Demonstratio Mathematica

Open Access

Research Article

Pooja Gupta* and Purshottam Narain Agrawal

Rate of convergence of Szász-beta operators based on q-integers

Abstract: The purpose of this paper is to establish the rate of convergence in terms of the weighted modulus of continuity and Lipschitz type maximal function for the q-Szász-beta operators. We also study the rate of A-statistical convergence. Lastly, we modify these operators using King type approach to obtain better approximation.

Keywords: Weighted modulus of continuity, Lipschitz type maximal function, A-statistical convergence.

MSC: 41A25, 26A15, 41A36.

1 Introduction

In the last decade, approximation of functions by linear methods of convergence has been extensively studied by making use of the q-calculus. Lupas [1] introduced a q-analogue of the classical Bernstein polynomials and established its approximation and shape preserving properties. Subsequently, Phillips [2] proposed another generalization of Bernstein polynomials and obtained the rate of convergence and Voronovskaja type asymptotic theorem. After that many researchers started working in this direction and introduced the q-analogues of several sequences of positive linear operators. We mention here some of the important papers in this direction (cf. [3, 4] and [2, 5–7] etc.). Gupta, Srivastava and Sahai [8] discussed the rate of convergence in simultaneous approximation by Szász beta operators. Recently, Gupta and Mahmudov [5] introduced the q-analogue of Szász -Mirakyan-beta type operators as follows: for every n∈ ℕ and q∈ (0, 1),the positive linear operator D_n^q is defined by

$$D_n^q(f(t), x) = \sum_{k=0}^{\infty} s_{n,k}^q(x) \, q^{k(k+1)/2} \int_0^{\infty/A} P_{n,k}^q(t)f(t)d_q t,$$

where

$$s_{n,k}^q(x) = \frac{([n]_q x)^k}{[k]_q!} q^{\frac{k(k-1)}{2}} \frac{1}{E([n]_q x)}$$

and

$$P_{n,k}^q(t) = \frac{1}{B_q(n, k+1)} \frac{t^k}{(1+t)_q^{n+k+1}} \,,$$

for $x \in [0, \infty)$ and for every real valued continuous function f on $[0, \infty)$. They also studied a local approximation theorem, degree of approximation for a Lipschitz class and the rate of convergence of these operators for a weighted space. Subsequently, Yüksel and Dinlemez [9] gave an alternate form of these operators as

*Corresponding Author: Pooja Gupta: Department of Mathematics, IIT Roorkee, India, E-mail: poojaguptaiitr@gmail.com
Purshottam Narain Agrawal: Department of Mathematics, IIT Roorkee, India, E-mail: pnappfma@gmail.com

follows: for $0 < q < 1$ and $\gamma > 0$, let $f \in C_\gamma[0, \infty) := \{f \in C[0, \infty) : |f(t)| \le M(1 + t^\gamma), \ for\ some\ M > 0\}$, endowed with the norm $||f||_\gamma = \sup\limits_{0 \le t < \infty} \dfrac{|f(t)|}{1 + t^\gamma}$, the q-Szász-Mirakjan-beta type operators are defined as

$$B_{n,q}(f, x) = \sum_{k=0}^{\infty} S_{n,k}^q(x) \int_0^{\infty/A} b_{n,k}^q(t) f(t) d_q t, \tag{1}$$

where

$$S_{n,k}^q(x) = ([n]_q x)^k \frac{e_q^{-[n]_q x}}{[k]_q!}$$

and

$$b_{n,k}^q(x) = \frac{q^{k^2} x^k}{B_q(k + 1, n)(1 + x)_q^{n+k+1}}.$$

They obtained a Voronovskaja type approximation theorem in the space of continuous functions weighted by $\rho(x) = 1 + x^2$. For a detailed account of other significant research in this direction we refer to [10, 11]. For the definitions and notations of q calculus one can consult [12].

In the present paper, we study the rate of convergence of the operators defined by (1) in the same weighted space, the Voronovskaja type theorem for A-statistical convergence, the rate of A-statistical convergence in terms of the modulus of continuity. Finally to obtain better approximation, a modification of these operators is given and a local approximation theorem is derived using King type approach. The paper is organised as follows: Section 2 is devoted to some definitions and auxiliary results and in section 3 we prove the main results of the paper.

2 Preliminaries

For simplicity, we define $F_s^-(n)_q = \prod_{i=0}^{s}[n - i]_q$ and $F_s^+(n)_q = \prod_{i=0}^{s}[n + i]_q$.

Lemma 1. *[9] Let $e_m(t) = t^m$, $m = 0, 1, 2, 3, 4$. Then for every $q \in (0, 1)$, we have*
1. $B_{n,q}(e_0, x) = 1$.
2. $B_{n,q}(e_1, x) = \dfrac{[n]_q}{q^2 F_0^-(n-1)_q} x + \dfrac{1}{q F_0^-(n-1)_q}$,
3. $B_{n,q}(e_2; x) = \dfrac{[n]_q^2}{q^6 F_1^-(n-1)_q} x^2 + \dfrac{[2]^2[n]_q}{q^5 F_1^-(n-1)_q} x + \dfrac{[2]_q}{q^3 F_1^-(n-1)_q}$,
4. $B_{n,q}(e_3, x) = \dfrac{[n]_q^3}{q^{12} F_2^-(n-1)_q} x^3 + \dfrac{([5]_q + q[2]_q^2)[n]_q^2}{q^{11} F_2^-(n-1)_q x^2} + \dfrac{([2]_q^2[4]_q + q^2[2]_q)[n]_q}{q^9 F_2^-(n-1)_q} x + \dfrac{[2]_q[3]_q}{q^6 F_2^-(n-1)_q}$,
5. $B_{n,q}(e_4, x) = \dfrac{[n]_q^4}{q^{20} F_3^-(n-1)_q} x^4 + \dfrac{([7]_q + q[5]_q + q^2[2]_q^2)[n]_q^3}{q^{19} F_3^-(n-1)_q} x^3 + \dfrac{([5]_q[6]_q + q[2]_q^2[6]_q + q^2[2]_q^2[4]_q + q^4[2]_q)[n]_q^2}{q^{17} F_3^-(n-1)_q} x^2$

