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# Second order evolution equations with nonlocal conditions

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**Abstract:** In this paper, we shall establish sufficient conditions for the existence of solutions for second order semilinear functional evolutions equation with nonlocal conditions in Fréchet spaces. Our approach is based on the concepts of Hausdorff measure, noncompactness and Tikhonoff's fixed point theorem. We give an example for illustration.

**Keywords:** Second order differential equations, mild solution, evolution system, Fréchet space, nonlocal condition, Hausdorff's measures of noncompactness.

**MSC:** 34G20.

## 1 Introduction

Our main goal in this paper is to investigate the existence of solutions to the following nonlocal initial value problem for the second order evolution equation

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in J := [0, \infty), \tag{1}$$

$$y(0) = g(y),\ y'(0) = h(y), \tag{2}$$

where $\{A(t)\}_{0\le t<+\infty}$ is a family of linear closed operators from $E$ into $E$, that generate an evolution system of linear bounded operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $0 \le s \le t < +\infty$, $f : J \times E \to E$ is a Carathéodory function, $g, h : C(J; E) \to E$ are given functions, and $(E, |\cdot|)$ is a real Banach space.

Evolution equations arise in many areas of applied mathematics [1, 2]. This type of equations have received a lot of attention in recent years [3]. There are many results concerning second-order differential equations, see for example Fattorini [4], and Travis and Webb [5]. Useful for the study of abstract second order equations is the existence of an evolution system $U(t, s)$ for the homogenous equation

$$y''(t) = A(t)y(t), \quad \text{for } t \ge 0.$$

For this purpose there are many techniques to show the existence of $U(t, s)$ which have been developed by Kozak [6]. On the other hand, recently there has been an increasing interest in studying the abstract non-autonomous second order initial value problem

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in [0, T], \tag{3}$$

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$$y(0) = y_0, \; y'(0) = y_1. \tag{4}$$

The reader is referred to [7–10] and the references mentioned in these works. The pioneering work on evolution initial value problems with nonlocal conditions is due to Byszewski. As pointed out by Byszewski [11–14]. the study of evolution initial value problems with nonlocal conditions is of significance since they have applications in problems in physics and other areas of applied mathematics. Several authors have investigated the problem of nonlocal initial conditions for different classes of abstract differential equations in Banach spaces, for example, we refer the reader to [15–21] and the references therein.

In this paper we use the technique of measures of noncompactness. It is well known that this method provides an excellent tool for establishing the existence of solutions of nonlinear differential equations. This technique works fruitfully for both integral and differential equations. More details are found in Akhmerov *et al.* [22], Alv́ares [23], Banaś and Goebel [24], Guo *et al.* [25], Olszowy [26–28], Olszowy and Wędrychowicz [29], and the references therein.

Motivated by the above-mentioned works, we derive some sufficient conditions for the existence of solutions of second order semilinear functional evolution equations with nonlocal conditions in Fréchet spaces. Our results are achieved by applying the Hausdorff measure of noncompactness and fixed point theorem.

The work is organized as follows: In Section two, we recall some definitions and facts about evolution systems. In Section three, we give the existence of mild solutions to the problem (1)-(2). In Section four we present an example to illustrate our main result.

## 2 Preliminaries

Let $C(J, E)$ be the Fréchet space of all continuous functions $y$, mapping $J$ into $E$, equipped with the family of seminorms

$$\|y\|_T = \sup\{|y(t)|, t \in [0, T], \; T \geq 0\}.$$

In what follows, let $\{A(t), \; t \geq 0\}$ be a family of closed linear operators on the Banach space $E$ with domain $D(A(t))$ which is dense in $E$ and independent of $t$.

In this work the existence of solutions the problem (1)-(2) is related to the existence of an evolution operator $U(t, s)$ for the following homogeneous problems

$$y''(t) = A(t)y(t), \qquad t \in J. \tag{5}$$

This concept of evolution operator has been developed by Kozak [6].

