}\right) \\
&< \varepsilon,
\end{aligned}$$

which shows that the mapping H is continuous. To conclude, we must show that $H(\Omega_M)$ is precompact. Let $\{x^l\}_{l\in\mathbb{Z}_+}$ be a sequence in Ω_M. Then, for each fixed $l \in \mathbb{Z}_+$, $\{x^l(t)\}_{t\in\mathbb{Z}}$ is a bounded sequence and by the *Bolzano-Weierstrass Theorem,* $\{x^l(t)\}_{t\in\mathbb{Z}}$ has a convergent subsequence $\{x^l(t_k)\}$. By repeating the diagonalization process for each $l \in \mathbb{Z}_+$, we can construct a convergent subsequence $\{x^{l_k}\}_{l_k\in\mathbb{Z}_+}$ of $\{x^l\}_{l\in\mathbb{Z}_+}$ in Ω_M. By continuity of H, we obtain $\{H(x^l)\}_{l\in\mathbb{Z}_+}$ has a convergent subsequence in $H(\Omega_M)$. This means, $H(\Omega_M)$ is precompact. The proof follows by applying Schauder's fixed point theorem which ensures that is there exists a $x \in \Omega_M$ such that $\left(Hx^\theta\right)(t) = x(t)$ for all $t \in \mathbb{Z}_+$. □

In preparation for the next example, we present the existence result of [23] for almost periodic solutions of the system (2).

Theorem 7 ([23, Theorem 2.3])**.** *Suppose that f, F, and p are almost periodic functions and the following conditions are satisfied:*

i. $\left\|p(n) + f(n,0)\right\| \leq L$ *for all* $n \in \mathbb{Z}$, *where* L *is a positive constant*

ii. *There exists a positive constant* γ *such that*

$$\lim_{n-m\to\infty} \frac{1}{n-m} \sum_{j=m}^{n-1} e(j) = -\gamma < 0,$$

where $e : \mathbb{Z} \to \mathbb{R}$ *is a function*

iii. *For all* $(n,x) \in \mathbb{Z} \times \mathbb{R}^l$,

$$\left[x-y, f(n,x) - f(n,y) + \sum_{m=-\infty}^{n} F(n,m)x(m) - \sum_{m=-\infty}^{n} F(n,m)y(m)\right] \leq e(n)\|x-y\|,$$

where $[x,y] = h^{-1}(\|x+hy\| - \|x\|)$ *for* $h > 0$.

Then, the system (2) has a unique almost periodic solution.

It is clear that if the function $f(n,x(n))$ in the system (2) is specifically chosen to be $\left(A(n) - I\right)x(n)$ where I indicates the identity matrix, then the equation (2) reduces the system (15).

Example 1. *Consider the infinite delayed abstract system*

$$x(t+1) = A(t)x(t) + \sum_{k=-\infty}^{t} B(t,k)x(k) + g(t), \tag{18}$$

where A and B are certain matrix functions satisfying (A1-A3) and $\|I + hA\| \geq 1 + hU_B$ for a positive constant h, where U_B is as in A3. Then, [23, Theorem 2.3] is not sufficient to ensure the existence of an almost periodic *solution of (18). To see this, we focus on the conditions (ii) and (iii) of Theorem 7. Consider*

$$\left[x-y, A(t)x-A(t)y+\sum_{k=-\infty}^{t}B(t,k)x(k)-\sum_{k=-\infty}^{t}B(t,k)y(k)\right]$$
$$=h^{-1}\left(\left\|x-y+hA\left(x-y\right)+h\sum_{k=-\infty}^{t}B(t,k)\left(x(k)-y(k)\right)\right\|-\|x-y\|\right). \tag{19}$$

By using (19), it may be deduced that

$$h^{-1}\left(\|(I+hA)\left(x-y\right)\|-\left\|h\sum_{k=-\infty}^{t}B(t,k)\left(x(k)-y(k)\right)\right\|-\|x-y\|\right)$$
$$\le\left[x-y, A(t)x-A(t)y+\sum_{k=-\infty}^{t}B(t,k)x(k)-\sum_{k=-\infty}^{t}B(t,k)y(k)\right],$$

and

$$0\le h^{-1}\left(\|(I+hA)\|-hU_B-1\right)\|x-y\|\le\left[x-y, A(t)x-A(t)y+\sum_{k=-\infty}^{t}B(t,k)x(k)-\sum_{k=-\infty}^{t}B(t,k)y(k)\right].$$

Thus, under the condition $\|I+hA\|\ge 1+hU_B$ there is no function $e:\mathbb{Z}\to\mathbb{R}$ satisfying (ii) and (iii) of Theorem 7. That is, [23, Theorem 2.3] is insufficient for almost periodic solutions of (18). Unlike the conditions (ii) and (iii) of Theorem 7, we use a more general concept called discrete exponential dichotomy in our main result. If the homogeneous part of the equation (18) admits an exponential dichotomy, then Theorem 6 implies that the infinite delayed Volterra difference system has an almost periodic solution.

