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A note on range-kernel uncomplementation

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Abstract: This note exhibits a Banach-space operator such that neither the range nor the kernel is complemented both for the operator and its adjoint.

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1 Introduction

By a subspace we mean a *closed* linear manifold of a normed space. Every linear manifold of a normed space has an algebraic complement which is a linear manifold not necessarily closed. A subspace is complemented if it has a subspace as an algebraic complement. Every subspace of a Hilbert space is complemented. This is not the case in a Banach space. Banach-space operators with complemented range and kernel play a crucial role in many aspects of operator theory, especially in Fredholm theory, being the main feature behind the difference between Hilbert-space and Banach-space approaches for dealing with Fredholm operators [1, 2].

It is known that compact perturbations of left or right semi-Fredholm (in particular, of invertible) operators, as well as continuous projections, acting on an arbitrary Banach space have complemented kernel and complemented closed range, and also that the class of all operators with complemented kernel and complemented closure of range is algebraically and topologically large. This is summarized in Lemma 3.2. The main result of this note exhibits a Banach-space operator whose closed range and kernel are not complemented, both for the operator itself as well as for its normed-space adjoint — Theorem 4.1.

The paper is organized as follows. Section 2 deals with notation and terminology, including the concepts of upper-lower and left-right semi-Fredholmness. Section 3 considers the classes $\Gamma[\mathcal{X}]$ and $\Delta[\mathcal{X}]$ of operators T on a Banach space $\mathcal{X}$ (i.e., operators in $\mathcal{B}[\mathcal{X}]$) for which closure of range, $\mathcal{R}(T)^-$, and kernel, $\mathcal{N}(T)$, are both complemented, or are both uncomplemented, respectively. It is shown in Lemma 3.1 that the collection $\Theta(\mathcal{B}[\mathcal{X}])$ of all classes of operators in $\mathcal{B}[\mathcal{X}]$ for which $\mathcal{R}(T)^-$ is complemented if and only if $\mathcal{N}(T)$ is complemented coincides (as expected) with the power set of the union $\Gamma[\mathcal{X}] \cup \Delta[\mathcal{X}]$. Lemma 3.2 and Corollary 3.1 (on range-kernel complementation for normed-space adjoints) close the section. Section 4 focuses on range-kernel uncomplementation, where Lemma 4.1 deals with complemented subspaces and their direct sum with the null space, and the main result appears in Theorem 4.1. All Propositions in Sections 2, 3, 4 are well-known results, which are applied throughout the text. Since these are used quite frequently, those propositions are stated in full (whose proofs are always addressed to current literature).

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2 Notation and Terminology

Our notation and terminology are quite standard. Throughout the paper $\mathcal{X}$ will stand for a normed space (or a Banach space, when completeness is necessary). A closed linear manifold of $\mathcal{X}$ (closed in the norm topology of $\mathcal{X}$) will be referred to as a subspace of $\mathcal{X}$. Let $\mathcal{M}^-$ denote the closure of a linear manifold $\mathcal{M}$ of $\mathcal{X}$, which is a subspace of $\mathcal{X}$, and let $\mathcal{B}[\mathcal{X}]$ denote the normed algebra of all operators on $\mathcal{X}$; that is, of all bounded linear (i.e., continuous linear) transformations of $\mathcal{X}$ into itself. For any operator $T \in \mathcal{B}[\mathcal{X}]$, let $\mathcal{N}(T) = T^{-1}(\{0\})$ denote its kernel, which is a subspace (i.e., a closed linear manifold) of $\mathcal{X}$, and let $\mathcal{R}(T) = T(\mathcal{X})$ denote its range, which is a linear manifold of $\mathcal{X}$.

For every linear manifold $\mathcal{M}$ of any normed space $\mathcal{X}$ there exists another linear manifold $\mathcal{N}$ such that $\mathcal{X} = \mathcal{M} + \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$, where $\mathcal{N}$ and $\mathcal{M}$ are referred to as algebraic complements of each other. The codimension of $\mathcal{M}$ is the (invariant) dimension of any algebraic complement of it: $\operatorname{codim} \mathcal{M} = \dim \mathcal{N}$. A subspace $\mathcal{M}$ of a normed space $\mathcal{X}$ is *complemented* if it has a subspace as an algebraic complement. In other words, a *closed* linear manifold $\mathcal{M}$ of a normed space $\mathcal{X}$ is complemented if there is a *closed* linear manifold $\mathcal{N}$ of $\mathcal{X}$ such that $\mathcal{M}$ and $\mathcal{N}$ are algebraic complements (i.e., such that $\mathcal{M} + \mathcal{N} = \mathcal{X}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$). In this case $\mathcal{M}$ and $\mathcal{N}$ are *complementary subspaces*. Equivalently, a subspace is complemented if and only if it is the range of a continuous projection (see e.g. [1, Remark 1.1]). A normed space is *complemented* if every subspace of it is complemented. If a Banach space is complemented, then it is isomorphic (i.e., topologically isomorphic) to a Hilbert space [3] (see also [4]). Thus complemented Banach spaces are identified with Hilbert spaces — only Hilbert spaces (up to an isomorphism) are complemented. However, an uncomplemented subspace of an uncomplemented Banach space may be isomorphic to a Hilbert space [5]. For a thorough presentation of results along this line see [6].

Definition 2.1. (See e.g. [7, Definition 16.1]). Let $\mathcal{X}$ be a Banach space and consider the following classes of operators on $\mathcal{X}$.

$$\Phi_+[\mathcal{X}] = \big\{T \in \mathcal{B}[\mathcal{X}] \colon \mathcal{R}(T) \text{ is closed and } \dim \mathcal{N}(T) < \infty\big\}$$

is the class of *upper semi-Fredholm* operators from $\mathcal{B}[\mathcal{X}]$, and

$$\Phi_-[\mathcal{X}] = \big\{T \in \mathcal{B}[\mathcal{X}] \colon \mathcal{R}(T) \text{ is closed and } \operatorname{codim} \mathcal{R}(T) < \infty\big\}$$

is the class of *lower semi-Fredholm* operators from $\mathcal{B}[\mathcal{X}]$. Set

$$\Phi[\mathcal{X}] = \Phi_+[\mathcal{X}] \cap \Phi_-[\mathcal{X}],$$

which is the class of *Fredholm* operators from $\mathcal{B}[\mathcal{X}]$.