    $+ \dfrac{([2]_q^2[4]_q[5]_q + q^2[2]_q[5]_q + q^3[2]_q[3]_q)[n]_q}{q^{14} F_3^-(n-1)_q} x + \dfrac{[2]_q[3]_q[5]_q}{q^{10} F_3^-(n-1)_q}$.

Consequently,

$$B_{n,q}((t - x); x) = \frac{([n]_q - q^2[n-1]_q)x + q}{q^2[n-1]_q}, n > 1$$

and

$$
\begin{aligned}
B_{n,q}((t - x)^2; x) &= \left(\frac{[n]_q^2}{q^6[n-1]_q[n-2]_q} + 1 - \frac{2[n]_q}{q^2[n-1]_q} \right) x^2 + \left(\frac{[2]_q^2[n]_q}{q^5[n-1]_q[n-2]_q} - \frac{2}{q[n-1]_q} \right) x \\
&\quad + \frac{[2]_q}{q^3[n-1]_q[n-2]_q}, n > 2 \\
&= \gamma_{n,q}(x), (say).
\end{aligned}
$$

Now let us consider the following Lipschitz-type space [13]:

$$Lip_M^\star(r) := \left\{ f \in C_\gamma[0, \infty) : |f(t) - f(x)| \le M \frac{|t-x|^r}{(t+x)^{\frac{r}{2}}} ; t \ge 0 \ and \ x \in (0, \infty) \right\},$$

where M is a positive constant and $0 < r \le 1$. Erençein [14] obtained the rate of convergence of the generalized Baskakov operators for functions belonging to the space $Lip_M^\star(r)$. We also study such a kind of result for the operators defined in (1).

Lemma 2. *For all $x \ge 0$ and $n > 2$, we have*

$$B_{n,q}(|t-x|; x) \le \sqrt{\gamma_{n,q}(x)}.$$

Proof. We may write $B_{n,q}(|t-x|; x) = \sum_{k=0}^\infty S_{n,k}^q(x) \int_0^\infty b_{n,k}^q(t)|t-x|d_qt.$

Applying the Cauchy-Schwarz inequality we obtain

$$B_{n,q}(|t-x|; x) \le \sum_{k=0}^\infty S_{n,k}^q(x) \left(\int_0^\infty b_{n,k}^q(t)(t-x)^2 d_qt \right)^{\frac{1}{2}} \left(\int_0^\infty b_{n,k}^q(t)d_qt \right)^{\frac{1}{2}}.$$

Again, applying the Cauchy-Schwarz inequality and Lemma 1 we find that

$$B_{n,q}(|t-x|; x) \quad \le \quad \left(\sum_{k=0}^\infty S_{n,k}^q(x) \int_0^\infty b_{n,k}^q(t)d_qt \right)^{\frac{1}{2}} \left(\sum_{k=0}^\infty S_{n,k}^q(x) \int_0^\infty b_{n,k}^q(t)(t-x)^2 d_qt \right)^{\frac{1}{2}}$$

$$= \quad \sqrt{B_{n,q}((t-x)^2; x)} = \sqrt{\gamma_{n,q}(x)}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3 Main Results

3.1 Direct Results

Theorem 1. *Let $0 < r \le 1$ and $f \in Lip_M^\star(r)$. Then for all $x > 0$ and $n > 2$, we have*

$$|B_{n,q}(f; x) - f(x)| \le M \left(\frac{\gamma_{n,q}(x)}{x} \right)^{\frac{r}{2}},$$

where $\gamma_{n,q}(x)$ is defined as in Lemma 1.

Proof. By our definition for f$\in Lip_M^\star(1)$, using Lemma 2 we have

$$|B_{n,q}(f; x) - f(x)| \quad \le \quad \sum_{k=0}^\infty S_{n,k}^q(x) \int_0^\infty b_{n,k}^q(t)|f(t) - f(x)|d_qt$$

$$\le \quad M \sum_{k=0}^\infty S_{n,k}^q(x) \int_0^\infty b_{n,k}^q(t) \frac{|t-x|}{(t+x)^{r/2}} d_qt$$

$$\le \quad \frac{M}{(x)^{\frac{r}{2}}} \sum_{k=0}^\infty S_{n,k}^q(x) \int_0^\infty b_{n,k}^q(t)|t-x|d_qt$$