As an implementation of our results, we present the following numerical example:

Example 2. *Let us set $A(t)=\frac{1}{h}I_{2\times 2}$,*

$$B(t,k)=\begin{bmatrix}\frac{1}{2h}\left(\frac{1}{4}\left(\sin\left(\frac{\pi}{2}k\right)+\sin\left(\frac{\pi}{2}k\sqrt{2}\right)\right)\right)^{t-k} & 0\\ 0 & \frac{1}{2h}\exp(k-t)\end{bmatrix},\ k\le t,$$

and

$$g(t)=\begin{bmatrix}g_1(t)\\ g_2(t)\end{bmatrix}$$

in (15) where $g_1(t), g_2(t)$ are any periodic or almost periodic functions defined on $\mathbb{Z}$, abstract Banach space is $\mathcal{X}=\mathbb{R}$ and h is any positive integer.We have $0<\sup_{t\in\mathbb{Z}_+}\sum_{k=-\infty}^{t}\|B(t,k)\|<\frac{1}{h}<\infty$ or equivalently, $0<\max_{1\le i\le 2}N_i<\frac{1}{h}<\infty$ where

$$N_i:=\sum_{k=-\infty}^{t}\sum_{j=1}^{n}\left|b_{ij}(t,k)\right|,\ i=1,2.$$

Since $b_{11}(t,k)=\frac{1}{2h}\left(\frac{1}{4}\left(\sin\left(\frac{\pi}{2}k\right)+\sin\left(\frac{\pi}{2}k\sqrt{2}\right)\right)\right)^{t-k}$ is an almost periodic function (see [24]) and $b_{22}(t,k)=\frac{1}{2h}\exp(k-t)$ is a convolution type function, the conditions (A1-A4) are satisfied and we guarantee the existence of an almost periodic solution by Theorem 6. However, Theorem 7 is insufficient to ensure the existence of an almost periodic solution of the system (15), since $\|I+hA\|\ge 1+hU_B$ for any positive integer h (see Example 1).

As it is discussed in [4], if $\sum_{k=-\infty}^{-1}B(t,k)\theta(k)=0$, then the infinite delay Volterra system (15) reduces to the nonconvolution type Volterra system

$$y(t+1)=A(t)y(t)+\sum_{k=0}^{t}B(t,k)y(k)+g(t),\ t\in\mathbb{Z}_+. \tag{20}$$

For any matrix valued function $\tilde{B}(t,k)$ satisfying the conditions A2 and A3, if we set $B(t,k) := \tilde{B}(t,k)u_0(k)$, where u_0 is discrete Heaviside function, then $\sum_{k=-\infty}^{-1} B(t,k)\theta(k) = 0$ and we obtain the reduced system (20). This means, the existence of an almost periodic solution of the system (20) can be proven similar to Theorem 6. Hence, if x^θ is an almost periodic solution of the system (15), then $x(t) = x^\theta(t)$, $t \in \mathbb{Z}_+$ is an almost periodic solution of the system (20). This along with Theorem 6 leads to the following result providing a solution for the open problem proposed by S. Elaydi (see [5]) in 2009.

Theorem 8. *Assume A1-A4 and consider the equation*

$$y(t+1) = A(t)y(t) + \sum_{k=0}^{t} B(t,k)y(k) + g(t), \ \ t \in \mathbb{Z}_+. \tag{21}$$

If the following assumptions hold:
A5 *The matrix function $\tilde{B}(t,k)$ is discrete almost periodic in t and k for $t, k \in \mathbb{Z}$*
A6 *The matrix function $B(t,k)$ is defined as $B(t,k) = \tilde{B}(t,k)u_0(k)$*
A7 $\sup_{t\in\mathbb{Z}_+} \sum_{k=0}^{t} \|B(t,k)\| < \infty$ *and nonzero*

then the system (21) has an almost periodic solution.

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