Definition 2.2. (See e.g. [8, Section 5.1]). Let $\mathcal{X}$ be a Banach space and consider the following classes of operators on $\mathcal{X}$.

$$\begin{aligned}\mathcal{F}_\ell[\mathcal{X}] &= \big\{T \in \mathcal{B}[\mathcal{X}] \colon T \text{ is left essentially invertible}\big\}\\ &= \big\{T \in \mathcal{B}[\mathcal{X}] \colon S\,T = I + K \text{ for some } S \in \mathcal{B}[\mathcal{X}] \text{ and some compact } K \in \mathcal{B}[\mathcal{X}]\big\}\end{aligned}$$

is the class of *left semi-Fredholm* operators from $\mathcal{B}[\mathcal{X}]$, and

$$\begin{aligned}\mathcal{F}_r[\mathcal{X}] &= \big\{T \in \mathcal{B}[\mathcal{X}] \colon T \text{ is right essentially invertible}\big\}\\ &= \big\{T \in \mathcal{B}[\mathcal{X}] \colon TS = I + K \text{ for some } S \in \mathcal{B}[\mathcal{X}] \text{ and some compact } K \in \mathcal{B}[\mathcal{X}]\big\}\end{aligned}$$

is the class of *right semi-Fredholm* operators from $\mathcal{B}[\mathcal{X}]$. Set

$$\mathcal{F}[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}] \cap \mathcal{F}_r[\mathcal{X}] = \big\{T \in \mathcal{B}[\mathcal{X}] \colon T \text{ is essentially invertible}\big\},$$

which is the class of *Fredholm* operators from $\mathcal{B}[\mathcal{X}]$, and

$$\mathcal{SF}[\mathcal{X}] = \mathcal{F}_\ell[\mathcal{X}] \cup \mathcal{F}_r[\mathcal{X}],$$

which is the class of *semi-Fredholm* operators from $\mathcal{B}[\mathcal{X}]$.

For a collection of relations among $\Phi_+[\mathcal{X}]$, $\Phi_-[\mathcal{X}]$, $\mathcal{F}_\ell[\mathcal{X}]$, and $\mathcal{F}_r[\mathcal{X}]$ see e.g. [1, Section 3]. In particular, the well-known identity

$$\Phi[\mathcal{X}] = \mathcal{F}[\mathcal{X}].$$

The classes $\Phi_+[\mathcal{X}]$ and $\Phi_-[\mathcal{X}]$ are open in $\mathcal{B}[\mathcal{X}]$ (see e.g. [7, Proposition 16.11]), and so are the classes $\mathcal{F}_\ell[\mathcal{X}]$ and $\mathcal{F}_r[\mathcal{X}]$ (see e.g. [9, Proposition XI.2.6]). For a collection of standard results involving $\mathcal{F}_\ell[\mathcal{X}]$, $\mathcal{F}_r[\mathcal{X}]$, $\mathcal{F}[\mathcal{X}]$ and $\mathcal{SF}[\mathcal{X}]$ see e.g. [8, Section 5.1] (also [10, Problem 181]).

Definition 2.3. Let $\mathcal{X}$ be any normed space and define the following classes of operators on $\mathcal{X}$.

$$\Gamma_R[\mathcal{X}] = \big\{T \in \mathcal{B}[\mathcal{X}]\colon\ \mathcal{R}(T)^- \text{ is a complemented subspace of } \mathcal{X}\big\},$$
$$\Gamma_N[\mathcal{X}] = \big\{T \in \mathcal{B}[\mathcal{X}]\colon\ \mathcal{N}(T) \text{ is a complemented subspace of } \mathcal{X}\big\}.$$

Left and upper, as well as right and lower, semi-Fredholm operators are linked by range and kernel complementation, respectively, as follows.

Proposition 2.1. *Let $\mathcal{X}$ be a Banach space.*

$$\begin{aligned}\mathcal{F}_\ell[\mathcal{X}] &= \Phi_+[\mathcal{X}] \cap \Gamma_R[\mathcal{X}] \\ &= \big\{T \in \Phi_+[\mathcal{X}]\colon\ \mathcal{R}(T) \text{ is a complemented subspace of } \mathcal{X}\big\}.\end{aligned}$$

$$\begin{aligned}\mathcal{F}_r[\mathcal{X}] &= \Phi_-[\mathcal{X}] \cap \Gamma_N[\mathcal{X}] \\ &= \big\{T \in \Phi_-[\mathcal{X}]\colon\ \mathcal{N}(T) \text{ is a complemented subspace of } \mathcal{X}\big\}.\end{aligned}$$

Proof. [7, Theorems 16.14, 16.15] (since $\mathcal{R}(T)^- = \mathcal{R}(T)$ if $T \in \Phi_+[\mathcal{X}] \cup \Phi_-[\mathcal{X}]$). □

That is, $T \in \mathcal{F}_\ell[\mathcal{X}]$ if and only if $T \in \Phi_+[\mathcal{X}]$ and $\mathcal{R}(T)$ (which is closed by Definition 2.1 so that $\mathcal{R}(T) = \mathcal{R}(T)^-$) is complemented, and $T \in \mathcal{F}_r[\mathcal{X}]$ if and only if $T \in \Phi_-[\mathcal{X}]$ and $\mathcal{N}(T)$ is complemented.

3 Range-Kernel Complementation

We will be dealing with operators for which closure of range and kernel are either both complemented or both uncomplemented. We begin by describing these two classes of operators. Let $\mathcal{X}$ be a normed space and set

$$\begin{aligned}\Gamma[\mathcal{X}] &= \Gamma_R[\mathcal{X}] \cap \Gamma_N[\mathcal{X}] \\ &= \big\{T \in \mathcal{B}[\mathcal{X}]\colon\ \mathcal{R}(T)^- \text{ and } \mathcal{N}(T) \text{ are complemented subspaces of } \mathcal{X}\big\},\end{aligned}$$

the class of operators on $\mathcal{X}$ for which closure of range and kernel are complemented. (Operators with this property are sometimes called *inner regular* [11, Section 0] — see also [12,Theorem 3.8.2].) Clearly $\Gamma_R[\mathcal{X}] = \Gamma_N[\mathcal{X}]$ if and only if $\Gamma_R[\mathcal{X}] = \Gamma_N[\mathcal{X}] = \Gamma[\mathcal{X}]$. On the other hand consider the complement of the union $\Gamma_R[\mathcal{X}] \cup \Gamma_N[\mathcal{X}]$,

$$\begin{aligned}\Delta[\mathcal{X}] &= \mathcal{B}[\mathcal{X}]\backslash(\Gamma_R[\mathcal{X}] \cup \Gamma_N[\mathcal{X}]) \\ &= \big\{T \in \mathcal{B}[\mathcal{X}]\colon\ \mathcal{R}(T)^- \text{ and } \mathcal{N}(T) \text{ are not complemented subspaces of } \mathcal{X}\big\},\end{aligned}$$

so that

$$\Gamma[\mathcal{X}] \cap \Delta[\mathcal{X}] = \varnothing.$$

Let $\mathcal{T}[\mathcal{X}] \subseteq \mathcal{B}[\mathcal{X}]$ be an arbitrary class of operators such that the collection of operators with complemented closure of range coincides with the collection of operators with complemented kernel. That is, let $\mathcal{T}[\mathcal{X}]$ be a class of operators for which

$$\Gamma_R[\mathcal{X}] \cap \mathcal{T}[\mathcal{X}] = \Gamma_N[\mathcal{X}] \cap \mathcal{T}[\mathcal{X}].$$