Applying Hölder inequality two times with $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$ we obtain

$$
\begin{aligned}
|B_{n,q}(f;x) - f(x)| &\leq \frac{M}{(x)^{\frac{r}{2}}} \left\{ \sum_{k=0}^{\infty} S_{n,k}^{q}(x) \left(\int_{0}^{\infty} b_{n,k}^{q}(t)(|t-x|^{r})^{p} d_{q}t \right)^{\frac{1}{p}} \left(\int_{0}^{\infty} b_{n,k}^{q}(t) d_{q}t \right)^{\frac{1}{q}} \right\} \\
&\leq \frac{M}{(x)^{\frac{r}{2}}} \left\{ \left(\sum_{k=0}^{\infty} S_{n,k}^{q}(x) \int_{0}^{\infty} b_{n,k}^{q}(t)(|t-x|^{r})^{p} d_{q}t \right)^{\frac{1}{p}} \left(\sum_{k=0}^{\infty} S_{n,k}^{q}(x) \int_{0}^{\infty} b_{n,k}^{q}(t) d_{q}t \right)^{\frac{1}{q}} \right\} \\
&= \frac{M}{(x)^{\frac{r}{2}}} (B_{n,q}(|t-x|^{2};x))^{\frac{r}{2}} (B_{n,q}(1;x))^{\frac{1}{q}} \\
&= M \left(\frac{\gamma_{n,q}(x)}{x} \right)^{\frac{r}{2}}.
\end{aligned}
$$

which completes the proof. □

In order to establish the weighted approximation properties of the operators (1), we define the space $C_{2}^{\star}[0,\infty)$ as follows:

$$
C_{2}^{\star}[0,\infty) := \{ f \in C_{2}[0,\infty) : \lim_{x \to \infty} \frac{|f(x)|}{1+x^2} < \infty \}.
$$

For some other important papers in this area please see [15, 16].

Theorem 2. *Let* $0 < q_n < 1$ *and* $A > 0$. *Then for each* $f \in C_{\gamma}[0,\infty)$, *the sequence* $B_{n,q_n}(f;x)$ *converges to* f *uniformly on* $[0,A]$ *if and only if* $\lim_{n\to\infty} q_n = 1$.

Proof. The proof of the theorem follows along the lines of the proof of Theorem 1 in [17]. Hence the details are omitted. □

Theorem 3. *Let* $0 < q_n < 1$ *and* $q_n \to 1$, *as* $n \to \infty$. *Then for each* $f \in C_{2}^{\star}[0,\infty)$ *and* $\alpha > 0$, *we have*

$$
\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|B_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} = 0.
$$

Proof. Let $x_0 \in [0,\infty)$ be arbitrary but fixed. Then

$$
\begin{aligned}
\sup_{x\in[0,\infty)} \frac{|B_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} &\leq \sup_{x\leq x_0} \frac{|B_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} + \sup_{x> x_0} \frac{|B_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} \\
&\leq \|B_{n,q_n}(f) - f\|_{C[0,x_0]} + \|f\|_2 \sup_{x>x_0} \frac{|B_{n,q_n}(1+t^2;x)|}{(1+x^2)^{1+\alpha}} + \sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}} \\
&= I_1 + I_2 + I_3, \ say. \quad\quad\quad (2)
\end{aligned}
$$

Since $|f(x)| \leq \|f\|_2(1+x^2)$, we have $\sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}} \leq \frac{\|f\|_2}{(1+x_0^2)^{\alpha}}$.

Let $\epsilon > 0$ be arbitrary. We can choose x_0 to be so large that

$$
\frac{\|f\|_2}{(1+x_0^2)^{\alpha}} < \frac{\epsilon}{6}. \quad\quad\quad (3)
$$

In view of Theorem 2, there exists a $n_1 \in \mathbb{N}$ such that

$$
\|f_2\| \frac{|B_{n,q_n}(1+t^2;x)|}{(1+x^2)^{1+\alpha}} < \frac{(1+x^2)\|f\|_2}{(1+x^2)^{1+\alpha}} + \frac{\epsilon}{3}, \ \forall n \geq n_1.
$$

Hence

$$
\|f_2\| \sup_{x>x_0} \frac{|B_{n,q_n}(1+t^2;x)|}{(1+x^2)^{1+\alpha}} < \frac{\|f\|_2}{(1+x_0^2)^{\alpha}} + \frac{\epsilon}{3}, \ \forall n \geq n_1. \quad\quad\quad (4)
$$

Thus, combining (3) and (4)

$$I_2 + I_3 < \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} = \frac{2\epsilon}{3}, \quad \forall n \geq n_1 \tag{5}$$

Using Theorem 2, we can see that the first term of the inequality (2) implies that

$$||B_{n,q_n}(f) - f||_{C[0,x_0]} < \frac{\epsilon}{3}, \forall n \geq n_2. \tag{6}$$

Let $n_0 = max(n_1, n_2)$. Then, combining (2), (5) and (6) we get the desired result.

For $f \in C_2^\star[0, \infty)$, the weighted modulus of continuity is defined as

$$\Omega_2(f, \delta) = \sup_{x \geq 0} \sup_{0 < |h| \leq \delta} \frac{|f(x + h) - f(x)|}{1 + (x + h)^2}.$$

$\square$

Lemma 3. *[18] If $f \in C_2^\star[0, \infty)$, then*
1. $\Omega_2(f, \delta)$ *is a monotone increasing function of δ,*
2. $\lim_{\delta \to 0^+} \Omega_2(f, \delta) = 0$;
3. *for any $\lambda \in [0, \infty)$, $\Omega_2(f, \lambda\delta) \leq (1 + \lambda)\Omega_2(f, \delta)$.*

Theorem 4. *If $f \in C_2^\star[0, \infty)$, then for sufficiently large n we have*

$$|B_{n,q_n}(f; x) - f(x)| \leq K(1 + x^{2+\lambda})\Omega_2(f, \delta_n), \quad x \in [0, \infty),$$

where $\lambda \geq 1$, $\delta_n = \max\{\alpha_n, \beta_n, \gamma_n, \}$, $\alpha_n, \beta_n, \gamma_n$ being

$$\left(\frac{[n]_{q_n}^2}{q_n^6[n-1]_{q_n}[n-2]_{q_n}} + 1 - \frac{2[n]_{q_n}}{q_n^2[n-1]_{q_n}} \right), \left(\frac{[2]_{q_n}^2[n]_{q_n}}{q_n^5[n-1]_{q_n}[n-2]_{q_n}} - \frac{2}{q_n[n-1]_{q_n}} \right)$$

and

$$\frac{[2]_{q_n}}{q_n^3[n-1]_{q_n}[n-2]_{q_n}}$$

respectively and K is a positive constant independent of f and n.

