Research Article Open Access

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Approximation by q-analogue of modified Jakimovski-Leviatan-Stancu type operators

https://doi.org/10.1515/dema-2017-0019
Received February 16, 2017; accepted June 21, 2017

Abstract: In this paper, we introduce the q-analogue of the Jakimovski-Leviatan type modified operators introduced by Atakut with the help of the q-Appell polynomials. We obtain some approximation results via the well-known Korovkin's theorem for these operators. We also study convergence properties by using the modulus of continuity and the rate of convergence of the operators for functions belonging to the Lipschitz class. Moreover, we study the rate of convergence in terms of modulus of continuity of these operators in a weighted space.

Keywords: q-analogue of Jakimovski-Leviatan operators; modulus of continuity; rate of approximation; K-functional; weighted spaces

MSC: 41A10, 41A25, 41A36

1 Introduction and Preliminaries

During the last two decades, applications of q-calculus have emerged as a new area in the field of approximation theory. Lupaş [1] was the first who introduced the q-analogue of the well known Bernstein polynomials and investigated its approximating and shape-preserving properties. In 1997 Phillips [2] considered another q-analogue of the classical Bernstein polynomials. Subsequently, many authors have introduced q-generalizations of various operators and investigated several approximation properties (see, e.g., [3–13]).

Jakimovski and Leviatan [14] in 1969 introduced a new type of operators P_n by using Appell polynomials as follows. Let $g(u) = \sum_{n=0}^{\infty} a_n u^n,\ g(1) \neq 0$ be an analytic function in the disk $|u| < r$ $(r > 1)$ and $p_k(x) = \sum_{i=0}^{k} a_i \frac{x^{k-i}}{(k-i)!}$ $(k \in \mathbb{N})$ be the Appell polynomials defined by the identity

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1.1}$$

We consider the class of functions of exponential type which are defined on the semi-axis and satisfy the property $|f(x)| \leq \kappa e^{\vartheta x}$ for some finite constants $\kappa,\ \vartheta > 0$ and denote the set of such functions by $E[0, \infty)$. In [14], the authors considered the sequence of operators P_n, with

$$P_n(f; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.2}$$

for $f \in E[0, \infty)$ and established several approximation properties of these operators.
If $g(1) = 1$ in (1.1) we get $p_k(x) = \frac{x^k}{k!}$, and we recover the well-known classical Favard-Szász operators defined

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by

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right). \tag{1.3}$$

Recently (see [15]), the q-Appell polynomials Jakimovski-Leviatan type operators have been defined by

$$K_n^q(f;x) = \frac{E_q(-[n]_q x)}{A_q(1)}\sum_{k=0}^{\infty}\frac{A_{k,q}([n]_q x)}{[k]_q!}f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.4}$$

where the q-Appell polynomials [16] are defined by means of generating function $A_q(t)$,

$$A_q(t) = \sum_{n=0}^{\infty}A_{n,q}\frac{t^n}{[n]_q!},\ A_q(1) \neq 0,$$

which is an analytic function in the disk $|t| < r$ $(r > 1)$, and

$$A_{n,q}(x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k,q}x^k\ (n \in \mathbb{N}),$$

$$A_q(t)e_q(tx) = \sum_{n=0}^{\infty}A_{n,q}(x)\frac{t^n}{[n]_q!}\quad (0 < q < 1).$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ be the set of complex numbers, real numbers and positive integers respectively, and let $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and $\mathbb{R}^+ = [0,\infty)$.

Also for $f \in L_1[0,\infty)$ the integral type Jakimovski-Leviatan operators given by Atakut (see [17]) are defined as follows:

$$L_n^{\star}(f;x) = \frac{e^{-a_n(x)}}{g(1)}\sum_{k=0}^{\infty}P_k(a_n(x))\frac{b_n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\frac{1}{1-q}}e^{-b_n t}t^{\lambda+k}f(t)\mathrm{d}t, \tag{1.5}$$

where $\lambda > 0$ and $\{a_n\}$, $\{b_n\}$ are increasing and unbounded sequences of positive numbers such that $\lim_{n\to\infty}\frac{1}{n} = 0$, $\frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right)$.

Motivated by the above discussed work, we define q-generalization of operators (1.5) (see [17]).

We now present some basic definitions and notations of the q-calculus which are used in this paper.

Definition 1.1. For $|q| < 1$, the basic (or q-) number $[\lambda]_q$ is defined by

$$[\lambda]_q = \begin{cases} \dfrac{1-q^{\lambda}}{1-q} & (\lambda \in \mathbb{C}) \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases} \tag{1.6}$$

Definition 1.2. For $|q| < 1$, the basic (or q-) the q-factorial $[n]_q!$ is defined by

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \prod\limits_{k=1}^{n}[k]_q & (n \in \mathbb{N}). \end{cases} \tag{1.7}$$

Definition 1.3. For $|q| < 1$, the generalized basic (or q-) binomial coefficient $\begin{bmatrix} \lambda \\ n \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} \lambda \\ n \end{bmatrix}_q = \frac{\left(q^{-\lambda};q\right)_n}{(q;q)_n}\left(-q^{\lambda}\right)^n q^{-\binom{n}{2}}\qquad (\lambda \geq n;\ n \in \mathbb{N}_0). \tag{1.8}$$

For $q, \nu \in \mathbb{C}$ $(|q| < 1)$, the basic (or q-) shifted factorial $(\lambda; q)_\nu$ is defined by (see, e.g.,[18–20])

$$(\lambda; q)_\nu = \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\nu+j}} \right) \qquad (|q| < 1;\ \lambda \geq n,\ \nu \in \mathbb{C}), \tag{1.9}$$

so that

$$(\lambda; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{j=0}^{n-1} \left(1 - \lambda q^j\right) & (n \in \mathbb{N}) \end{cases}$$

and

$$(\lambda; q)_\infty := \prod_{j=0}^{\infty} \left(1 - \lambda q^j\right) \qquad (|q| < 1;\ \lambda \geq n). \tag{1.10}$$

Definition 1.4. For $|q| < 1$, the basic (or q-) exponential function $e_q(z)$ of the first kind is defined by

$$e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} = \frac{1}{(z; q)_\infty}. \tag{1.11}$$

Definition 1.5. For $|q| < 1$, the basic (or q-) exponential function $E_q(z)$ of the second kind is defined by

$$E_q(z) := \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{z^k}{(q; q)_k} = (-z; q)_\infty. \tag{1.12}$$

Remark 2. It is easily seen by applying the definitions (1.11) and (1.12) that

$$\lim_{q \to 1} \{e_q\left((1 - q)z\right)\} = e^z = \lim_{q \to 1} \{E_q\left((1 - q)z\right)\} \text{and} \qquad e_q(z) \cdot E_q(-z) = 1. \tag{1.13}$$