**Definition 2.1.** *A family $U$ of bounded operators $U(t, s) : E \to E$, $(t, s) \in \Delta := \{(t, s) \in J \times J : s \leq t\}$ is called an evolution operator of the equation (5) if the following conditions hold:*

$(e_1)$ *For any $x \in E$ the map $(t, s) \longmapsto U(t, s)x$ is continuously differentiable and*

- (a) *for any $t \in J$, $U(t, t) = 0$,*
- (b) *for all $(t, s) \in \Delta$ and for any $x \in E$, $\frac{\partial}{\partial t}U(t, s)x\big|_{t=s} = x$ and $\frac{\partial}{\partial s}U(t, s)x\big|_{t=s} = -x$.*

$(e_2)$ *For all $(t, s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}U(t, s)x \in D(A(t))$, the map $(t, s) \longmapsto U(t, s)x$ is of class $C^2$ and*

- (a) $\frac{\partial^2}{\partial t^2}U(t, s)x = A(t)U(t, s)x,$
- (b) $\frac{\partial^2}{\partial s^2}U(t, s)x = U(t, s)A(s)x,$
- (c) $\frac{\partial^2}{\partial s \partial t}U(t, s)x\big|_{t=s} = 0.$

$(e_3)$ *For all* $(t,s) \in \Delta$, *then* $\frac{\partial}{\partial s} U(t,s)x \in D(A(t))$, *there exist* $\frac{\partial^3}{\partial t^2 \partial s} U(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t} U(t,s)x$ *and*

(a) $\frac{\partial^3}{\partial t^2 \partial s} U(t,s)x = A(t)\frac{\partial}{\partial s}(t)U(t,s)x.$
*Moreover, the map* $(t,s) \longmapsto A(t)\frac{\partial}{\partial s}(t)U(t,s)x$ *is continuous,*

(b) $\frac{\partial^3}{\partial s^2 \partial t} U(t,s)x = \frac{\partial}{\partial t} U(t,s)A(s)x.$

Throughout this paper, we will use the following definition of the concept of Hausdorff measure of noncompactness [24].

**Definition 2.2.** *The Hausdorff measure of noncompactness* $\mu$ *is defined by*

$$\mu(D) = inf\,\{r > 0, D \text{ can be covered by a finite number of balls with radius } r\}$$

*for a bounded set* $D$ *in any Banach space* $X$.

**Lemma 2.3.** *[24] Let* $X$ *be a Banach space and* $C, D \subset X$ *be bounded, then the following properties hold:*

$(i_1)$ $\mu(D) = 0$ *if only if* $D$ *is relatively compact,*
$(i_2)$ $\mu(\overline{D}) = \mu(D)$ ; $\overline{D}$ *the closure of* $D$,
$(i_3)$ $\mu(C) \le \mu(D)$ *when* $C \subset D$,
$(i_4)$ $\mu(C + D) \le \mu(C) + \mu(D)$ *where* $C + D = \{x \mid x = y + z; y \in C; z \in D\}$,
$(i_5)$ $\mu(aD) = |a|\mu(D)$ *for any* $a \in \mathbb{R}$,
$(i_6)$ $\mu(ConvD) = \mu(D)$; *where* $ConvD$ *is the convex hull of* $D$,
$(i_7)$ $\mu(C \cup D) = \max(\mu(C), \mu(D))$,
$(i_8)$ $\mu(C \cup \{x\}) = \mu(D)$ *for any* $x \in E$.

Denote by $\omega^T(y, \epsilon)$ the modulus of continuity of $y$ on the interval $[0, T]$, i.e.,

$$\omega^T(y,\epsilon) = \sup\left\{\left|y(t) - y(s)\right|; t, s \in [0, T], |t - s| \le \epsilon\right\}.$$