Proof. From the definition of $\Omega_2(f, \delta)$ and Lemma 3, we have

$$\begin{aligned}
|f(t) - f(x)| &\leq (1 + (x + |t - x|^2))\left(1 + \frac{|t - x|}{\delta}\right)\Omega_2(f, \delta) \\
&\leq (1 + (2x + t)^2)\left(1 + \frac{|t - x|}{\delta}\right)\Omega_2(f, \delta) \\
&= \phi_x(t)\left(1 + \frac{\psi_x(t)}{\delta}\right)\Omega_2(f, \delta),
\end{aligned}$$

where $\phi_x(t) = 1 + (2x + t)^2$ and $\psi_x(t) = |t - x|$. Thus, from the definition (1) $|B_{n,q_n}(f; x) - f(x)| \leq \left(B_{n,q_n}(\phi_x; x) + \frac{B_{n,q_n}(\phi_x\psi_x; x)}{\delta_n} \right)\Omega_2(f, \delta_n)$.

Now, applying the Cauchy -Schwarz inequality to the second term on the right hand side of the above inequality we get:

$$|B_{n,q_n}(f; x) - f(x)| \leq \left(B_{n,q_n}(\phi_x; x) + \frac{1}{\delta_n}\sqrt{B_{n,q_n}(\phi_x^2; x)}\sqrt{B_{n,q_n}(\psi_x^2; x)} \right)\Omega_2(f, \delta_n). \tag{7}$$

From Lemma 1

$$\frac{1}{1+x^2}B_{n,q_n}(1+t^2;x) = \frac{1}{1+x^2} + \frac{1}{1+x^2}\left(\frac{[n]_{q_n}^2 x^2 + q_n[2]_{q_n}^2[n]_{q_n}x + q_n^3[2]_{q_n}}{q_n^6[n-1]_{q_n}[n-2]_{q_n}}\right)$$

$$= \frac{1}{1+x^2} + \left(\frac{[n]_{q_n}^2}{q_n^6[n-1]_{q_n}[n-2]_{q_n}}\right)\frac{x^2}{1+x^2}$$

$$+ \left(\frac{[2]_{q_n}^2[n]_{q_n}}{q_n^5[n-1]_{q_n}[n-2]_{q_n}}\right)\frac{x}{1+x^2} + \frac{[2]_{q_n}}{q_n^3[n-1]_{q_n}[n-2]_{q_n}}\frac{1}{1+x^2}$$

$$\leq 1 + C_1, \tag{8}$$

for sufficiently large n, where C_1 is a positive constant.

From (8) there exists a positive constant K_1 such that $B_{n,q_n}(\phi_x;x) \leq K_1(1+x^2)$, for sufficiently large n, K_1 being $1 + C_1$.

Proceeding similarly, $\frac{1}{1+x^4}B_{n,q_n}(1+t^4;x) \leq 1 + C_2$, for sufficiently large n, where C_2 is a positive constant. So, there exists a positive constant K_2 such that $\sqrt{B_{n,q_n}(\phi_x^2;x)} \leq K_2(1+x^2)$ where $x \in [0,\infty)$ and n is large enough. Also, we get

$$B_{n,q_n}(\psi_x^2;x) = \left\{\frac{[n]_{q_n}^2}{q_n^6[n-1]_{q_n}[n-2]_{q_n}} + 1 - \frac{2[n]_{q_n}}{q_n^2[n-1]_{q_n}}\right\}x^2$$

$$+\left\{\left(\frac{[2]_{q_n}^2[n]_{q_n}}{q_n^5[n-1]_{q_n}[n-2]_{q_n}} - \frac{2}{q_n[n-1]_{q_n}}\right)x + \frac{[2]_{q_n}}{q_n^3[n-1]_{q_n}[n-2]_{q_n}}\right\}$$

$$= \alpha_n x^2 + \beta_n x + \gamma_n.$$

Hence from (7) we have

$$|B_{n,q_n}(f;x) - f(x)| \leq (1+x^2)\left(K_1 + \frac{1}{\delta_n}K_2\sqrt{\alpha_n x^2 + \beta_n x + \gamma_n}\right)\Omega_2(f,\delta_n).$$

If we take $\delta_n^2 = \max\{\alpha_n, \beta_n, \gamma_n\}$, then we get

$$|B_{n,q_n}(f;x) - f(x)| \leq (1+x^2)(K_1 + K_2\sqrt{x^2+x+1})\Omega_2(f,\delta_n),$$

$$\leq K_3(1+x^{2+\lambda})\Omega_2(f,\delta_n),$$

for sufficiently large n and $x \in [0,\infty)$. Hence, the proof is completed.   □

Let $C_B[0,\infty)$ denote the space of bounded and uniformly continuous functions endowed with the norm $||f|| = \sup_{x\in[0,\infty)}|f(x)|$.