Definition 1.6. For $0 < |q| < 1$, the q- analog of the derivative, denoted by D_q, is defined by (see, e.g., [21])

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad x \neq 0. \tag{1.14}$$

If $f'(0)$ exists, then $D_q f(0) = f'(0)$. As q tends to 1^-, the q-derivative reduces to the usual derivative. Clearly, if f is differentiable, then

$$\lim_{q \to 1-} D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x} = \frac{\mathrm{d}f(x)}{\mathrm{d}x}, \quad x \neq 0. \tag{1.15}$$

It is easy to check the q-analog of Leibniz's rule

$$D_q[f(x)g(x)] = f(qx)D_q g(x) + g(x)D_q f(x). \tag{1.16}$$

Proposition. ([21]) For $0 < |q| < 1$ and $n \in \mathbb{N}$,

$$D_q(1 + x)_q^n = [n]_q(1 + qx)_q^{n-1} \tag{1.17}$$

and

$$D_q \left\{ \frac{1}{(1 + x)_q^n} \right\} = -\frac{[n]_q}{(1 + x)_q^{n+1}}. \tag{1.18}$$

Definition 1.7. For $0 < |q| < 1$, q-analog of integration, is defined by (see, e.g., [22])

$$\int_0^1 f(x)\mathrm{d}_q x = (1 - q) \sum_{i=0}^{\infty} f(q^i)q^i, \tag{1.19}$$

which reduces to $\int_0^1 f(x)\mathrm{d}x$ in the case of $q \to 1^-$. More generally, the q-Jackson integral from 0 to $a \in \mathbb{R}$ can be defined by (see, e.g., [23, 24])

$$\int_0^a f(x)\mathrm{d}_q x = a(1-q)\sum_{i=0}^{\infty} f(aq^i)q^i, \tag{1.20}$$

provided the sum converges absolutely. The q-Jackson integral on a general interval $[a, b]$ may be defined by (see, e.g., [23, 24])

$$\int_a^b f(x)\mathrm{d}_q x = \int_0^b f(x)\mathrm{d}_q x - \int_0^a f(x)\mathrm{d}_q x. \tag{1.21}$$

The q-Jackson integral and q-derivative are related by the fundamental theorem of quantum calculus which can be restated as follows, (see, e.g., ([24], p. 73)).

Definition 1.8. For $|q| < 1$, the basic (or q-) Gamma function $\Gamma_q(z)$ is defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z} \qquad (|q| < 1;\ z \in \mathbb{C}), \tag{1.22}$$

so that

$$\lim_{q\to 1}\{\Gamma_q(z)\} = \Gamma(z)$$

in terms of the familiar (Euler's) Gamma function $\Gamma(z)$.

Here the two representations are based on the following remarkable function (see [25, p. 15])

$$K(A;t) = A^{t-1}\frac{(-\frac{q}{A};q)_\infty}{(-\frac{q^t}{A};q)_\infty}\frac{(-A;q)_\infty}{(-Aq^{1-t};q)_\infty} \qquad (t \in \mathbb{R}), \tag{1.23}$$

where

$$\Gamma_q(\alpha)\} = \int_0^{\frac{1}{1-q}} x^{\alpha-1}E_q(q(1-q)x)\mathrm{d}_q x \qquad (\alpha > 0) \tag{1.24}$$

and

$$\Gamma_q(\alpha)\} = K(A;\alpha)\int_0^{\frac{\infty}{A(1-q)}} x^{\alpha-1}e_q(-(1-q)x)\mathrm{d}_q x \qquad (\alpha > 0). \tag{1.25}$$

Remark 3. In terms of the basic (or q-) Gamma function $\Gamma_q(z)$ defined by (1.22), the generalized basic (or q-) binomial coefficient $\begin{bmatrix}\lambda\\ \nu\end{bmatrix}_q$ in (1.8) can by extended to the following form:

$$\begin{aligned}\begin{bmatrix}\lambda\\ \nu\end{bmatrix}_q &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\nu+1)\Gamma_q(\nu+1)}\\ &= \frac{\left(q^{\lambda-\nu+1}, q^{\nu+1};q\right)_\infty}{\left(q, q^{\lambda+1}\right)_\infty} \qquad (|q| < 1;\ \lambda, \nu \in \mathbb{C}).\end{aligned} \tag{1.26}$$

2 Construction of Operators and Auxiliary results

Appell polynomials were introduced by Appell in 1880 (see [26]). In 1967, Al-Salam [27] introduced the family of q-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^{\infty}$, and studied some of their properties.

In this paper, we define q-generalization of Jakimovski-Leviatan operators defined by (1.5) as follows:

$$L^{\star}_{n,q}(f;x)=\frac{1}{g_q(1)e_q(a_{[n]_q}(x))}\sum_{k=0}^{\infty}P_{k,q}(a_{[n]_q}(x))Q_n(t), \tag{2.1}$$

where

$$Q_n(t)=\frac{b_n^{\lambda+k+1}}{\Gamma_q(\lambda+k+1)}\int_0^{\frac{1}{1-q}}E_q(-b_nqt)t^{\lambda+k}f(t)\mathrm{d_q}t$$

for $x\in[0,\infty)$, $q_n\in(0,1]$ and with the same notations $\{a_{[n]_{q_n}}\}$ and $\{b_{[n]_{q_n}}\}$ given increasing and unbounded sequences of positive numbers such that

$$\lim_{n\to\infty}\frac{1}{[n]_{q_n}}=0,\quad \frac{a_{[n]_{q_n}}}{b_{[n]_{q_n}}}=1+O\left(\frac{1}{b_{[n]_{q_n}}}\right). \tag{2.2}$$

It is easy to verify that if $q_n\to 1$, these operators turn into the classical one.
Here we also introduce q-analogue of modified Jakimovski-Leviatan-Stancu type operators and obtain better approximation results. Let $\alpha,\beta\in\mathbb{R}$ such that $0\le\alpha\le\beta$. Then for $x\in[0,\infty)$, $q\in(0,1]$,

$$S^{\star}_{n,q}(f;x)=\frac{1}{g_q(1)e_q(a_{[n]_q}(x))}\sum_{k=0}^{\infty}P_{k,q}(a_{[n]_q}(x))Q^{\alpha,\beta}_{n,q}(t), \tag{2.3}$$

where

$$Q^{\alpha,\beta}_{n,q}(t)=\frac{b_{[n]_q}^{\lambda+k+1}}{\Gamma_q(\lambda+k+1)}\int_0^{\frac{1}{1-q}}E_q(-b_{[n]_q}qt)t^{\lambda+k}f\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)\mathrm{d_q}t.$$

If we take $\alpha=\beta=0$ in (2.3), then the operators $S^{\star}_{n,q}(f;x)$ reduce to operators defined by (2.1).