Equivalently, $\mathcal{T}[\mathcal{X}]$ is any class of operators from $\mathcal{B}[\mathcal{X}]$ such that

$$\Gamma_R[\mathcal{X}] \cap \mathcal{T}[\mathcal{X}] = \Gamma_N[\mathcal{X}] \cap \mathcal{T}[\mathcal{X}] = \Gamma[\mathcal{X}] \cap \mathcal{T}[\mathcal{X}].$$

Let $\Theta(\mathcal{B}[\mathcal{X}])$ stand for the collection of all these classes. In other words, let $\wp(\mathcal{B}[\mathcal{X}])$ stand for the power set of $\mathcal{B}[\mathcal{X}]$ (the collection of all classes of operators from $\mathcal{B}[\mathcal{X}]$), and consider the subcollection $\Theta(\mathcal{B}[\mathcal{X}]) \subseteq \wp(\mathcal{B}[\mathcal{X}])$ of all classes $\mathcal{T}[\mathcal{X}]$ of operators from $\mathcal{B}[\mathcal{X}]$ for which $\Gamma_R[\mathcal{X}] \cap \mathcal{T}[\mathcal{X}] = \Gamma_N[\mathcal{X}] \cap \mathcal{T}[\mathcal{X}]$:

$$\begin{aligned}\Theta(\mathcal{B}[\mathcal{X}]) &= \big\{\mathcal{T}[\mathcal{X}] \in \wp(\mathcal{B}[\mathcal{X}])\colon\ \Gamma_R[\mathcal{X}] \cap \mathcal{T}[\mathcal{X}] = \Gamma_N[\mathcal{X}] \cap \mathcal{T}[\mathcal{X}]\big\}\\ &= \big\{\mathcal{T}[\mathcal{X}] \in \wp(\mathcal{B}[\mathcal{X}])\colon\ \forall\ T \in \mathcal{T}[\mathcal{X}],\\ &\qquad \mathcal{R}(T)^- \text{ is complemented if and only if } \mathcal{N}(T) \text{ is complemented}\big\}.\end{aligned}$$

Since $\Gamma[\mathcal{X}] \cap \Gamma_R[\mathcal{X}] = \Gamma[\mathcal{X}] \cap \Gamma_N[\mathcal{X}] = \Gamma[\mathcal{X}]$, and $\Delta[\mathcal{X}] \cap \Gamma_R[\mathcal{X}] = \Delta[\mathcal{X}] \cap \Gamma_N[\mathcal{X}] = \varnothing$,

$$\Gamma[\mathcal{X}] \cup \Delta[\mathcal{X}] \in \Theta(\mathcal{B}[\mathcal{X}]).$$

Lemma 3.1. $\Theta(\mathcal{B}[\mathcal{X}]) = \wp(\Gamma[\mathcal{X}] \cup \Delta[\mathcal{X}])$.

Proof. Observe:

$$\Gamma[\mathcal{X}] \cup \Delta[\mathcal{X}] \text{ is a maximum in } \Theta(\mathcal{B}[\mathcal{X}])$$

in the inclusion ordering of $\mathcal{B}[\mathcal{X}]$. Indeed, take any class $\mathcal{T}[\mathcal{X}] \in \Theta(\mathcal{B}[\mathcal{X}])$ so that, for every $T \in \mathcal{T}[\mathcal{X}]$, $\mathcal{R}(T)^-$ is complemented if and only if $\mathcal{N}(T)$ is complemented. Hence either both $\mathcal{R}(T)^-$ and $\mathcal{N}(T)$ are complemented, or both $\mathcal{R}(T)^-$ and $\mathcal{N}(T)$ are not complemented. This means $\mathcal{T}[\mathcal{X}] \subseteq \Gamma[\mathcal{X}] \cup \Delta[\mathcal{X}]$. Since $\Gamma[\mathcal{X}] \cup \Delta[\mathcal{X}]$ lies in $\Theta(\mathcal{B}[\mathcal{X}])$, the above statement holds true. Equivalently, $\mathcal{T}[\mathcal{X}] \in \Theta(\mathcal{B}[\mathcal{X}]) \Longrightarrow \mathcal{T}[\mathcal{X}] \subseteq \Gamma[\mathcal{X}] \cup \Delta[\mathcal{X}]$, and so $\mathcal{T}[\mathcal{X}] \in \Theta(\mathcal{B}[\mathcal{X}]) \iff \mathcal{T}[\mathcal{X}] \subseteq \Gamma[\mathcal{X}] \cup \Delta[\mathcal{X}]$; that is,

$$\Theta(\mathcal{B}[\mathcal{X}]) = \big\{\mathcal{T}[\mathcal{X}] \in \wp(\mathcal{B}[\mathcal{X}])\colon\ \mathcal{T}[\mathcal{X}] \subseteq \Gamma[\mathcal{X}] \cup \Delta[\mathcal{X}]\big\},$$

which means $\Theta(\mathcal{B}[\mathcal{X}]) = \wp(\Gamma[\mathcal{X}] \cup \Delta[\mathcal{X}])$. □

The following proposition is required for proving the next lemma. It is an immediate consequence of Definition 2.2 since the class of all compact operators is an ideal in $\mathcal{B}[\mathcal{X}]$.

Proposition 3.1. *The class of all compact perturbations of left semi-Fredholm, right semi-Fredholm, semi-Fredholm, and Fredholm operators coincides with the class of all left semi-Fredholm, right semi-Fredholm, semi-Fredholm, and Fredholm operators, respectively.*

Proof. See e.g. [8, Theorem 5.6]. □

Classes of operators in $\Theta(\mathcal{B}[\mathcal{X}])$ restricted to subclasses of $\Gamma[\mathcal{X}]$ are summarized in Lemma 3.2 below, which contains auxiliary results that will be required in the sequel. In particular, it shows that $\Gamma[\mathcal{X}]$ is topologically and algebraically large in the sense that it includes an open group from $\mathcal{B}[\mathcal{X}]$.

Let $\mathcal{X}$ be a Banach space and consider the following classes of operators.