For $f \in C_B[0,\infty)$, the Lipschitz-type maximal function of order η introduced by Lenze [19] is defined as

$$\widetilde{\omega}_\eta(f,x) = sup_{t\neq x, t\in[0,\infty)}\frac{|f(t)-f(x)|}{|t-x|^\eta}, \quad x \in [0,\infty)$$

and

$$\eta \in (0,1].$$

Theorem 5. *Let $f \in C_B[0,\infty)$. Then, for all $x \in [0,\infty)$ we have*

$$|B_{n,q}(f;x) - f(x)| \leq \widetilde{\omega}(f,x)(\gamma_{n,q}(x))^{\frac{\eta}{2}}.$$

Proof. From the above definition of Lipschitz-type maximal function of order η we have

$$|B_{n,q}(f;x) - f(x)| \leq \widetilde{\omega}_n(f,x)B_{n,q}(|t-x|^\eta;x).$$

Applying Hölder's inequality with $p = \frac{2}{\eta}$ and $\frac{1}{q} = 1 - \frac{1}{p}$, we get

$$|B_{n,q}(f;x) - f(x)| \leq \widetilde{\omega}(f,x)B_{n,q}((t-x)^2;x)^{\frac{\eta}{2}} \leq \widetilde{\omega}(f,x)(\gamma_{n,q}(x))^{\eta/2}.$$

Thus, the proof is completed.   □

Let $0 < \alpha \leq 1$ and let D be a subset of the interval $[0, \infty)$. Then by $Lip_M(D, \alpha)$, we denote the space of all functions satisfying the condition

$$|f(y) - f(x)| \leq M|y - x|^\alpha, \text{ for } y \in \overline{D} \text{ and } x \in [0, \infty), \tag{9}$$

where $\overline{D}$ denotes the closure of D in $[0, \infty)$.

Theorem 6. *Let D be a subset of $[0, \infty)$, $q \in (0, 1)$ and $\alpha \in (0, 1]$. Then, for every $n \in N, x \in [0, \infty)$ and $f \in C_B[0, \infty) \bigcap Lip_M(D, \alpha)$, we have*

$$|B_{n,q}(f; x) - f(x)| \leq M\{(\gamma_{n,q}(x))^{\alpha/2} + 2(d(x, D))^\alpha\},$$

where M is a positive constant; $d(x, D)$ is the distance between x and D defined as $d(x, D); = \inf\{|y - x| : y \in D\}$; and $\gamma_{n,q}$ is as defined in Lemma 1.

Proof. Let $\overline{D}$ denote the closure of D in $[0, \infty)$ and $x \in [0, \infty)$ be fixed. Then, there exists a point $x_0 \in \overline{D}$ such that $|x - x_0| = d(x, D)$. Using the triangle inequality we get

$$|f(y) - f(x)| \leq |f(y) - f(x_0) + |f(x) - f(x_0)|.$$

Hence it follows from (9) that

$$
\begin{aligned}
|B_{n,q}(f; x) - f(x)| \quad &\leq \quad B_{n,q}(|f(y) - f(x)|; x) \\
&\leq \quad B_{n,q}(|f(y) - f(x_0)|; x) + |f(x) - f(x_0)|) \\
&\leq \quad M\{B_{n,q}(|y - x_0|^\alpha; x) + (d(x, D))^\alpha\} \\
&= \quad M\{B_{n,q}(|y - x|^\alpha; x) + 2(d(x, D))^\alpha\}.
\end{aligned}
$$

Using the Cauchy-Bunyakowsky-Schwarz inequality for positive linear operators we find that

$$|B_{n,q}(f; x) - f(x)| \leq M\{B_{n,q}((y - x)^2; x)^{\alpha/2} + 2(d(x, D))^\alpha\}.$$

This completes the proof. □

3.2 A-Statistical Convergence

Let $A = (a_{nk})$ be a non- negative infinite summability matrix. For a given sequence $x := (x_n)$, the A-transform of x denoted by $Ax : (Ax)_n$ is defined as

$$(Ax)_n = \sum_{k=1}^\infty a_{nk} x_k,$$

provided the series converges for each n. A is said to be regular if $lim_n(Ax)_n = L$ whenever $lim_n(x)_n = L$. Then $x = (x_n)$ is said to be A-statistically convergent to L i.e. $st_A - lim_n(x)_n = L$ if for every $\epsilon > 0$, $lim_n \Sigma_{k:|x_k-L|\geq\epsilon} a_{nk} = 0$. If we replace A by C_1, then A is a Cesaro matrix of order 1 and A-statistical convergence is reduced to statistical convergence. Similarly, if A=I, the identity matrix, then A-statistical convergence is called ordinary convergence. Kolk [20] proved that statistical convergence is better than ordinary convergence. Many researchers have made significant contributions in this area. We refer the reader to some of the important papers in this direction (cf. [17],[21–23] and [7, 24–27] etc.). Let $q_n \in (0, 1)$ be a sequence such that

$$st_A - lim_n q_n = 1, \quad st_A - lim_n q_n^n = \lambda, \quad (\lambda < 1) \tag{10}$$

$$and \ st_A - lim_n \frac{1}{[n]_{q_n}} = 0.$$

Theorem 7. *Let $A = (a_{nk})$ be a non negative regular summability matrix and (q_n) be a sequence in $(0, 1)$ satisfying (10). Let the operators $B_{n,q_n}, n \in N$, be defined as in (1). Then for each function $f \in C_2[0, \infty)$, we have*

$$st_A - lim_n||B_{n,q_n}(f; .) - f||_\rho = 0,$$

where $\rho(x) = 1 + x^{2+\lambda}, \lambda > 0$

Proof. By Bohman Korovkin type theorem ([28],Theorem 3, p.191) it is sufficient to prove that

$$st_A - lim_n||B_{n,q_n}(e_i;.) - e_i||_2 = 0,$$

where $e_i(x) = x^i$, $i = 0, 1, 2$. From Lemma 1, $st_A - lim_n||B_{n,q_n}(e_0;.) - e_0||_2 = 0$ holds.