Lemma 2.1. *Let $L^{\star}_{n,q}(\cdot\,;\cdot)$ be the operators given by* (2.1). *Then for all $x\in[0,\infty)$, $q\in(0,1)$ and each $n\in\mathbb{N}$, we have the following identities:*

(1) $L^{\star}_{n,q}(1;x)=1$

(2) $L^{\star}_{n,q}(t;x)=\left(\frac{a_{[n]_q}}{b_{[n]_q}}\right)x+\frac{1}{b_{[n]_q}}\left(\lambda+1+\frac{g'_q(1)}{g_q(1)}\right)$

(3) $L^{\star}_{n,q}(t^2;x)=\left(\frac{a_{[n]_q}}{b_{[n]_q}}\right)^2x^2+\frac{2a_{[n]_q}}{b^2_{[n]_q}}\left(\lambda+2+\frac{g'_q(1)}{g_q(1)}\right)x+\frac{1}{b^2_{[n]_q}}\left((\lambda+1)(\lambda+2)+2(\lambda+2)\frac{g'_q(1)}{g_q(1)}+\frac{g''_q(1)}{g_q(1)}\right)$

Proof. Using (1.1), it can be easily seen that

$$\sum_{k=0}^{\infty}P_{k,q}(a_{[n]_q}x)=g_q(1)e_qa_{[n]_q}x, \tag{2.4}$$

$$\sum_{k=0}^{\infty}kP_{k,q}(a_{[n]_q}x)=\left(g'_q(1)+a_{[n]_q}g_q(1)x\right)e_qa_{[n]_q}x, \tag{2.5}$$

$$\sum_{k=0}^{\infty}k^2P_{k,q}(a_{[n]_q}x)=\left(g''_q(1)+2a_{[n]_q}g'_q(1)x+g'_q(1)+a^2_{[n]_q}g_q(1)x^2\right)e_qa_{[n]_q}x. \tag{2.6}$$

(1) For $f\equiv 1$

$$\begin{aligned}L^{\star}_{n,q}(1;x)&=\frac{e_q(-a_{[n]_q}(x))}{g_q(1)}\sum_{k=0}^{\infty}P_{k,q}(a_{[n]_q}(x))\frac{b_{[n]_q}^{\lambda+k+1}}{\Gamma_q(\lambda+k+1)}\int_0^{\frac{1}{1-q}}E_q(-b_{[n]_q}qt)t^{\lambda+k}\mathrm{d_q}t\\&=\frac{e_q(-a_{[n]_q}(x))}{g_q(1)}g_q(1)e_q(a_{[n]_q}(x))\frac{1}{\Gamma_q(\lambda+k+1)}\int_0^{\frac{1}{1-q}}E_q(-qt)t^{\lambda+k}\mathrm{d_q}t\\&=1.\end{aligned}$$

(2) For $f \equiv t$

$$
\begin{aligned}
L^{\star}_{n,q}(t;x) &= \frac{e_q(-a_{[n]_q}(x))}{g_q(1)} \sum_{k=0}^{\infty} P_{k,q}(a_{[n]_q}(x)) \frac{b_{[n]_q}^{\lambda+k+1}}{\Gamma_q(\lambda+k+1)} \int_0^{\frac{1}{1-q}} E_q(-b_{[n]_q} qt) t^{\lambda+k+1} \mathrm{d_q} t \\
&= \frac{e_q(-a_{[n]_q}(x))}{g_q(1)} \sum_{k=0}^{\infty} P_{k,q}(a_{[n]_q}(x)) \frac{1}{b_{[n]_q}\Gamma_q(\lambda+k+1)} \int_0^{\frac{1}{1-q}} E_q(-qt) t^{\lambda+k+1} \mathrm{d_q} t \\
&= \frac{e_q(-a_{[n]_q}(x))}{g_q(1)} \sum_{k=0}^{\infty} P_{k,q}(a_{[n]_q}(x)) \frac{\lambda+k+1}{b_{[n]_q}\Gamma_q(\lambda+k+2)} \int_0^{\frac{1}{1-q}} E_q(-qt) t^{\lambda+k+1} \mathrm{d_q} t \\
&= \frac{(\lambda+1)e_q(-a_{[n]_q}(x))}{b_{[n]_q} g_q(1)} \sum_{k=0}^{\infty} P_{k,q}(a_{[n]_q}(x)) + \frac{e_q(-a_{[n]_q}(x))}{b_{[n]_q} g_q(1)} \sum_{k=0}^{\infty} k P_{k,q}(a_{[n]_q}(x)) \\
&= \frac{(\lambda+1)e_q(-a_{[n]_q}(x))}{b_{[n]_q} g_q(1)} g_q(1) e_q(a_{[n]_q}x) + \frac{e_q(-a_{[n]_q}(x))}{b_{[n]_q} g_q(1)} \left(g_q'(1) + a_{[n]_q} g_q(1) x \right) e_q(a_{[n]_q}x) \\
&= \frac{a_{[n]_q}}{b_{[n]_q}} x + \frac{1}{b_{[n]_q}} \left(\lambda + 1 + \frac{g_q'(1)}{g_q(1)} \right).
\end{aligned}
$$

(3) For $f \equiv t^2$

$$
\begin{aligned}
L^{\star}_{n,q}(t^2;x) &= \frac{e_q(-a_{[n]_q}(x))}{g_q(1)} \sum_{k=0}^{\infty} P_{k,q}(a_{[n]_q}(x)) \frac{b_{[n]_q}^{\lambda+k+1}}{\Gamma_q(\lambda+k+1)} \int_0^{\frac{1}{1-q}} E_q(-b_{[n]_q} qt) t^{\lambda+k+2} \mathrm{d_q} t \\
&= \frac{e_q(-a_{[n]_q}(x))}{g_q(1)} \sum_{k=0}^{\infty} P_{k,q}(a_{[n]_q}(x)) \frac{1}{b_{[n]_q}^2 \Gamma_q(\lambda+k+1)} \int_0^{\frac{1}{1-q}} E_q(-qt) t^{\lambda+k+2} \mathrm{d_q} t \\
&= \frac{e_q(-a_{[n]_q}(x))}{g_q(1)} \sum_{k=0}^{\infty} P_{k,q}(a_{[n]_q}(x)) \frac{(\lambda+k+1)(\lambda+k+2)}{b_{[n]_q}^2 \Gamma_q(\lambda+k+3)} \int_0^{\frac{1}{1-q}} E_q(-qt) t^{\lambda+k+2} \mathrm{d_q} t \\
&= \frac{(\lambda+1)(\lambda+2) e_q(-a_{[n]_q}(x))}{b_{[n]_q}^2 g_q(1)} \sum_{k=0}^{\infty} P_{k,q}(a_{[n]_q}(x)) \\
&\quad + \frac{(2\lambda+3) e_q(-a_{[n]_q}(x))}{b_{[n]_q}^2 g_q(1)} \sum_{k=0}^{\infty} k P_{k,q}(a_{[n]_q}(x)) + \frac{e_q(-a_{[n]_q}(x))}{b_{[n]_q}^2 g_q(1)} \sum_{k=0}^{\infty} k^2 P_{k,q}(a_{[n]_q}(x)) \\
&= \frac{(\lambda+1)(\lambda+2)}{b_{[n]_q}^2} + \frac{(2\lambda+3)}{b_{[n]_q}^2} \left(\frac{g_q'(1)}{g_q(1)} + a_{[n]_q} x \right) \\
&\quad + \frac{1}{b_{[n]_q}^2} \left(\frac{g_q''(1)}{g_q(1)} + \frac{g_q'(1)}{g_q(1)} + \left(\frac{2 g_q'(1)}{g_q(1)} + 1 \right) a_{[n]_q} x + a_{[n]_q}^2 x^2 \right).
\end{aligned}
$$