Moreover, let us put

$$\omega^T(D,\epsilon) = \sup\left\{\omega^T(y,\epsilon); y \in D\right\},$$

$$\omega_0^T(D) = \lim_{\epsilon \to 0} \sup \omega^T(D,\epsilon).$$

**Lemma 2.4.** *[30, Lemma 2.6] If* $\{D_n\}_{n=0}^{+\infty}$ *is a sequence of nonempty, bounded and closed subsets of* $E$, *such that* $D_{n+1} \subset D_n (n = 0, 1, 2, \ldots)$ *and if* $\lim_{n\to\infty} \mu(D_n) = 0$ *for each* $n \in \mathbb{N}$, *then the intersection*

$$D_\infty = \cap_{n=0}^{+\infty} D_n$$

*is nonempty and compact.*

**Lemma 2.5.** *[31] If* $B$ *is a bounded subset of Banach space, then for each* $\varepsilon > 0$ *there is a sequence* $\{b_n\}_{n=0}^{\infty} \subset B$, *such that*

$$\mu(B) \le 2\mu(\{b_n\}_{n=0}^{\infty}) + \varepsilon.$$

We recall that a subset $B \subset L^1([0; T]; E)$ is uniformly integrable if there exists $\xi \in L^1([0; T]; \mathbb{R}^+)$, such that

$$\|x(s)\| \le \xi(s) \text{ for } x \in B \text{ and a.e. } s \in [0; T].$$

**Lemma 2.6.** *[32] If $\{B_n\}_{n=0}^{\infty} \subset L^1([0;T], E)$ is uniformly integrable, then the function $t \to \mu(\{B_n(t)\}_{n=0}^{\infty})$ is measurable and*

$$\mu\left\{\int_0^t B_n(s))ds\right\}_{n=0}^{\infty} \leq 2\int_0^t \mu(\{B_n(s)\}_{n=0}^{\infty})ds, \qquad t \in [0;T].$$

**Lemma 2.7.** *[31] Assume that a set $X \subset C([0;T], E)$ is bounded, then*

$$\sup_{t\in[0,T]} \mu(X(t)) \leq \mu(X([0,T])) \leq \omega_0^T(X) + \sup_{t\in[0,T]} \mu(X(t)),$$

$$\sup_{t\in[0,T]} \mu(X(t)) \leq \eta(X) \leq \omega_0^T(X) + \sup_{t\in[0,T]} \mu(X(t)),$$

*where*

$$X(t) = \{x(t) : x \in X\} \qquad t \in [0,T],$$

$$X([0,T]) = \{x(s) : x \in X, s \in [0,T]\}$$

*and $\eta$ is a measure of noncompactness in $C([0,T], E)$.*

**Theorem 2.8** (Tykhonoff fixed point theorem)**.** *[33] Let $F$ be a locally convex space, $\mathcal{K}$ a compact convex subset of $F$ and $N : \mathcal{K} \to \mathcal{K}$ a continuous map. Then $N$ has at least one fixed point in $\mathcal{K}$.*

# 3 Main result

**Definition 3.1.** *A function $y \in C(J, E)$ is called a mild solution to the problem (1)-(2) if $y$ satisfies the integral equation*

$$y(t) = -\frac{\partial}{\partial s}U(t,0)g(y) + U(t,0)h(y) + \int_0^t U(t,s)f(s,y(s))ds. \tag{6}$$

To prove our results we introduce the following conditions:

$(H_1)$ There exists a constant $M \geq 1$, such that:

$$\|U(t,s)\|_{B(E)} \leq M, (t,s) \in \Delta.$$

$(H_2)$ There exists a constant $\tilde{M} \geq 0$, such that:

$$\|\frac{\partial}{\partial s}U(t,s)\|_{B(E)} \leq \tilde{M}, (t,s) \in \Delta.$$

$(H_3)$ There exist an integrable function $p : J \to \mathbb{R}_+$ and a continuous nondecreasing function $\psi : [0,\infty) \to (0;\infty)$, such that:

$$\left|f(t,u)\right| \leq p(t)\psi(|u|) \text{ for a.e } t \in J \text{ and each } u \in E.$$

$(H_4)$ There exists a locally integrable function $\sigma : J \to \mathbb{R}_+$, such that for any nonempty bounded set $D \subset E$ we have :

$$\mu(f(t,D)) \leq \sigma(t)\mu(D) \text{ for a.e } t \in J.$$

$(H_5)$ $g, h : C(J,E) \to E$ are continuous mappings and

$$\sup_{y\in D}|g(y)| < \infty, \quad \sup_{y\in D}|h(y)| < \infty$$

for any nonempty bounded set $D \subset C(J,E)$.