(i) $\mathcal{K}[\mathcal{X}]$: the ideal of all compact operators from $\mathcal{B}[\mathcal{X}]$.
(ii) $\mathcal{G}[\mathcal{X}]$: the group of all invertible operators in $\mathcal{B}[\mathcal{X}]$ (with an inverse in $\mathcal{B}[\mathcal{X}]$).
(iii) $(\mathcal{G} + \mathcal{K})[\mathcal{X}]$: the essentially invertible operators in $\mathcal{B}[\mathcal{X}]$ (the collection of all operators of the form $G + K$ where $G \in \mathcal{G}[X]$ and $K \in \mathcal{K}[\mathcal{X}]$).
(iv) $\mathcal{F}[\mathcal{X}]$: the class of all Fredholm operators from $\mathcal{B}[\mathcal{X}]$.
(v) $(\mathcal{F} + \mathcal{K})[\mathcal{X}]$: the collection of all compact perturbations of Fredholm operators in $\mathcal{B}[\mathcal{X}]$ (operators of the form $F + K$ where $F \in \mathcal{F}[X]$ and $K \in \mathcal{K}[\mathcal{X}]$).
(vi) $\mathcal{SF}[\mathcal{X}]$: the class of all semi-Fredholm operators from $\mathcal{B}[\mathcal{X}]$.
(vii) $(\mathcal{SF}+\mathcal{K})[\mathcal{X}]$: the collection of all compact perturbations of semi-Fredholm operators in $\mathcal{B}[\mathcal{X}]$ (operators of the form $F+K$ where $F \in \mathcal{SF}[X]$ and $K \in \mathcal{K}[\mathcal{X}]$).
(viii) $\mathcal{E}[\mathcal{X}]$: the set of all projections in $\mathcal{B}[\mathcal{X}]$ (the collection of all linear, continuous, idempotent (i.e., $E = E^2$) operators on $\mathcal{X}$).

Lemma 3.2. *Let $\mathcal{X}$ be a Banach space. The above classes of operators from $\mathcal{B}[\mathcal{X}]$ share the following properties.*

(a) $(\mathcal{G}+\mathcal{K})[\mathcal{X}] \subseteq (\mathcal{F}+\mathcal{K})[\mathcal{X}] \subseteq (\mathcal{SF}+\mathcal{K})[\mathcal{X}] \subseteq \Gamma[\mathcal{X}]$,

(b) *if $\mathcal{X}$ is a reflexive Banach space with a Schauder basis, then $\mathcal{K}[\mathcal{X}] \subseteq \Gamma[\mathcal{X}]$,*

(c) $\mathcal{E}[\mathcal{X}] \subseteq \Gamma[\mathcal{X}]$ *but* $\mathcal{E}[\mathcal{X}] \not\subseteq (\mathcal{SF}+\mathcal{K})[\mathcal{X}] \cup \mathcal{K}[\mathcal{X}]$,

(d) *$\Gamma[\mathcal{X}]$ includes an open group in $\mathcal{B}[\mathcal{X}]$.*

Proof. By Proposition 3.1, (iv) and (v), and (vi) and (vii), are equivalent:

$$(\mathcal{F}+\mathcal{K})[\mathcal{X}] = \mathcal{F}[\mathcal{X}] \quad \text{and} \quad (\mathcal{SF}+\mathcal{K})[\mathcal{X}] = \mathcal{SF}[\mathcal{X}].$$

Since $\Phi_+[\mathcal{X}] \subseteq \Gamma_N[\mathcal{X}]$ and $\Phi_-[\mathcal{X}] \subseteq \Gamma_R[\mathcal{X}]$ (see e.g. [1, Lemma 3.1]), it follows by Proposition 2.1 that $\mathcal{F}_\ell[\mathcal{X}] \cup \mathcal{F}_r[\mathcal{X}] \subseteq \Gamma_N[\mathcal{X}] \cap \Gamma_R[\mathcal{X}]$. That is,

$$\mathcal{SF}[\mathcal{X}] \subseteq \Gamma[\mathcal{X}].$$

Moreover, $\mathcal{G}[\mathcal{X}] \subseteq \mathcal{F}[\mathcal{X}] \subseteq \mathcal{SF}[\mathcal{X}]$ trivially. Thus we get (a):

$$(\mathcal{G}+\mathcal{K})[\mathcal{X}] \subseteq (\mathcal{F}+\mathcal{K})[\mathcal{X}] \subseteq (\mathcal{SF}+\mathcal{K})[\mathcal{X}] \subseteq \Gamma[\mathcal{X}],$$

and so all classes in (ii) to (viii) lie in $\Gamma[\mathcal{X}]$. In fact, since the null operator $O \in \mathcal{B}[\mathcal{X}]$ is compact, the above chain of inclusions trivially ensures

$$\mathcal{G}[\mathcal{X}] \subseteq \mathcal{F}[\mathcal{X}] \subseteq \mathcal{SF}[\mathcal{X}] \subseteq \Gamma[\mathcal{X}].$$

On the other hand, since the null operator $O \in \mathcal{K}[\mathcal{X}]$ is not in $\mathcal{SF}[\mathcal{X}]$, that chain of inclusions does not imply $\mathcal{K}[\mathcal{X}] \subseteq \Gamma[\mathcal{X}]$. But such an inclusion holds if $\mathcal{X}$ is a reflexive Banach space with a Schauder basis [2, Corollary 5.1] (see also [1, Theorem 2.1(f)] for a partial result along this line). This is item (b). There are, however, subclasses of $\Gamma[\mathcal{X}]$ consisting of operators that are not included in $(\mathcal{SF}+\mathcal{K})[\mathcal{X}] \cup \mathcal{K}[\mathcal{X}]$. For instance, let $E \in \mathcal{B}[\mathcal{X}]$ be a projection. Thus $\mathcal{R}(E)$ and $\mathcal{N}(E)$ are complementary subspaces of $\mathcal{X}$, and conversely, if $\mathcal{M}$ and $\mathcal{N}$ are complementary subspaces of a Banach space $\mathcal{X}$, then the (unique) projection $E\colon \mathcal{X} \to \mathcal{X}$ with $\mathcal{R}(E) = \mathcal{M}$ and $\mathcal{N}(E) = \mathcal{N}$ is continuous (i.e., $E \in \mathcal{B}[\mathcal{X}]$ — see e.g. [13, Theorem 3.2.14 and Corollary 3.2.15] or [14, Problem 4.35]). Therefore,