□

Lemma 2.2. *Let $S^{\star}_{n,q}(\cdot\,;\cdot)$ be the operators given by (2.3). Then for all $x \in [0,\infty)$, $q \in (0,1)$ and each $n \in \mathbb{N}$, we have the following identities:*

1° $S^{\star}_{n,q}(1;x) = 1$;

2° $S^{\star}_{n,q}(t;x) = \left(\frac{[n]_q}{[n]_q+\beta}\right) \left(\frac{a_{[n]_q}}{b_{[n]_q}}\right) x + \frac{1}{b_{[n]_q}} \left(\frac{[n]_q}{[n]_q+\beta}\right) \left(\lambda + 1 + \frac{g_q'(1)}{g_q(1)}\right) + \frac{\alpha}{[n]_q+\beta}$,

$$
\begin{aligned}
3^\circ\ S^\star_{n,q}(t^2;x) &= \left(\frac{[n]_q}{[n]_q+\beta}\right)^2\left(\frac{a_{[n]_q}}{b_{[n]_q}}\right)^2 x^2 \\
&+ \left(\frac{2a_{[n]_q}}{b_{[n]_q}}\right)\left(\frac{[n]_q}{[n]_q+\beta}\right)\left(\frac{1}{b_{[n]_q}}\frac{[n]_q}{[n]_q+\beta}\left(\lambda+2+\frac{g_q'(1)}{g_q(1)}\right)+\frac{\alpha}{[n]_q+\beta}\right)x \\
&+ \frac{1}{b^2_{[n]_q}}\left(\frac{[n]_q}{[n]_q+\beta}\right)^2\left((\lambda+1)(\lambda+2)+2(\lambda+2)\frac{g_q'(1)}{g_q(1)}+\frac{g_q''(1)}{g_q(1)}\right) \\
&+ \frac{1}{b_{[n]_q}}\left(\frac{2[n]_q\alpha}{[n]_q+\beta}\right)\left(\lambda+1+\frac{g_q'(1)}{g_q(1)}\right)+\left(\frac{\alpha}{[n]_q+\beta}\right)^2.
\end{aligned}
$$

Proof. From (2.3) and Lemma 2.1, we have

$$S^\star_{n,q}(t;x)=\frac{[n]_q}{[n]_q+\beta}L^\star_{n,q}(t;x)+\frac{\alpha}{[n]_q+\beta}L^\star_{n,q}(1;x),$$

$$S^\star_{n,q}(t^2;x)=\left(\frac{[n]_q}{[n]_q+\beta}\right)^2L^\star_{n,q}(t^2;x)+\frac{2[n]_q\alpha}{([n]_q+\beta)^2}L^\star_{n,q}(t;x)+\left(\frac{\alpha}{[n]_q+\beta}\right)^2L^\star_{n,q}(1;x).$$

Hence it can be easily proved. □

It is also interesting to find $L^\star_{n,q}((t-x)^j;x)$ and $S^\star_{n,q}((t-x)^j;x)$ for $j=1,2$.

Lemma 2.3. *Let the operators $L^\star_{n,q}(\cdot\,;\cdot)$ be given by* (2.1). *Then for each $x\geq 0$, $q\in(0,1)$, we have*

$$L^\star_{n,q}(t-x;x)=\left(\frac{a_{[n]_q}}{b_{[n]_q}}-1\right)x+\frac{1}{b_{[n]_q}}\left(\lambda+1+\frac{g_q'(1)}{g_q(1)}\right)$$

$$
\begin{aligned}
L^\star_{n,q}((t-x)^2;x) &= \left(\frac{a_{[n]_q}}{b_{[n]_q}}-1\right)^2x^2+\frac{2}{b_{[n]_q}}\left(\left(\frac{a_{[n]_q}}{b_{[n]_q}}-1\right)\left(\lambda+1+\frac{g_q'(1)}{g_q(1)}\right)+\frac{a_{[n]_q}}{b_{[n]_q}}\right)x \\
&+ \frac{1}{b^2_{[n]_q}}\left((\lambda+1)(\lambda+2)+2(\lambda+2)\frac{g_q'(1)}{g_q(1)}+\frac{g_q''(1)}{g_q(1)}\right).
\end{aligned}
$$

Proof. From the linearity property we have

$$L^\star_{n,q}(t-x;x)=L^\star_{n,q}(t;x)-xL^\star_{n,q}(1;x)$$

$$L^\star_{n,q}((t-x)^2;x)=L^\star_{n,q}(t^2;x)-2xL^\star_{n,q}(t;x)+x^2L^\star_{n,q}(1;x).$$