$(H_6)$ There exist $L_i > 0$ $(i = 1, 2)$, such that:

$$\mu(g(D)) \leq L_1\eta(D),$$

and

$$\mu(h(D)) \leq L_2\eta(D),$$

for any nonempty bounded set $D \subset C(J, E)$.

$(H_7)$ There exists a constant $R > 0$, such that:

$$\tilde{M} \sup_{y\in B_R} |g(y)| + M(\sup_{y\in B_R} |h(y)| + \psi(R)\, \|p\|_{L^1}) \leq R,$$

where $B_R$ is the closed ball in $C(J; E)$, centered at zero and with radius $R$.

Consider the operators $N_i : C(J, E) \to C(J, E)(i = 1, 2, 3)$, defined by

$$(N_1y)(t) = -\frac{\partial}{\partial s}U(t, 0)g(y),$$

$$(N_2y)(t) = U(t, 0)h(y),$$

$$(N_3y)(t)) = \int_0^t U(t, s)f(s, y(s))ds.$$

**Lemma 3.2.** *[34] Assume that the hypotheses $(H_1) - (H_7)$ hold and $D \subset C(J; E)$ is a bounded set. Then*

$$\omega_0^T(N_1(D)) \leq 2M\mu(g(D)),$$

$$\omega_0^T(N_2(D)) \leq 2M\mu(h(D)),$$

$$\omega_0^T(N_3(D)) \leq 2M \int_0^T \mu(f(s, D(s))ds.$$

**Theorem 3.3.** *Assume that the hypotheses $(H_1) - (H_7)$ are satisfied. If*

$$3\tilde{M}L_1 + 3ML_2 + \frac{6}{\tau} < 1, \ \tau > 6, \tag{7}$$

*then the problem (1)-(2) admits at least one mild solution.*

*Proof.* Consider the operator $N : C(J, E) \to C(J, E)$, defined by

$$(Ny)(t) = -\frac{\partial}{\partial s}U(t, 0)g(y) + U(t, 0)h(y) + \int_0^t U(t, s)f(s, y(s))ds.$$

We define

$$D = B_R = \left\{y \in C(J, E) : \|y\|_T \leq R\right\}.$$

The set $B_R$ is non-empty, convex and closed.

Now, for $t \in [0, T]$, $T > 0$, from $(H_1) - (H_3)$ and $(H_7)$ we have

$$\begin{aligned}
\left|(Ny)(t)\right| &\leq \left\|\frac{\partial}{\partial s}U(t,0)\right\|_{B(E)} |g(y)| + \left\|U(t,s)\right\|_{B(E)} |h(y)| + \left\|U(t,s)\right\|_{B(E)} \int_0^t p(s)\psi(|y(s)|)ds \\
&\leq \tilde{M}|g(y)| + M|h(y)| + M\int_0^t p(s)\psi(R)ds \\
&\leq \tilde{M}\sup_{y\in B_R}|g(y)| + M(\sup_{y\in B_R}|h(y)| + \psi(R)\,\|p\|_{L^1}) \\
&\leq R. \qquad (8)
\end{aligned}$$

Equation (8) ensures that the operator $N$ maps the set $B_R$ into itself.

**Step 1.** $N$ is continuous.

Let $(y_n)_{n\in N}$ be a sequence in $B_R$, such that $y_n \to y$ in $B_R$.

For $t \in [0, T]$, $T \geq 0$ we have

$$\begin{aligned}
\left|(Ny_n)(t) - (Ny)(t)\right| &\leq |g(y_n) - g(y)| + |h(y_n) - h(y)| \\
&+ M\int_0^t \left|f(s, y_n(s)) - f(s, y(s))\right| ds.
\end{aligned}$$

Since the functions $g$, $h$ are continuous and $f$ is Carathéodory, the Lebesgue dominated convergence theorem implies that

$$\|Ny_n - Ny\|_T \to 0 \quad \text{as } n \to +\infty.$$

So $N$ is continuous.