Again, using Lemma 1, we have

$$||B_{n,q_n}(e_1;.) - e_1||_2 \quad \le \quad sup_{x\in[0,\infty)}\left\{\frac{x}{1+x^2}\left|\frac{[n]_{q_n}}{[n-1]_{q_n}q_n^2}-1\right|+\frac{1}{1+x^2}\frac{1}{q_n[n-1]_{q_n}}\right\}$$

$$\le \quad \left|\frac{[n]_{q_n}}{[n-1]_{q_n}q_n^2}-1\right|+\frac{1}{q_n[n-1]_{q_n}}. \tag{11}$$

For each $\epsilon > 0$, let us define the following sets:

$$G : \quad = \quad \left\{k : ||B_{k,q_k}(e_1;.) - e_1|| \ge \epsilon\right\},$$

$$G_1 : \quad = \quad \left\{k : \left|\frac{[k]_{q_k}}{q_k^2[k-1]_{q_k}}\right| - 1 \ge \frac{\epsilon}{2}\right\}$$

$$G_2 : \quad = \quad \left\{k : \frac{1}{q_k[k-1]_{q_k}} \ge \frac{\epsilon}{2}\right\},$$

which yields $G \subseteq G_1 \bigcup G_2$ in view of (11) and therefore for all $n \in N$ we have

$$\sum_{k\in G} a_{nk} \le \sum_{k\in G_1} a_{nk} + \sum_{k\in G_2} a_{nk}.$$

Hence $st_A - lim_n||B_{n,q_n}(e_2;.) - e_2||_2 = 0$. Proceeding similarly,

$$||B_{n,q_n}(e_2;.) - e_2||_2 \quad \le \quad \left|\frac{[n]_{q_n}^2}{q_n^6[n-1]_{q_n}[n-2]_{q_n}}-1\right|+\frac{[2]_{q_n}^2[n]_{q_n}}{q_n^5[n-1]_{q_n}[n-2]_{q_n}}+\frac{[2]_{q_n}}{q_n^3[n-1]_{q_n}}.$$

Now, let us define the following sets:

$$M : \quad = \quad \left\{k : ||B_{k,q_k}(e_2;.) - e_2|| \ge \epsilon\right\},$$

$$M_1 : \quad = \quad \left\{k : \left|\frac{[k]_{q_k}^2}{q_k[k-1]_{q_k}[k-2]_{q_k}}-1\right| \ge \frac{\epsilon}{3}\right\}$$

$$M_2 : \quad = \quad \left\{k : \frac{[2]_{q_k}^2[k]_{q_k}}{q_k^5[k-1]_{q_k}[k-2]_{q_k}} \ge \frac{\epsilon}{3}\right\}$$

$$M_3 : \quad = \quad \left\{k : \frac{[2]_{q_k}}{q_k^3[k-1]_{q_k}} \ge \frac{\epsilon}{3}\right\},$$

then we obtain $M \subseteq M_1 \bigcup M_2 \bigcup M_3$, which implies that

$$\sum_{k\in G} a_{nk} \le \sum_{k\in M_1} a_{nk} + \sum_{k\in M_2} a_{nk} + \sum_{k\in M_3} a_{nk}.$$

Hence, taking the limit as $n \to \infty$ we get $st_A - lim_n||B_{n,q_n}(e_2;.) - e_2||_2 = 0$. This completes the proof of the theorem. □

Theorem 8. *Let $A = (a_{jn})$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0, 1)$ satisfying (10). Assume that D is a compact set in $[0, \infty)$. Then for every $f \in C_B[0, \infty) \bigcap Lip_M(D, \alpha)$ with $M > 0$ and $\alpha \in (0, 1]$, we have*

$$st_A - lim_n||B_{n,q_n}(f) - f||_D = 0.$$

Proof. Let $x \in D$ be fixed. Then from Theorem 6, for each $x \in D$

$$|B_{n,q_n}(f;x) - f(x)| \leq M(\gamma_{n,q}(x))^{\alpha/2} \leq M\left(\left|\frac{[n]_q^2}{q^6[n-1]_q[n-2]_q} + 1 - \frac{2[n]_q}{q^2[n-1]_q}\right|x^2\right.$$
$$+ \left.\left|\frac{[2]_q^2[n]_q}{q^5[n-1]_q[n-2]_q}\right|x + \left|\frac{[2]_q}{q^3[n-1]_q[n-2]_q}\right|\right)^{\alpha/2}. \tag{12}$$