It can be easily proved from Lemma 2.2. □

Lemma 2.4. *Let the operators $S^\star_{n,q}(\cdot\,;\cdot)$ be given by* (2.3). *Then for each $x\geq 0$, $q\in(0,1)$, we have*

$$S^{\star}_{n,q}(t-x;x) = \left(\frac{[n]_q}{[n]_q+\beta} \frac{a_{[n]_q}}{b_{[n]_q}} - 1 \right) x + \left(\frac{[n]_q}{[n]_q+\beta} \right) \left(\frac{1}{b_{[n]_q}} \left(\lambda + 1 + \frac{g'_q(1)}{g_q(1)} \right) \right)$$

$$\begin{aligned} S^{\star}_{n,q}((t-x)^2;x) &= \left(\frac{a_{[n]_q}}{b_{[n]_q}} \frac{[n]_q}{[n]_q+\beta} - 1 \right)^2 x^2 \\ &+ \frac{2}{b_{[n]_q}} \frac{[n]_q}{[n]_q+\beta} \left\{ \left(\frac{a_{[n]_q}}{b_{[n]_q}} \frac{[n]_q}{[n]_q+\beta} - 1 \right) \left(\lambda + 1 + \frac{g'_q(1)}{g_q(1)} \right) \right. \\ &+ \left. \frac{a_{[n]_q}}{b_{[n]_q}} \frac{[n]_q}{[n]_q+\beta} + \alpha \left(\frac{a_{[n]_q}}{[n]_q+\beta} - \frac{b_{[n]_q}}{[n]_q} \right) \right\} x \\ &+ \frac{1}{b^2_{[n]_q}} \left(\frac{[n]_q}{[n]_q+\beta} \right)^2 \left((\lambda+1)(\lambda+2) + 2(\lambda+2) \frac{g'_q(1)}{g_q(1)} + \frac{g''_q(1)}{g_q(1)} \right) \\ &+ \frac{1}{b_{[n]_q}} \left(\frac{2[n]_q \alpha}{[n]_q+\beta} \right) \left(\lambda + 1 + \frac{g'_q(1)}{g_q(1)} \right) + \left(\frac{\alpha}{[n]_q+\beta} \right)^2 . \end{aligned}$$

3 Main Results

We obtain the Korovkin type and weighted Korovkin type approximation theorems for our operators defined by (2.1).

Let $C_B(\mathbb{R}^+)$ be the set of all bounded and continuous functions on $\mathbb{R}^+$, which is a linear normed space with

$$\|f\|_{C_B} = \sup_{x \ge 0} |f(x)|.$$

Let

$$C_\zeta[0,\infty) := \left\{ f \in C[0,\infty) : |f(t)| \le M(1+t)^\zeta \text{ for some } M > 0 \right\}.$$

$$H := \left\{ f \in C[0,\infty) : \frac{f(x)}{1+x^2} \quad \text{is convergent as } x \to \infty \right\}.$$

Theorem 3.1. *Let $q = q_n$ satisfy $0 < q_n < 1$ with $\lim_{n\to\infty} q_n = 1$ and $S^{\star}_{n,q}(\cdot\,;\cdot)$ be the operators defined by (2.3). Then for any function $f \in C_\zeta[0,\infty) \cap H$, $\zeta \ge 2$ and $x \in [0,\infty)$*

$$\lim_{n\to\infty} S^{\star}_{n,q_n}(f;x) = f(x)$$

uniformly on each compact subset of $[0,\infty)$.

Proof. The proof is based on Lemma 2.2 and well known Korovkin's theorem regarding the convergence of a sequence of linear positive operators. So it is enough to prove the conditions

$$\lim_{n\to\infty} S^{\star}_{n,q_n}((t^j;x) = x^j, \quad j = 0,1,2, \quad \text{as } n \to \infty$$

uniformly on $[0,\infty]$.
Clearly $\frac{1}{[n]_{q_n}} \to 0$, $(n \to \infty)$ we have

$$\lim_{n\to\infty} S^{\star}_{n,q_n}(t;x) = x, \quad \lim_{n\to\infty} S^{\star}_{n,q_n}(t^2;x) = x^2.$$

This completes the proof. □

Following Gadžiev [28, 29] (see also [30–32]) we recall the weighted spaces of the functions on $\mathbb{R}^+$, as well as additional conditions under which the analogous theorem of Korovkin holds for such kind of functions.

Let $x \to \phi(x)$ be a continuous and strictly increasing function and $\varrho(x) = 1 + \phi^2(x)$, $\lim_{x\to\infty} \varrho(x) = \infty$. Let $B_\varrho(\mathbb{R}^+)$ be a set of functions defined on $\mathbb{R}^+$ and satisfying

$$|f(x)| \le M_f \varrho(x),$$

where M_f is a constant depending only on f. Its subset of continuous functions will be denoted by $C_\varrho(\mathbb{R}^+)$, i.e., $C_\varrho(\mathbb{R}^+) = B_\varrho(\mathbb{R}^+) \cap C(\mathbb{R}^+)$. It is well known that a sequence of linear positive operators $\{L_n\}_{n\ge1}$ maps $C_\varrho(\mathbb{R}^+)$ into $B_\varrho(\mathbb{R}^+)$ if and only if

$$|L_n(\varrho; x)| \le K\varrho(x),$$

where $x \in \mathbb{R}^+$ and K is a positive constant. Note that $B_\varrho(\mathbb{R}^+)$ is a normed space with the norm

$$\|f\|_\varrho = \sup_{x\ge0} \frac{|f(x)|}{\varrho(x)}.$$

Finally, let $C^0_\varrho(\mathbb{R}^+)$ be a subset of $C_\varrho(\mathbb{R}^+)$ such that the limit

$$\lim_{n\to\infty} \frac{f(x)}{\varrho(x)} = K_f$$

exists and is finite.

Let $B[0, 1]$ be the space of all bounded functions on $[0, 1]$ and $C[0, 1]$ be the space of all functions f continuous on $[0, 1]$ equipped with norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|, \quad f \in C[0, 1].$$

The famous Korovkin's theorem states as follows:

Theorem 3.2 (cf. [33]). *Let $\{L_n\}_{n\ge1}$ be the sequence of linear positive operators acting from $C[0, 1]$ into $B[0, 1]$ Then*

$$\lim_{n\to\infty} \|L_n(t^k; x) - x^k\|_\infty = 0 \; (k = 0, 1, 2).$$

if and only if for all $f \in C[0, 1]$

$$\lim_{n\to\infty} \|L_n(f(t); x) - f\|_\infty = 0.$$

Theorem 3.3. *Let $\{L_n\}_{n\ge1}$ be the sequence of linear positive operators acting from $C_\varrho(\mathbb{R}^+)$ into $B_\varrho(\mathbb{R}^+)$ satisfying the conditions*

$$\lim_{n\to\infty} \|L_n(\varphi^k(t); x) - \varphi^k(x)\|_\varrho = 0 \; (k = 0, 1, 2)$$

then for any function $f \in C^0_\varrho(\mathbb{R}^+)$,

$$\lim_{n\to\infty} \|L_n(f(t); x) - f\|_\varrho = 0.$$

Proof. For the completeness, we give some sketch of the proof for the version which will be used in our next result. Consider $\varphi(x) = x$, $\varrho(x) = 1 + x^2$, and

$$\|L_n(t^k; x) - x^k\|_\varrho = \sup_{x\ge0} \frac{|\, L_n(t^k; x) - x^k \,|}{1 + x^2}.$$