Consider the measure of noncompacteness $\mu^*(D)$, defined on the family of bounded subsets of the space $C(J, E)$ by

$$\mu^*(D) = \sup\left\{e^{-\tau\tilde{\sigma}(T)}(\omega_0^T(D) + \sup_{t\in[0,T]}\mu(D_n(t)); T \geq 0\right\},$$

where

$$\tilde{\sigma}(t) = M\int_0^t \sigma(s)ds, \qquad \tau > 6.$$

**Step 2.** $D_\infty = \cap_{n=0}^{+\infty} D_n$ is compact.

In the sequel, we consider the sequence of sets $\{D_n\}_{n=0}^{+\infty}$, defined by induction as follows :

$$D_0 = D = B_R,\ D_{n+1} = Conv(N(D_n)) \text{ for } n = 0, 1, 2, \cdots \text{ and } D_\infty = \cap_{n=0}^{+\infty} D_n.$$

This sequence is nondecreasing, i.e., $D_{n+1} \subset D_n$ for each $n$.

**Claim 1.** $\lim_{n\to+\infty} \mu^*(D_n) = 0$.

We know from Lemma 2.5 that for each $\varepsilon > 0$ there is a sequence of functions $\{W_k\}_{k=0}^{\infty} \subset (N_3 D_n)(s)$, such that

$$\mu(N_3 D_n)(s) \leq 2\mu(\{W_k\}_0^{\infty}) + \varepsilon.$$

This implies that there is a sequence $\{Q_k\}_{k=0}^{\infty} \subset W_k(s)$, such that

$$W_k = (N_3 Q_k)(s) \text{ for } k = 1, 2, \ldots$$

Using the properties of $\mu$, Lemma 2.5, Lemma 2.6 and assumptions $(H_4)$, $(H_5)$, we get

$$\begin{aligned}
\mu(D_{n+1}(t)) &= \mu(ConvN(D_n)(t)) \\
&= \mu((N_1D_n)(t)) + \mu((N_2D_n)(t)) + \mu((N_3D_n)(t)) \\
&= \mu(g(D_n)) + \mu(h(D_n)) + 2\mu(\{W_k\}_{k=0}^{\infty}) + \varepsilon \\
&= \mu(g(D_n)) + \mu(h(D_n)) + 2\mu(\{(N_3Q_k)(s)\}_{k=0}^{\infty}) + \varepsilon \\
&= \mu(g(D_n)) + \mu(h(D_n)) + 4\mu\left(\left\{\int_0^t U(t,s)f(s,(N_3Q_k)(s))ds\right\}_{k=0}^{\infty}\right) + \varepsilon \\
&\le \tilde{M}L_1\eta(D_n) + MK_2\eta(D_n) + 4M\int_0^t \sigma(s)\mu(\{Q_k(s)\}_{k=0}^{\infty})ds + \varepsilon \\
&\le \tilde{M}L_1\eta(D_n) + MK_2\eta(D_n) + 4M\int_0^t \sigma(s)\mu(D_n(s))ds + \varepsilon.
\end{aligned}$$

Since $\varepsilon$ is arbitrary, using Lemma 2.7 we obtain

$$\begin{aligned}
\mu(D_{n+1}(t)) &\le \tilde{M}L_1(\omega_0^T(D_n) + \sup_{t\in[0,T]}\mu(D_n(t))) \\
&+ ML_2(\omega_0^T(D_n) + \sup_{t\in[0,T]}\mu(D_n(t))) \\
&+ 4M\int_0^t \sigma(s)(\omega_0^T(D_n) + \sup_{s\in[0,T]}\mu(D_n(s)))ds.
\end{aligned} \tag{9}$$