$$\mathcal{E}[\mathcal{X}] \subseteq \Gamma[\mathcal{X}].$$

On the other hand let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M} \oplus \mathcal{N}$ be infinite-dimensional Banach spaces. If $E = I \oplus O = \left(\begin{smallmatrix} I & O \\ O & O \end{smallmatrix}\right)$ on $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$, then $E \in \mathcal{E}[\mathcal{X}]$ with $\mathcal{R}(E) = \mathcal{M} \oplus \{0\}$ and $\mathcal{N}(E) = \{0\} \oplus \mathcal{N}$ (and so they are complemented in $\mathcal{M} \oplus \mathcal{N}$). Since $\dim \mathcal{N} = \infty$, we get $\dim \mathcal{N}(E) = \infty$ and $\operatorname{codim} \mathcal{R}(E) = \infty$, and hence $E \notin \mathcal{SF}[\mathcal{X}]$ (cf. Definitions 2.1 and 2.2, and Proposition 2.1). The restriction $E|_{\mathcal{M}\oplus\{0\}}$ is isometrically isomorphic to the identity operator I on $\mathcal{M}$ (i.e., $E|_{\mathcal{M}\oplus\{0\}} \cong I\colon \mathcal{M} \to \mathcal{M}$). Since $\dim \mathcal{M} = \infty$, the identity on the infinite-dimensional space $\mathcal{M}$ is not compact, and so $E|_{\mathcal{M}\oplus\{0\}}$ is not compact, which implies that $E \notin \mathcal{K}[\mathcal{X}]$. Hence,

$$\mathcal{E}[\mathcal{X}] \not\subseteq \mathcal{SF}[\mathcal{X}] \cup \mathcal{K}[\mathcal{X}].$$

Since $(\mathcal{SF}+\mathcal{K})[\mathcal{X}] = \mathcal{SF}[\mathcal{X}]$ by Proposition 3.1, it follows that

$$\mathcal{E}[\mathcal{X}] \not\subseteq (\mathcal{SF}+\mathcal{K})[\mathcal{X}] \cup \mathcal{K}[\mathcal{X}],$$

completing the proof of (c). Since the group $\mathcal{G}[\mathcal{X}]$ is open in $\mathcal{B}[\mathcal{X}]$ (see e.g. [14, Problem 4.48(b)]), the class $\Gamma[\mathcal{X}]$ is algebraically and topologically large, thus (d). □

Let $\mathcal{X}^*$ stand for the dual of the normed space $\mathcal{X}$, let $T^* \in \mathcal{B}[\mathcal{X}^*]$ denote the normed-space adjoint of $T \in \mathcal{B}[\mathcal{X}]$, and set

$$\Gamma[\mathcal{X}^*] = \big\{S \in \mathcal{B}[\mathcal{X}^*]\colon\ \mathcal{R}(S)^- \text{ and } \mathcal{N}(S) \text{ are complemented subspaces of } \mathcal{X}^*\big\},$$

$$\Delta[\mathcal{X}^*] = \big\{S \in \mathcal{B}[\mathcal{X}^*]\colon\ \mathcal{R}(S)^- \text{ and } \mathcal{N}(S) \text{ are not complemented subspaces of } \mathcal{X}^*\big\}.$$

Proposition 3.2 gives a full account on how range-kernel complementedness travels between an operator and its adjoint.

Proposition 3.2. *Let $\mathcal{X}$ be a Banach space and take any operator $T \in \mathcal{B}[\mathcal{X}]$.*

(a_1) *If $\mathcal{R}(T)^-$ is complemented, then $\mathcal{N}(T^*)$ is complemented*:

$$T \in \Gamma_R[\mathcal{X}] \quad \Longrightarrow \quad T^* \in \Gamma_N[\mathcal{X}^*].$$

(a_2) *If $\mathcal{X}$ is reflexive and $\mathcal{N}(T^*)$ is complemented, then $\mathcal{R}(T)^-$ is complemented*:

$$\mathcal{X} \text{ reflexive and } \ T^* \in \Gamma_N[\mathcal{X}^*] \quad \Longrightarrow \quad T \in \Gamma_R[\mathcal{X}].$$

(b_1) *If $\mathcal{X}$ is reflexive and $\mathcal{R}(T^*)^-$ is complemented, then $\mathcal{N}(T)$ is complemented*:

$$\mathcal{X} \text{ reflexive and } \ T^* \in \Gamma_R[\mathcal{X}^*] \quad \Longrightarrow \quad T \in \Gamma_N[\mathcal{X}].$$

(b_2) *If $\mathcal{R}(T)$ is closed and $\mathcal{N}(T)$ is complemented, then $\mathcal{R}(T^*)$ complemented*:

$$\mathcal{R}(T) = \mathcal{R}(T)^- \text{ and } \ T \in \Gamma_N[\mathcal{X}] \ \Longrightarrow \ \mathcal{R}(T^*) = \mathcal{R}(T^*)^- \text{ and } \ T^* \in \Gamma_R[\mathcal{X}^*].$$

Proof. [2, Theorem 3.1] □

Corollary 3.1. *Let $\mathcal{X}$ is a Banach space and take $T \in \mathcal{B}[\mathcal{X}]$.*

(a_1) *If $T \in (\mathcal{G} + \mathcal{K})[\mathcal{X}]$, then $T^* \in (\mathcal{G} + \mathcal{K})[\mathcal{X}^*]$.*
(a_2) *If $T \in (\mathcal{F} + \mathcal{K})[\mathcal{X}]$, then $T^* \in (\mathcal{F} + \mathcal{K})[\mathcal{X}^*]$.*
(a_3) *If $T \in (\mathcal{SF} + \mathcal{K})[\mathcal{X}]$, then $T^* \in (\mathcal{SF} + \mathcal{K})[\mathcal{X}^*]$.*

(a) $(\mathcal{G} + \mathcal{K})[\mathcal{X}^*] \subseteq (\mathcal{F} + \mathcal{K})[\mathcal{X}^*] \subseteq (\mathcal{SF} + \mathcal{K})[\mathcal{X}^*] \subseteq \Gamma[\mathcal{X}^*]$.