Then for $k = 0, 1, 2$ it is easily proved that

$$\lim_{n\to\infty} \|L_n(t^k; x) - x^k\|_\varrho = 0.$$

Hence by using Korovkin's theorem, for any function $f \in C^0_\varrho(\mathbb{R}^+)$, we get

$$\lim_{n\to\infty} \|L_n(f(t); x) - f\|_\varrho = 0.$$

□

Theorem 3.4. *Let $q = q_n$ satisfy $0 < q_n < 1$ with $\lim_{n\to\infty} q_n = 1$ and $S^{\star}_{n,q_n}(\cdot;\cdot)$ be the operators defined by (2.3) and $\varrho(x) = 1 + x^2$. Then for $f \in C^0_\varrho(\mathbb{R}^+)$ we have*

$$\lim_{n\to\infty} \|S^{\star}_{n,q_n}(f;x) - f\|_\varrho = 0.$$

Proof. Using Theorem 3.3 for $\varphi(x) = x$ and $\varrho(x) = 1 + x^2$, we consider

$$\|S^{\star}_{n,q_n}(t^j;x) - x^j\|_\varrho = \sup_{x\geq 0} \frac{| S^{\star}_{n,q_n}(t^j;x) - x^j |}{1 + x^2},$$

for $j = 0, 1, 2$.
According to Lemma 2.2 for $j = 0$ it is obvious that $| S^{\star}_{n,q_n}(1;x) - 1 |= 0$, and therefore

$$\lim_{n\to\infty} \|S^{\star}_{n,q_n}(1;x) - 1\|_\varrho = 0.$$

For $j = 1$

$$\sup_{x\geq 0} \frac{| S^{\star}_{n,q_n}(t;x) - t |}{1 + x^2} \leq \left|\frac{a_{[n]_{q_n}}}{b_{[n]_{q_n}}} - 1\right| \sup_{x\geq 0} \frac{x}{1 + x^2} + \frac{1}{b_{[n]_{q_n}}}\left|\lambda + 1 + \frac{g'_{q_n}(1)}{g_{q_n}(1)}\right| \sup_{x\geq 0} \frac{1}{1 + x^2}.$$

Therefore

$$\lim_{n\to\infty} \|S^{\star}_{n,q_n}(t;x) - x\|_\varrho = 0.$$

For $j = 2$

$$\begin{aligned}\sup_{x\geq 0} \frac{| S^{\star}_{n,q_n}(t^2;x) - x^2 |}{1 + x^2} \leq{}& \left|\frac{a^2_{[n]_{q_n}}}{b^2_{[n]_{q_n}}} - 1\right| \sup_{x\geq 0} \frac{x^2}{1 + x^2} + \frac{2a_{[n]_{q_n}}}{b^2_{[n]_{q_n}}}\left|\lambda + 2 + \frac{g'_{q_n}(1)}{g_{q_n}(1)}\right| \sup_{x\geq 0} \frac{x}{1 + x^2}\\ &+ \frac{1}{b^2_{[n]_{q_n}}}\left|(\lambda + 1)(\lambda + 2) + 2(\lambda + 2)\frac{g''_{q_n}(1)}{g_{q_n}(1)} + \frac{g'_{q_n}(1)}{g_{q_n}(1)}\right| \sup_{x\geq 0} \frac{1}{1 + x^2}.\end{aligned}$$

Hence we have

$$\lim_{n\to\infty} \|S^{\star}_{n,q_n}(t^2;x) - x^2\|_\varrho = 0.$$

□

4 Rate of Convergence

Here we calculate the rate of convergence of operators (2.1) by means of modulus of continuity and Lipschitz type functions.

Let $f \in C_B[0,\infty]$ be the space of all bounded and continuous functions on $[0,\infty)$ and $x \geq 0$. Then for $\delta > 0$, the modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by

$$\omega(f,\delta) = \sup_{|t-x|\leq\delta} |f(t) - f(x)|, \quad t \in [0,\infty). \tag{4.1}$$

It is known that $\lim_{\delta\to 0+} \omega(f,\delta) = 0$ for $f \in C_B[0,\infty)$ and for any $\delta > 0$ one has

$$|f(t) - f(x)| \leq \left(\frac{|t - x|}{\delta} + 1\right) \omega(f,\delta). \tag{4.2}$$

In the sequel we use the following notations:

$$\delta^x_{n,q} = \sqrt{S^{\star}_{n,q}\left((t - x)^2;x\right)}, \tag{4.3}$$

where by using the Lemma 2.4, we have

$$\delta^x_{n,q}=\begin{cases}\left(\dfrac{a_{[n]_q}}{b_{[n]_q}}\dfrac{[n]_q}{[n]_q+\beta}-1\right)^2x^2\\ +\dfrac{2}{b_{[n]_q}}\dfrac{[n]_q}{[n]_q+\beta}\left(\left(\dfrac{a_{[n]_q}}{b_{[n]_q}}\dfrac{[n]_q}{[n]_q+\beta}-1\right)\left(\lambda+1+\dfrac{g'_q(1)}{g_q(1)}\right)\right)x\\ +\dfrac{2}{b_{[n]_q}}\dfrac{[n]_q}{[n]_q+\beta}\left(\dfrac{a_{[n]_q}}{b_{[n]_q}}\dfrac{[n]_q}{[n]_q+\beta}+\alpha\left(\dfrac{a_{[n]_q}}{[n]_q+\beta}-\dfrac{b_{[n]_q}}{[n]_q}\right)\right)x\\ +\dfrac{1}{b^2_{[n]_q}}\left(\dfrac{[n]_q}{[n]_q+\beta}\right)^2\left((\lambda+1)(\lambda+2)+2(\lambda+2)\dfrac{g'_q(1)}{g_q(1)}+\dfrac{g''_q(1)}{g_q(1)}\right)\\ +\dfrac{1}{b_{[n]_q}}\left(\dfrac{2[n]_q\alpha}{[n]_q+\beta}\right)\left(\lambda+1+\dfrac{g'_q(1)}{g_q(1)}\right)+\left(\dfrac{\alpha}{[n]_q+\beta}\right)^2;\\ \quad\text{for } 0<q<1,\ 0<\alpha<\beta,\ \ \alpha,\beta\in\mathbb{R}\\ \\ \left(\dfrac{a_n}{b_n}-1\right)^2x^2+\dfrac{2}{b_n}\left(\left(\dfrac{a_n}{b_n}-1\right)\left(\lambda+1+\dfrac{g'(1)}{g(1)}\right)+\dfrac{a_n}{b_n}\right)x\\ +\dfrac{1}{b_n^2}\left((\lambda+1)(\lambda+2)+2(\lambda+2)\dfrac{g'(1)}{g(1)}+\dfrac{g''(1)}{g(1)}\right);\qquad \text{for } q=1,\ \alpha=\beta=0.\end{cases}$$

Here when $q=1,\ \alpha=\beta=0$, then $\delta^x_{n,q}$ is reduced to $\delta^x_n=\sqrt{L^\star_n\left((t-x)^2;x\right)}$.