Since D is compact, the number $x_0 := \sup_{x \in D} x$ is finite. Hence, taking supremum over $x \in D$ on both sides of (12), we obtain that

$$||B_{n,q_n}(f) - (f)||_D \leq M\left(\left|\frac{[n]_q^2}{q^6[n-1]_q[n-2]_q} + 1 - \frac{2[n]_q}{q^2[n-1]_q}\right|x_0^2\right.$$
$$+ \left.\left|\frac{[2]_q^2[n]_q}{q^5[n-1]_q[n-2]_q}\right|x_0 + \left|\frac{[2]_q}{q^3[n-1]_q[n-2]_q}\right|\right)^{\alpha/2},$$
$$\leq M\left(\left|\frac{[n]_q^2}{q^6[n-1]_q[n-2]_q} + 1 - \frac{2[n]_q}{q^2[n-1]_q}\right|\right.$$
$$+ \left.\left|\frac{[2]_q^2[n]_q}{q^5[n-1]_q[n-2]_q}\right| + \left|\frac{[2]_q}{q^3[n-1]_q[n-2]_q}\right|\right)^{\alpha/2}(1 + x_0^2)^{\alpha/2}$$
$$= LM(\gamma_{n,q}(1))^{\alpha/2},$$

where $L = (1 + x_0^2)^{\frac{\alpha}{2}}$ which yields for every $\epsilon > 0$,

$$n \in N : ||B_{n,q_n}(f) - f||_D \geq \epsilon \subseteq \left\{n \in N : \gamma_{n,q}(1) \geq \left(\frac{\epsilon}{LM}\right)^{2/\alpha}\right\}$$

and hence

$$\sum_{n:||B_{n,q_n}(f)-f||_D \geq \epsilon} a_{jn} \leq \sum_{n:\gamma_{n,q}(1) \geq (\frac{\epsilon}{LM})^{2/\alpha}} a_{jn} = 0,$$

thus we get the required result. □

Now we will prove a Voronovskaja-type theorem for operators $B_{n,q}$ by taking a sequence (q_n) satisfying (10).

Lemma 4. *Let $A = (a_{jn})$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0, 1)$ satisfying* (10). *Then*

$$st_A \lim_{n\to\infty} [n]_{q_n} B_{n,q_n}((t-x;x)) = (2-\lambda)x + 1,$$
$$st_A \lim_{n\to\infty} [n]_{q_n} B_{n,q_n}((t-x)^2;x) = (2-\lambda)x^2 + 2x,$$
$$st_A \lim_{n\to\infty} [n]_{q_n}^2 B_{n,q_n}((t-x)^4;x) = (3\lambda^2 - 12\lambda + 12)x^4 + (24-12\lambda)x^3 + 12x^2,$$

uniformly with respect to $x \in [0, b]$, $(b > 0)$, where $\lambda = st_A - \lim q_n^n$.

Proof. From Lemma 1

$$\lim_{n\to\infty} [n]_{q_n} B_{n,q_n}(t-x) = \left(\frac{[n]_{q_n}^2}{q_n^2[n-1]_{q_n}} - [n]_{q_n}\right)x + \frac{[n]_{q_n}}{q_n[n-1]_{q_n}},$$

and

$$[n]_{q_n} B_{n,q_n}((t-x)^2;x) = \left(\frac{[n]_{q_n}^3}{q_n^6[n-1]_{q_n}[n-2]_{q_n}} + [n]_{q_n} - \frac{2[n]_{q_n}^2}{q_n^2[n-1]_{q_n}}\right)x^2$$
$$+ \left(\frac{[2]_{q_n}^2[n]_{q_n}^2}{q_n^5[n-1]_{q_n}[n-2]_{q_n}} - \frac{2[n]_{q_n}}{q_n[n-1]_{q_n}}\right)x + \frac{[2]_{q_n}[n]_{q_n}}{q_n^3[n-1]_{q_n}[n-2]_{q_n}},$$

Then, using (10) we get

$$st_A - \lim_{n\to\infty} [n]_{q_n} B_{n,q_n}((t-x);x) = (2-\lambda)x + 1,$$

and

$$st_A - \lim_{n\to\infty} [n]_{q_n} B_{n,q_n}((t-x)^2;x) = (2-\lambda)x^2 + 2x$$

uniformly with respect to $x \in [0, b]$. Again using Lemma 1, we obtain

$$
\begin{aligned}
B_{n,q_n}((t-x)^4;x) = & \left(\frac{[n]_{q_n}^6}{q_n^{20} F_3^-[n-1]_{q_n}} + [n]_{q_n}^2 - \frac{4[n]_{q_n}^5}{q_n^{12} F_2^-[n-1]_{q_n}} - \frac{4[n]_{q_n}^3}{q_n^2 F_0^-[n-1]_{q_n}} + \frac{6[n]_{q_n}^4}{q_n^6 F_1^-[n-1]_{q_n}} \right) x^4 \\
& + \left(\frac{([7]_{q_n} + q_n[5]_{q_n} + q_n^2[2]_{q_n}^2)[n]_{q_n}^5}{q_n^{19} F_3^-[n-1]_{q_n}} - \frac{4([5]_{q_n} + q_n[2]_{q_n}^2)[n]_{q_n}^5}{q_n^{11} F_2^-[n-1]_{q_n}} \right. \\
& \left. - \frac{4[n]_{q_n}^2}{q_n F_0^-[n-1]_{q_n}} + \frac{6[2]_{q_n}^2[n]_{q_n}^3}{q_n^5 F_1^-[n-1]_{q_n}} \right) x^3 \\
& + \left(\frac{([5]_{q_n}[6]_{q_n} + q_n[2]_{q_n}^2[6]_{q_n} + q_n^2[2]_{q_n}^2[4]_{q_n} + q_n^4[2]_{q_n})[n]_{q_n}^4}{q_n^{17} F_3^-[n-1]_{q_n}} \right. \\
& \left. - \frac{4([2]_{q_n}^2[4]_{q_n} + q_n^2[2]_{q_n})[n]_{q_n}^3}{q_n^9 F_2^-[n-1]_{q_n}} + \frac{6[2]_{q_n}^2[n]_{q_n}^3}{q_n^5 F_1^-[n-1]_{q_n}} \right) x^2 \\
& + \left(\frac{([4]_{q_n}[2]_{q_n}^2[5]_{q_n} + q_n^2[2]_{q_n}[5]_{q_n} + q_n^3[2]_{q_n}[3]_{q_n})[n]_{q_n}^3}{q_n^{14} F_3^-[n-1]_{q_n}} \right. \\
& \left. - \frac{4[2]_{q_n}[3]_{q_n}[n]_{q_n}^2}{q_n^6 F_2^-[n-1]_{q_n}} \right) x + \frac{[2]_{q_n}[3]_{q_n}[4]_{q_n}[n]_{q_n}^2}{q_n^{10} F_3^-[n-1]_{q_n}}.
\end{aligned}
$$