(b) *If $T \in \mathcal{K}[\mathcal{X}]$ and $\mathcal{X}$ is a reflexive Banach space with a Schauder basis, then $T^* \in \mathcal{K}[\mathcal{X}^*] \subseteq \Gamma[\mathcal{X}^*]$.*

(c) *If $T \in \mathcal{E}[\mathcal{X}]$, then $T^* \in \mathcal{E}[\mathcal{X}^*] \subseteq \Gamma[\mathcal{X}^*]$.*

(d_1) *If $T \in \Gamma[\mathcal{X}]$ and $\mathcal{R}(T)$ is closed, then $T^* \in \Gamma[\mathcal{X}^*]$.*
(d_2) *If $T^* \in \Gamma[\mathcal{X}^*]$ and $\mathcal{X}$ is reflexive, then $T \in \Gamma[\mathcal{X}]$.*

Proof. As it is well known, if $T \in \mathcal{B}[\mathcal{X}]$ is compact, invertible, Fredholm, or semi-Fredholm, then so is its normed-space adjoint $T^* \in \mathcal{B}[\mathcal{X}^*]$ (see e.g. [13, Theorem 3.4.15], [13, Proposition 3.2.5], [7, Theorem 16.4], and [8, Section 5.1], respectively), and the normed-space adjoint of the sum is the sum of the normed-space adjoints (see e.g. [13, Proposition 3,1,4]). Thus the results in (a_i) for $i = 1, 2, 3$ hold true, and so (a) holds by Lemma 3.2. Since reflexivity for $\mathcal{X}$ is equivalent to reflexivity for $\mathcal{X}^*$ (see e.g. [9, Theorem V.4.2]), since $\mathcal{X}^*$ has a Schauder basis whenever $\mathcal{X}$ has (see e.g. [13, Theorem 4.4.1]), and since if $E \in \mathcal{B}[\mathcal{X}]$ is a continuous projection and so is its adjoint $E^* \in \mathcal{B}[\mathcal{X}^*]$ (reason: $E^{*2} = E^{2*} = E^*$ — see e.g. [13, Proposition 3.1.10]), the results in (b) and (c) follow from Lemma 3.2. The results in (d_1) and (d_2) follow from Proposition 3.2. □

4 Range-Kernel Uncomplementation

Classes of operators T such that $T \in \Gamma[\mathcal{X}]$ and $T^* \in \Gamma[\mathcal{X}^*]$ where exhibited in Lemma 3.2 and Corollary 3.1. In this section we exhibit an operator $T \in \mathcal{B}[\mathcal{X}]$ for which $T \in \Delta[\mathcal{X}]$ and $T^* \in \Delta[\mathcal{X}^*]$.

For any pair of normed spaces $(\mathcal{X}, \|\cdot\|_X)$ and $(\mathcal{Y}, \|\cdot\|_Y)$ over the same scalar field, let $\mathcal{X} \oplus \mathcal{Y}$ denote the direct sum of $\mathcal{X}$ and $\mathcal{Y}$ equipped with any standard norm (e.g., $\|(x, y)\|_p = (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{p}}$, $p \geq 1$, or $\|(x, y)\|_\infty = \max\{\|x\|_X, \|y\|_Y\}$), so that if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, then so is $\mathcal{X} \oplus \mathcal{Y}$ (see e.g. [14, Example 4.E]). The following lemma will be used to prove Theorem 4.1.

Lemma 4.1. *If $\mathcal{M}$, $\mathcal{Y}$ and $\mathcal{Z}$ are subspaces of a Banach space $\mathcal{X}$ such that $\mathcal{Y}$ and $\mathcal{Z}$ include $\mathcal{M}$ (i.e., $\mathcal{M} \subseteq \mathcal{Y} \cap \mathcal{Z}$), then the following assertions*

(a) $\mathcal{M}$ *is complemented in* $\mathcal{X}$,
(a′) $\mathcal{M}\oplus\{0\}$ *is complemented in* $\mathcal{X}\oplus\mathcal{X}$,
(a″) $\{0\}\oplus\mathcal{M}$ *is complemented in* $\mathcal{X}\oplus\mathcal{X}$,

(b) $\mathcal{M}$ *is complemented in* $\mathcal{Y}$,
(b′) $\mathcal{M}\oplus\{0\}$ *is complemented in* $\mathcal{Y}\oplus\mathcal{Z}$,

(c) $\mathcal{M}$ *is complemented in* $\mathcal{Z}$,
(c′) $\{0\}\oplus\mathcal{M}$ *is complemented in* $\mathcal{Y}\oplus\mathcal{Z}$,

are pairwise equivalent.

Proof. **Part 1.** Let O denote the null operator on $\mathcal{X}$ (or its restriction to $\mathcal{Y}$ or to $\mathcal{Z}$). Suppose the subspace $\mathcal{M}$ is complemented in the Banach space $\mathcal{X}$. Then there exists a continuous projection $P\colon\mathcal{X}\to\mathcal{X}$ with $\mathcal{R}(P)=P(\mathcal{X})=\mathcal{M}$ (see e.g. [14, Problem 4.35(b)]). Since $\mathcal{R}(P)=\mathcal{M}\subseteq\mathcal{Y}\cap\mathcal{Z}$, set $P_Y=P|_{\mathcal{Y}}:\mathcal{Y}\to\mathcal{Y}$ and $P_Z=P|_{\mathcal{Z}}:\mathcal{Z}\to\mathcal{Z}$, the restrictions of P to the normed spaces $\mathcal{Y}$ and $\mathcal{Z}$, respectively. These are continuous projections both with range equal to $\mathcal{M}$. Indeed, P_Y is continuous (since it is the restriction of a continuous function P on the normed space $\mathcal{X}$ to the normed space $\mathcal{Y}\subseteq\mathcal{X}$), and $\mathcal{R}(P_Y)=P(\mathcal{Y})\subseteq P(\mathcal{X})=\mathcal{M}=P(\mathcal{M})\subseteq P(\mathcal{Y})$ (since $\mathcal{M}\subseteq\mathcal{Y}\subseteq\mathcal{X}$ and $P=P^2$), so that $\mathcal{R}(P_Y)=\mathcal{R}(P)=\mathcal{M}$, and hence P_Y is idempotent (since $(P_Y)^2=P|_{\mathcal{Y}}P|_{\mathcal{Y}}=P^2|_{\mathcal{Y}}=P_Y$) — similarly, P_Z is continuous, idempotent, and $\mathcal{R}(P_Z)=\mathcal{M}$. Thus, since $\mathcal{M}=\mathcal{R}(P_Y)=\mathcal{R}(P_Z)$ is a subspace of the normed spaces $\mathcal{Y}$ and $\mathcal{Z}$, it is complemented in $\mathcal{Y}$ and in $\mathcal{Z}$ as well (see e.g. [14, Problem 4.35(a)]). Thus (a) implies (b,c). Moreover, since P, P_Y and P_Z are continuous projections, then set

$$E^X=P\oplus O=\begin{pmatrix}P&\\&O\end{pmatrix}:\mathcal{X}\oplus\mathcal{X}\to\mathcal{X}\oplus\mathcal{X},$$

$$E_X=O\oplus P=\begin{pmatrix}O&\\&P\end{pmatrix}:\mathcal{X}\oplus\mathcal{X}\to\mathcal{X}\oplus\mathcal{X},$$

$$E^Y=P_Y\oplus O=\begin{pmatrix}P_Y&\\&O\end{pmatrix}:\mathcal{Y}\oplus\mathcal{Z}\to\mathcal{Y}\oplus\mathcal{Z},$$