Theorem 4.1. *Let $S^\star_{n,q}(\cdot;\cdot)$ be the operators defined by* (2.3). *Then for $f\in C_B[0,\infty)$, $\quad q\in(0,1)$ and $n\in\mathbb{N}$ we have*

$$|S^\star_{n,q}(f;x)-f(x)|\le 2\omega\left(f;\delta^x_{n,q}\right), \tag{4.4}$$

where $\delta^x_{n,q}$ is defined in (4.3).

Proof. For our sequence of positive linear operators $\{S^\star_{n,q}(.;.)\}$ we have

$$\begin{aligned}S^\star_{n,q}(f;x)-f(x) &= S^\star_{n,q}(f;x)-f(x)S^\star_{n,q}(1;x)\\ &= S^\star_{n,q}\left(f(t)-f(x);x\right)\\ &\le S^\star_{n,q}\left(|\,f(t)-f(x)\,|;x\right),\end{aligned}$$

since $S^\star_{n,q}(1;x)=1$. Using (4.1) and (4.2) we get

$$\begin{aligned}|S^\star_{n,q}(f;x)-f(x)| &\le S^\star_{n,q}\left(1+\frac{|\,t-x\,|}{\delta};x\right)\omega(f;\delta)\\ &= \left(1+\frac{1}{\delta}S^\star_{n,q}(|\,t-x\,|;x)\right)\omega(f;\delta).\end{aligned}$$

Now, applying Cauchy-Schwarz inequality we conclude that

$$S^\star_{n,q}(|\,t-x\,|;x)\le S^\star_{n,q}(1;x)^{\frac12}S^\star_{n,q}\left((t-x)^2;x\right)^{\frac12}$$

so that

$$|\,S^\star_{n,q}(f;x)-f(x)\,|\le\left(1+\frac{1}{\delta}S^\star_{n,q}\left((t-x)^2;x\right)^{\frac12}\right)\omega(f;\delta). \tag{4.5}$$

Finally, putting $\delta=\delta^x_{n,q}=\sqrt{S^\star_{n,q}\left((t-x)^2;x\right)}$ we get the assertion □

Remark 4.2. Choosing $\delta=\frac{1}{[n]_q+\beta}$ in (4.5) we obtain the following estimate

$$|S^\star_{n,q}(f;x)-f(x)|\le\left(1+([n]_q+\beta)\delta^x_{n,q}\right)\omega\left(f;\frac{1}{[n]_q+\beta}\right), \tag{4.6}$$

where $\delta^x_{n,q}$ is defined in (4.3).

Remark 4.3. For $q = 1$ and $\alpha = \beta = 0$ the corresponding estimate for the sequence of positive linear operators $\{S^{\star}_{n,q}\}$ defined by (2.3) is reduced to $\{L^{\star}_n\}$, which can take the form as

$$|L^{\star}_n(f;x) - f(x)| \leq 2\omega\left(f;\delta^x_n\right), \tag{4.7}$$

where $\delta^x_n = \sqrt{L^{\star}_n\left((t-x)^2;x\right)}$.

Now we give the rate of convergence of the operators $L^{\star}_{n,q}(f;x)$ defined in (2.1) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$. Let $f \in C_B[0,\infty)$, $M > 0$ and $0 < \nu \leq 1$. The class $Lip_M(\nu)$ is defined as

$$Lip_M(\nu) = \left\{f : |f(\zeta_1) - f(\zeta_2)| \leq M|\zeta_1 - \zeta_2|^{\nu} \ \ (\zeta_1, \zeta_2 \in [0,\infty))\right\}. \tag{4.8}$$

Theorem 4.4. *Let $S^{\star}_{n,q}(\cdot\,;\cdot)$ be the operators defined by* (2.3). *Then for each $f \in Lip_M(\nu)$, $(M > 0, \ \ 0 < \nu \leq 1)$ and $q \in (0,1)$ satisfying* (4.8) *we have*

$$|S^{\star}_{n,q}(f;x) - f(x)| \leq M\left(\delta^x_{n,q}\right)^{\nu/2}$$

where $\delta^x_{n,q}$ is defined in (4.3).

Proof. We prove it by using (4.8) and Hölder's inequality. First, as in the proof of Theorem 4.1, we have

$$\begin{aligned} |S^{\star}_{n,q}(f;x) - f(x)| &\leq |S^{\star}_{n,q}(f(t) - f(x);x)| \\ &\leq S^{\star}_{n,q}\left(|f(t) - f(x)|;x\right) \\ &\leq MS^{\star}_{n,q}\left(|t-x|^{\nu};x\right). \end{aligned}$$

Therefore

$$\begin{aligned} |S^{\star}_{n,q}(f;x) - f(x)| &\leq M\frac{1}{g_q(1)e_q(a_{[n]_q}(x))}\sum_{k=0}^{\infty}P_{k,q}(a_{[n]_q}(x))\frac{b^{\lambda+k+1}_{[n]_q}}{\Gamma_q(\lambda+k+1)}\int_0^{\frac{1}{1-q}}E_q(-b_{[n]_q}qt)t^{\lambda+k}|t-x|^{\nu}\mathrm{d_q}t \\ &= M\frac{1}{g_q(1)e_q(a_{[n]_q}(x))}\left(\sum_{k=0}^{\infty}P_{k,q}(a_{[n]_q}(x))\frac{b^{\lambda+k+1}_{[n]_q}}{\Gamma_q(\lambda+k+1)}\right)^{\frac{2-\nu}{2}} \\ &\times \left(P_{k,q}(a_{[n]_q}(x))\frac{b^{\lambda+k+1}_{[n]_q}}{\Gamma_q(\lambda+k+1)}\right)^{\frac{\nu}{2}}\int_0^{\frac{1}{1-q}}E_q(-b_{[n]_q}qt)t^{\lambda+k}|t-x|^{\nu}\mathrm{d_q}t \\ &\leq M\left(\frac{1}{g_q(1)e_q(a_{[n]_q}(x))}\sum_{k=0}^{\infty}P_{k,q}(a_{[n]_q}(x))\frac{b^{\lambda+k+1}_{[n]_q}}{\Gamma_q(\lambda+k+1)}\int_0^{\frac{1}{1-q}}E_q(-b_{[n]_q}qt)t^{\lambda+k}\mathrm{d_q}t\right)^{\frac{2-\nu}{2}} \\ &\times \left(\frac{1}{g_q(1)e_q(a_{[n]_q}(x))}\sum_{k=0}^{\infty}P_{k,q}(a_{[n]_q}(x))\frac{b^{\lambda+k+1}_{[n]_q}}{\Gamma_q(\lambda+k+1)}\int_0^{\frac{1}{1-q}}E_q(-b_{[n]_q}qt)t^{\lambda+k}|t-x|^2\mathrm{d_q}t\right)^{\frac{\nu}{2}} \\ &= M\left(S^{\star}_{n,q}(t-x)^2;x\right)^{\frac{\nu}{2}}. \end{aligned}$$