Now, applying Lemma 3.2 and using assumptions $(H_4)$, $(H_5)$ (see also [35]), we derive

$$\begin{aligned}
\omega_0^T(D_{n+1}) &= \omega_0^T(Conv(ND_n)) \\
&= \omega_0^T(N_1D_n) + \omega_0^T(N_2D_n) + \omega_0^T(N_3D_n)) \\
&\le \tilde{M}L_1\eta(D_n) + ML_2\eta(D_n) + 2M\int_0^t \sigma(s)\mu(D_n(s))ds \\
&\le 2\tilde{M}L_1(\omega_0^T(D_n) + \sup_{t\in[0,T]}\mu(D_n(t)) \\
&+ 2ML_2(\omega_0^T(D_n) + \sup_{t\in[0,T]}\mu(D_n(t))) \\
&+ 2M\int_0^t \sigma(t)(\omega_0^T(D_n) + \sup_{s\in[0,T]}\mu(D_n(s))ds.
\end{aligned} \tag{10}$$

From (9) and (10), we have

$$\begin{aligned}
&\omega_0^T(D_{n+1}) + \sup_{t\in[0,T]}\mu(D_{n+1}(t)) \\
&\le (3\tilde{M}L_1 + 3ML_2)(\omega_0^T(D_n) + \sup_{t\in[0,T]}\mu(D_n(t))) + M\int_0^t \sigma(s)(\omega_0^T(D_n) + \sup_{s\in[0,T]}\mu(D_n(s)))ds.
\end{aligned}$$

Then

$$\omega_0^T(D_{n+1}) + \sup_{t\in[0,T]} \mu(D_{n+1}(t))$$
$$\leq (3\tilde{M}K_1 + 3MK_2)(\omega_0^T(D_n) + \sup_{t\in[0,T]} \mu(D_n(t))) + 6M\int_0^t \sigma(s)e^{-\tau\tilde{\sigma}(t)}e^{\tau\tilde{\sigma}(t)}(\omega_0^T(D_n) + \sup_{s\in[0,T]} \mu(D_n(s)))ds.$$

We obtain

$$e^{-\tau\tilde{\sigma}(T)}(\omega_0^T(D_{n+1}) + \sup_{t\in[0,T]} \mu(D_n(t)))$$
$$\leq \left(3\tilde{M}K_1 + 3MK_2 + \frac{6}{\tau}\right)\sup\{e^{-\tau\tilde{\sigma}(T)}(\omega_0^T(D_n) + \sup_{t\in[0,T]} \mu(D_n(t))) : T \geq 0\}.$$

Hence, we get

$$\mu^*(D_{n+1}) \leq \left(3\tilde{M}K_1 + 3MK_2 + \frac{6}{\tau}\right)\mu^*(D_n).$$

By the method of mathematical induction, we can prove that

$$\mu^*(D_{n+1}) \leq \left(3\tilde{M}K_1 + 3MK_2 + \frac{6}{\tau}\right)^{n+1}\mu^*(D_0).$$

Hence, by (7), we get

$$\lim_{n\to+\infty} \mu^*(D_n) = 0.$$

Taking into account Lemma 2.4, we infer that $D_\infty = \cap_{n=0}^{+\infty} D_n$ is nonempty, convex and compact. Thus, by Tykhonoff's fixed point theorem, the operator $N : D_\infty \to D_\infty$ has at least one fixed point, which is a mild solution to problem (1)-(2). □

## 4 An example

Consider the following partial differential equation with nonlocal conditions

$$\begin{cases} \frac{\partial^2 z(t,\tau)}{\partial t^2} &= \frac{\partial^2 z(t,\tau)}{\partial \tau^2} + a(t)\frac{\partial z(t,\tau)}{\partial t} \\ & +f_1(t, z(t,\tau)), \qquad t \in J,\ \tau \in [0,\pi], \\ z(t,0) = & z(t,\pi) = 0 \qquad t \in J, \\ z(0,\tau) = & \int_0^{+\infty} g_1(t, z(t,\tau))dt, \qquad \tau \in [0,\pi], \\ \frac{\partial}{\partial t}z(0,\tau) = & \int_0^{+\infty} h_1(t, z(t,\tau))dt, \qquad \tau \in [0,\pi] \end{cases} \tag{11}$$