$$E_Z=O\oplus P_Z=\begin{pmatrix}O&\\&P_Z\end{pmatrix}:\mathcal{Y}\oplus\mathcal{Z}\to\mathcal{Y}\oplus\mathcal{Z},$$

to get continuous projections on the normed spaces $\mathcal{X}\oplus\mathcal{X}$ and $\mathcal{Y}\oplus\mathcal{Z}$ with ranges $\mathcal{M}\oplus\{0\}$ and $\{0\}\oplus\mathcal{M}$, and so $\mathcal{M}\oplus\{0\}$ and $\{0\}\oplus\mathcal{M}$ are (closed) subspaces of the normed spaces $\mathcal{X}\oplus\mathcal{X}$ and $\mathcal{Y}\oplus\mathcal{Z}$, which are complemented in $\mathcal{X}\oplus\mathcal{X}$ and in $\mathcal{Y}\oplus\mathcal{Z}$ (see e.g. [14, Problem 4.35(a)]). Thus (a) implies (a′,a″,b′,c′). Hence,

$$\text{(a)}\quad\Longrightarrow\quad\text{(a′), (a″), (b), (b′), (c), (c′).}$$

Since $\mathcal{Y}$ and $\mathcal{Z}$ are subspaces of the Banach space $\mathcal{X}$, they are Banach spaces, and so the same argument that shows that (a) implies (a′,a″) also shows that

$$\text{(b)}\quad\Longrightarrow\quad\text{(b′)}\quad\text{and}\quad\text{(c)}\quad\Longrightarrow\quad\text{(c′).}$$

Part 2. Since $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Banach spaces, the direct sums $\mathcal{X}\oplus\mathcal{X}$ and $\mathcal{Y}\oplus\mathcal{Z}$ are again Banach spaces. Since $\mathcal{M}$ is a subspace of the Banach space $\mathcal{X}$ (thus a Banach space itself) it follows that $\mathcal{M}\oplus\{0\}$ and $\{0\}\oplus\mathcal{M}$ are Banach spaces, and so (closed) subspaces of the Banach spaces $\mathcal{X}\oplus\mathcal{X}$ and $\mathcal{Y}\oplus\mathcal{Z}$. If $\mathcal{M}\oplus\{0\}$ is complemented in the Banach space $\mathcal{Y}\oplus\mathcal{Z}$, then there exists a continuous projection $Q^Y:\mathcal{Y}\oplus\mathcal{Z}\to\mathcal{Y}\oplus\mathcal{Z}$ with $\mathcal{R}(Q^Y)=\mathcal{M}\oplus\{0\}$ (see e.g. [14, Problem 4.35(b)]), so that $Q^Y(\mathcal{N}\oplus\mathcal{R})=\mathcal{M}\oplus\{0\}$ if $\mathcal{M}\subseteq\mathcal{N}\subseteq\mathcal{Y}$ and $\mathcal{R}\subseteq\mathcal{Z}$. Let $J^Y\colon\mathcal{Y}\to\mathcal{Y}\oplus\{0\}$ be the natural embedding of $\mathcal{Y}$ onto $\mathcal{Y}\oplus\{0\}$ (which is an isometric isomorphism with $\mathcal{R}(J^Y)=\mathcal{Y}\oplus\{0\}$ and $\mathcal{R}((J^Y)^{-1})=\mathcal{Y}$). Set

$$F^Y=(J^Y)^{-1}Q^YJ^Y\colon\mathcal{Y}\to\mathcal{Y},$$

which is a continuous projection with range $\mathcal{M}$ (i.e., $\mathcal{R}(F^Y)=\mathcal{M}$). In fact $(F^Y)^2=(J^Y)^{-1}Q^YJ^Y(J^Y)^{-1}Q^YJ^Y=(J^Y)^{-1}Q^YJ^Y=F^Y$, $\|F^Y\|=\|(J^Y)^{-1}Q^YJ^Y\|=\|Q^Y\|$ (since J^Y and $(J^Y)^{-1}$ are isometries), and $\mathcal{R}(F^Y)=F^Y(\mathcal{Y})=$

$(J^Y)^{-1}Q^YJ^Y(\mathcal{Y}) = J^{Y-1}Q^Y(\mathcal{Y}\oplus\{0\}) = J^{Y-1}(\mathcal{M}\oplus\{0\}) = \mathcal{M}$. Thus $\mathcal{M}$ is complemented in the normed $\mathcal{Y}$ space (see e.g. [14, Problem 4.35(a)]), and therefore

$$(\mathrm{b}') \quad \Longrightarrow \quad (\mathrm{b}).$$

Similarly, if $\{0\}\oplus\mathcal{M}$ is complemented in the Banach space $\mathcal{Y}\oplus\mathcal{Z}$, then take the continuous projection $Q_Z : \mathcal{Y}\oplus\mathcal{Z}\to\mathcal{Y}\oplus\mathcal{Z}$ with $\mathcal{R}(Q_Z) = \{0\}\oplus\mathcal{M}$, consider the natural embedding J_Z: $\mathcal{Z}\to\{0\}\oplus\mathcal{Z}$ of $\mathcal{Z}$ onto $\{0\}\oplus\mathcal{Z}$, and set

$$F_Z = J_Z^{-1}Q_Z\,J_Z : \mathcal{Z}\to\mathcal{Z},$$

a continuous projection with $\mathcal{R}(F_Z) = \mathcal{M}$. Hence $\mathcal{M}$ is complemented in $\mathcal{Z}$, so

$$(\mathrm{c}') \quad \Longrightarrow \quad (\mathrm{c}).$$

Replacing $\mathcal{Y}$ and $\mathcal{Z}$ with $\mathcal{X}$ it follows, in particular, that if $\mathcal{M}\oplus\{0\}$ or $\{0\}\oplus\mathcal{M}$ is complemented in the Banach space $\mathcal{X}\oplus\mathcal{X}$, then $\mathcal{M}$ is complemented in $\mathcal{X}$. Thus

$$(\mathrm{a}') \quad \Longrightarrow \quad (\mathrm{a}) \quad \text{and} \quad (\mathrm{a}'') \quad \Longrightarrow \quad (\mathrm{a}),$$

which completes the proof. □

The next result (due to H.P. Rosenthal [15, Theorem 6]) seems to have been the first example of an injective ($\mathcal{N}(W) = \{0\}$) Banach-space operator $W\in\mathcal{B}[\mathcal{X}]$ with a closed range ($\mathcal{R}(W) = \mathcal{R}(W)^-$) which is not complemented.