This completes the proof. □

Let

$$C^2_B(\mathbb{R}^+) = \left\{g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+)\right\}, \tag{4.9}$$

with the norm

$$||g||_{C^2_B(\mathbb{R}^+)} = ||g||_{C_B(\mathbb{R}^+)} + ||g'||_{C_B(\mathbb{R}^+)} + ||g''||_{C_B(\mathbb{R}^+)}, \tag{4.10}$$

also

$$||g||_{C_B(\mathbb{R}^+)} = \sup_{x\in\mathbb{R}^+} |\,g(x)|. \tag{4.11}$$

Theorem 4.5. *Let $S^{\star}_{n,q}(\cdot\,;\cdot)$ be the operator defined in* (2.3). *Then for any $g \in C^2_B(\mathbb{R}^+)$, $x \geq 0$ and $q \in (0,1)$, we have*

$$|S^{\star}_{n,q}(f;x) - f(x)| \leq \frac{1}{2}\delta^x_{n,q}(2+\delta^x_{n,q})\|g\|_{C^2_B(\mathbb{R}^+)},$$

where $\delta^x_{n,q}$ is defined in (4.3).

Proof. Let $g \in C^2_B(\mathbb{R}^+)$. Then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(t) = g(x) + g'(x)(t-x) + g''(\psi)\frac{(t-x)^2}{2},$$

where ψ is between x and t, from which it follows

$$|g(t) - g(x)| \leq M_1|t-x| + \frac{1}{2}M_2(t-x)^2,$$

where by using the result of (4.10) and (4.11) we have

$$M_1 = \sup_{x\in\mathbb{R}^+}|g'(x)| = \|g'\|_{C_B(\mathbb{R}^+)} \leq \|g\|_{C^2_B(\mathbb{R}^+)},$$

$$M_2 = \sup_{x\in\mathbb{R}^+}|g''(x)| = \|g''\|_{C_B(\mathbb{R}^+)} \leq \|g\|_{C^2_B(\mathbb{R}^+)},$$

again from 4.10, thus we have

$$|g(t) - g(x)| \leq \left(|t-x| + \frac{1}{2}(t-x)^2\right)\|g\|_{C^2_B(\mathbb{R}^+)},$$

since

$$|S^{\star}_{n,q}(g,x) - g(x)| = |S^{\star}_{n,q}(g(t)-g(x);x)| \leq S^{\star}_{n,q}(|g(t)-g(x)|;x),$$

and also

$$S^{\star}_{n,q}(|t-x|;x) \leq S^{\star}_{n,q}\left((t-x)^2;x\right)^{\frac{1}{2}} = \delta^x_{n,q}$$

we get

$$\begin{aligned}|S^{\star}_{n,q}(g;x) - g(x)| &\leq \left(S^{\star}_{n,q}(|t-x|;x) + \frac{1}{2}S^{\star}_{n,q}((t-x)^2;x)\right)\|g\|_{C^2_B[0,\infty)}\\ &\leq \frac{1}{2}\delta^x_{n,q}(2+\delta^x_{n,q})\|g\|_{C^2_B[0,\infty)}.\end{aligned}$$

This completes the proof. □

The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf_{C^2_B(\mathbb{R}^+)}\left\{\left(\|f-g\|_{C_B(\mathbb{R}^+)} + \delta\|g''\|_{C^2_B(\mathbb{R}^+)}\right) : g \in \mathcal{W}^2\right\}, \tag{4.12}$$

where

$$\mathcal{W}^2 = \left\{g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+)\right\}. \tag{4.13}$$

There exits a positive constant $C > 0$ such that $K_2(f,\delta) \leq C\omega_2(f,\delta^{1/2})$, $\delta > 0$, where the second order modulus of continuity is given by

$$\omega_2(f,\delta^{1/2}) = \sup_{0<h<\delta^{1/2}}\sup_{x\in\mathbb{R}^+}|f(x+2h) - 2f(x+h) + f(x)|. \tag{4.14}$$

Theorem 4.6. *Let $S^{\star}_{n,q}(.\,;\,.)$ be the operators defined in* (2.3). *Then for $x \geq 0$, $q \in (0,1)$, $n \in \mathbb{N}$ and $f \in C_B(\mathbb{R}^+)$ we have*

$$|S^{\star}_{n,q}(f;x) - f(x)| \leq 2M\left\{\omega_2\left(f;\sqrt{\Delta^x_{n,q}}\right) + \min(1,\Delta^x_{n,q})\|f\|_{C_B(\mathbb{R}^+)}\right\},$$

where M is a positive constant, $\Delta^x_{n,q} = \frac{\delta^x_{n,q}(2+\delta^x_{n,q})}{4}$ is given in (4.3) *and $\omega_2(f;\delta)$ is the second order modulus of continuity of the function f defined in* (4.14).

Proof. We prove this by using our previous result

$$\begin{aligned}
|S^{\star}_{n,q}(f;x)-f(x)| &\le |S^{\star}_{n,q}(f-g;x)|+|S^{\star}_{n,q}(g;x)-g(x)|+|f(x)-g(x)| \\
&\le 2\|f-g\|_{C_B(\mathbb{R}^+)}+\frac{\delta^x_{n,q}}{2}(2+\delta^x_{n,q})\|g\|_{C^2_B(\mathbb{R}^+)} \\
&\le 2\left(\|f-g\|_{C_B(\mathbb{R}^+)}+\frac{\delta^x_{n,q}}{4}(2+\delta^x_{n,q})\|g\|_{C^2_B(\mathbb{R}^+)}\right)
\end{aligned}$$

By taking infimum over all $g \in C^2_B(\mathbb{R}^+)$ and by using (4.12), we get

$$|S^{\star}_{n,q}(f;x)-f(x)| \le 2K_2\left(f;\frac{\delta^x_{n,q}(2+\delta^x_{n,q})}{4}\right).$$

Now for an absolute constant $M > 0$ in [34] we use the relation

$$K_2(f;\delta) \le M\{\omega_2(f;\sqrt{\delta})+\min(1,\delta)\|f\|\}.$$

This completes the proof. □

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