where $a : J \to \mathbb{R}$ is a Hölder continuous function and $h_1, g_1 : J \times R \to \mathbb{R}$ are given functions.
Let $E = L^2([0,\pi],\mathbb{C})$ be the space of 2-integrable functions from $[0,\pi]$ into $\mathbb{R}$, and let $H^2([0,\pi],\mathbb{C})$ be the Sobolev space of functions $x : [0,\pi] \to \mathbb{R}$, such that $x'' \in L^2([0,\pi],\mathbb{C})$. We consider the operator $A_1 y(\tau) = y''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R},\mathbb{C})$, which is an infinitesimal generator of strongly continuous cosine function $C(t)$ on $E$. Moreover, we take $A_2(t)y(s) = a(t)y'(s)$, defined on $H^1([0,\pi],\mathbb{C})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$ which, generates an evolution operator $U$, defined by

$$U(t,s) = \sum_{n\in\mathbb{Z}} z_n(t,s)\langle x, w_n\rangle w_n,$$

where $z_n$ is a solution to the following scalar initial value problem

$$\begin{cases} z''(t) = -n^2 z(t) + ina(t)z(t) \\ z(0) = 0, \qquad z'(0) = 1. \end{cases}$$

Set

$$w(t)(\tau) = z(t,\tau),\ t \geq 0,\ \tau \in [0,\pi],$$

$$f(t,z(t,\tau)) = f_1(t,z(t,\tau)),$$

$$g(z)(\tau) = \int_0^{+\infty} g_1(t,z(t,\tau))dt, \tau \in [0,\pi],$$

$$h(z)(\tau) = \int_0^{+\infty} h_1(t,z(t,\tau))ds, \tau \in [0,\pi].$$

We now assume that:

(1) The map $f$ is Carathéodory and satisfies conditions $(H_3)$, $(H_4)$.
(2) The maps $g$ and $h$ satisfy the Carathéodory conditions and there exist functions $\varrho_i \in L^2(J)$ $(i = 1, 2)$ such that

$$|g_1(t;s_2) - g_1(t;s_1)| \le \varrho_1(t)|s_2 - s_1| \text{ for a.e. } t, s \in \mathbb{R}; \tag{12}$$

and

$$|h_1(t;s_2) - h_1(t;s_1)| \le \varrho_2(t)|s_2 - s_1| \text{ for a.e. } t, s \in \mathbb{R}; \tag{13}$$

Next, let us observe that, in view of (12) and (13), the mappings $g$ and $h$ fulfil the inequalities

$$\|g(t;z_2) - g(t;z_1)\| \le \left(\int_0^T \varrho_1^2(t)dt\right)^{\frac{1}{2}} \|z_2 - z_1\|,$$

and

$$\|h(t;z_2) - h(t;z_1)\| \le \left(\int_0^T \varrho_2^2(t)dt\right)^{\frac{1}{2}} \|z_2 - z_1\|.$$

Hence, reasoning similarly as in the proof of Claim 1 and using Lemma 2.7, we infer that for any $D \subset C(J;E)$

$$\begin{aligned} \mu(g(D)) &\le 4\left(\int_0^T \varrho_1^2(t)dt\right)^{\frac{1}{2}} \sup_{t\in J} \mu(D(t)) \\ &\le 4\left(\int_0^T \varrho_1^2(t)dt\right)^{\frac{1}{2}} \eta(D), \end{aligned}$$

and

$$\begin{aligned} \mu(h(D)) &\le 4\left(\int_0^T \varrho_2^2(t)dt\right)^{\frac{1}{2}} \sup_{t\in J} \mu(D(t)) \\ &\le 4\left(\int_0^T \varrho_2^2(t)dt\right)^{\frac{1}{2}} \eta(D). \end{aligned}$$

These show that the maps $g$ and $h$ satisfy conditions $(H_5)$ and $(H_6)$ with the constants

$$L_i = \left(\int_0^T \varrho_i^2(t)dt\right)^{\frac{1}{2}}, \; i = 1, 2.$$

Problem (11) can be written in the abstract form (1)-(2) with $A(t)$ and $f$ defined above. The existence of mild solutions can be deduced from an application of Theorem 3.3.

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