Proposition 4.1. *For every $p\in(2,\infty)$ there exists a proper (closed) subspace $\mathcal{M}$ of ℓ_+^p which is not complemented and is the range of a topological isomorphism $W : \ell_+^p\to\mathcal{R}(W) = \mathcal{M}\subset\ell_+^p$, so that there exists an operator $W\in\mathcal{B}[\ell_+^p]$ for which $\mathcal{R}(W) = \mathcal{R}(W)^-$ is not complemented.*

Proof. [15, Theorem 6]. □

Operators in $\Gamma[\mathcal{X}]$ whose adjoints are in $\Gamma[\mathcal{X}^*]$ were discussed in Section 3. Now it is exhibited an operator in $\Delta[\mathcal{X}]$ whose adjoint is in $\Delta[\mathcal{X}^*]$. Precisely, an operator with uncomplemented closed range and kernel whose the adjoint has uncomplemented closed range and kernel. The proof uses an argument borrowed from [16, Example 6] plus Proposition 4.1, Lemma 4.1 and Proposition 3.2.

Theorem 4.1. *There exists a Banach space operator $T\in\mathcal{B}[\mathcal{X}]$ such that $T\in\Delta[\mathcal{X}]$ and $T^*\in\Delta[\mathcal{X}^*]$.*

Proof. Consider the reflexive Banach space $\mathcal{X} = \ell_+^p$ for an arbitrary $p\in(2,\infty)$. According to Proposition 4.1 there exists a (proper, closed) uncomplemented subspace $\mathcal{M}$ of $\mathcal{X}$ which is the range of a topological isomorphism $W : \mathcal{X}\to\mathcal{M} = \mathcal{R}(W)\subset\mathcal{X}$. Now take the null operator $O\in\mathcal{B}[\mathcal{M}]$, and consider the operator $T\in\mathcal{B}[\mathcal{X}\oplus\mathcal{M}]$ (acting on the Banach space $\mathcal{X}\oplus\mathcal{M}$ obtained by the direct sum of the Banach spaces $\mathcal{X}$ and $\mathcal{M}$ equipped with any standard norm inherited from the norm of $\mathcal{X}$) given by

$$T = W\oplus O = \begin{pmatrix} W & \\ & O\end{pmatrix} : \mathcal{X}\oplus\mathcal{M}\to\mathcal{X}\oplus\mathcal{M},$$

and hence

$$\mathcal{R}(T) = \mathcal{R}(W)\oplus\mathcal{R}(O) = \mathcal{M}\oplus\{0\}\ \subset\mathcal{X}\oplus\mathcal{M},$$

$$\mathcal{N}(T) = \mathcal{N}(W)\oplus\mathcal{N}(O) = \{0\}\oplus\mathcal{M}\ \subset\ \mathcal{X}\oplus\mathcal{M},$$

where $\mathcal{R}(T) = \mathcal{M}\oplus\{0\}$ is a subspace of $\mathcal{X}\oplus\mathcal{M}$, so that $\mathcal{R}(T) = \mathcal{R}(T)^-$.

(a) Suppose $\mathcal{R}(T) = \mathcal{M}\oplus\{0\}$ is complemented in $\mathcal{X}\oplus\mathcal{M}$. Since $\mathcal{M}$ is a subspace of the Banach space $\mathcal{X}$, set $\mathcal{Y} = \mathcal{X}$ and $\mathcal{Z} = \mathcal{M}$ in Lemma 4.1 (b′)⇒(b), so that $\mathcal{M}$ is complemented in $\mathcal{X}$, which contradicts the fact that $\mathcal{M} = \mathcal{R}(W)$ is not complemented in $\mathcal{X}$. Therefore $\mathcal{R}(T) = \mathcal{M}\oplus\{0\}$ is not complemented in $\mathcal{X}\oplus\mathcal{M}$.

(b) Similarly, suppose $\mathcal{N}(T) = \{0\}\oplus\mathcal{M}$ is complemented in $\mathcal{X}\oplus\mathcal{M}$. The same argument (using Lemma 4.1 (c′)⇒(c)) ensures that $\mathcal{M}$ is complemented in $\mathcal{X}$, which is again a contradiction. So $\mathcal{N}(T) = \{0\}\oplus\mathcal{M}$ is not complemented in $\mathcal{X}\oplus\mathcal{M}$.

Outcome. Both $\mathcal{R}(T)^-$ and $\mathcal{N}(T)$ are not complemented, which means $T \in \Delta[\mathcal{X}]$.

Moreover, since $\mathcal{X}$ is reflexive, Proposition 3.2(a_2,b_1) thus ensures that both $\mathcal{N}(T^*)$ and $\mathcal{R}(T^*)^- = \mathcal{R}(T^*)$ are not complemented, which means $T^* \in \Delta[\mathcal{X}^*]$. □

References

[1] Kubrusly C. S., Duggal B. P., Upper-lower and left-right semi-Fredholmness, Bull. Belg. Math. Soc. Simon Stevin, 2016, 23, 217–233
[2] Kubrusly C. S., Range-kernel complementation (to appear)
[3] Lindenstrauss J., Tzafriri L., On the complemented subspaces problem, Israel J. Math., 1971, 9, 263–269
[4] Kalton N. J., The complemented subspace problem revisited, Studia Math., 2008, 188, 223–257
[5] Bennett G., Dor L. E., Goodman V., Johnson W. B., Newman, C. M., On uncomplemented subspaces of L_p, $1 < p < 2$, Israel J. Math., 1977, 26, 178–187
[6] Plichko A. M., Yost D., Complemented and uncomplemented subspaces of Banach spaces, Extracta Math., 2000, 15, 335–371
[7] Müller V., Spectral theory of linear operators – and spectral systems in Banach algebras, 2nd ed., Birkhäuser, Basel, 2007
[8] Kubrusly C. S., Spectral theory of operators on Hilbert spaces, Birkhäuser-Springer, New York, 2012
[9] Conway J. B., A course in functional analysis, 2nd ed., Springer, New York, 1990
[10] Halmos P. R., A Hilbert space problem book, 2nd ed., Springer, New York, 1982
[11] Duggal B. P., Kubrusly C. S., Perturbation of Banach space operators with a complemented range, Glasgow Math. J., 2017, 59, 659–671
[12] Harte R., Invertibility and singularity for bounded linear operators, Marcel Dekker, New York, 1988
[13] Megginson R., An introduction to Banach space theory, Springer, New York, 1998
[14] Kubrusly C. S., The elements of operator theory, Birkhäuser-Springer, New York, 2011
[15] Rosenthal H. P., On the subspaces of $L^p (p > 2)$ spanned by sequences of independent random variables, Israel J. Math., 1970, 8, 273–303
[16] Campbell S. L., Faulkner G. D., Operators on Banach spaces with complemented ranges, Acta Math. Acad. Sci. Hungar., 1980, 35, 123–128