Hence using (10) we get

$$st_A - \lim_{n\to\infty} [n]_{q_n}^2 B_{n,q_n}((t-x)^4;x) = (3\lambda^2 - 12\lambda + 12)x^4 + (24 - 12\lambda)x^3 + 12x^2,$$

uniformly with respect to x $\in [0, b]$. $\qquad\qquad\square$

Theorem 9. *(Voronovskaja-type theorem) Let $A = (a_{jn})$ be a nonnegative regular summability matrix, and let (q_n) be a sequence from $(0, 1)$ satisfying* (10). *Then for every $f \in C_2^\star[0, \infty)$ such that*

$$f', f'' \in C_2^\star[0, \infty),$$

we have

$$st_A - \lim_{n\to\infty} [n]_{q_n}(B_{n,q_n}(f;x) - f(x)) = ((2-\lambda)x + 1)f'(x) + ((2-\lambda)x^2 + 2x)\frac{f''(x)}{2},$$

uniformly with respect to $x \in [0, b]$, $b > 0$.

Proof. Let $f, f', f'' \in C_2^\star[0, \infty)$. For each $x \in [0, b]$, let us define a function $\psi(y) := \psi(y, x)$ by

$$\psi(y) = \begin{cases} \frac{f(y) - f(x) - (y-x)f'(x) - \frac{1}{2}(y-x)^2 f''(x)}{(y-x)^2} & \text{if } y \neq x \\ 0 & \text{if } y = x. \end{cases}$$

Then, by our hypothesis we have $\psi(x, x) = 0$ and the function $\psi(., x)$ belongs to $C_2^\star[0, \infty)$. We may write

$$f(y) = f(x) + (y-x)f'(x) + \frac{(y-x)^2}{2}f''(x) + (y-x)^2\psi(y, x).$$

Now,

$$
\begin{aligned}
[n]_{q_n}(B_{n,q_n}(f;x) - f(x)) = & \; f'(x)[n]_{q_n}B_{n,q_n}((t-x);x) + \frac{f''(x)}{2}[n]_{q_n}B_{n,q_n}((t-x)^2;x) \\
& + [n]_{q_n}B_{n,q_n}((t-x)^2\psi(y,x);x). \qquad\qquad (13)
\end{aligned}
$$

If we apply the Cauchy-Schwarz inequality for the last term on the right-hand side of the last equality, we get

$$[n]_{q_n}|B_{n,q_n}(t-x)^2\psi;x)| \le ([n]_{q_n}^2 B_{n,q_n}((t-x)^4;x))^{\frac{1}{2}} B_{n,q_n}(\psi^2;x))^{\frac{1}{2}} \tag{14}$$

Let $\eta(y,x) := \psi^2(y,x)$. Then $\eta(x,x) = 0$ and $\eta(.x) \in C_2^\star[0,\infty)$. Then, it follows from the proof of Theorem 6 that

$$st_A - \lim_{n\to\infty} B_{n,q_n}(\psi^2;x) = st_A - \lim_{n\to\infty} B_{n,q_n}(\eta(y,x);x)) = \eta(x,x) = 0, \tag{15}$$

uniformly with respect to $x \in [0,b](b > 0)$. Now, by (14), (15) and Lemma 4, we obtain that

$$st_A - \lim_{n\to\infty} [n]_{q_n} B_{n,q_n}((t-x)^2\psi;x) = 0,$$

uniformly with respect to $x \in [0,b]$. Combining the equations (13) – (15) and Lemma 4, we get

$$st_A - \lim_{n\to\infty} [n]_{q_n}(B_{n,q_n}(f;x) - f(x)) = ((2-\lambda)x + 1)f'(x) + \frac{((2-\lambda)x^2 + 2x)f''(x)}{2},$$

uniformly with respect to $x \in [0,b]$. This completes the proof. $\square$

3.3 Rate of A-statistical Approximation

Let $A = (a_{jn})$ be a nonnegative regular summability matrix and let (b_j) be a positive non-increasing sequence. Following [29], we say that a sequence $x = (x_n)$ is A-statistically convergent to the number L with the rate of $o(b_j)$ if for every $\varepsilon > 0$, $lim_j \frac{1}{b_j} \sum_{n:|x_n|\ge\varepsilon} a_{jn} < \infty$ then x is A-statistically bounded with the rate of $O(b_n)$ and it is denoted by $x_n = st_A - O(b_n)$, as $n \to \infty$.

The sequence $x = (x_n)$ is A-statistically convergent to L with the rate of $o_\mu(b_n)$, denoted by $x_n - L = st_A - o_\mu(b_n)$,(as $n \to \infty$), if for every $\epsilon > 0$,$lim_j \Sigma_{n:|x_n-L|\ge\epsilon b_n} = 0$. Finally, the sequence $x = (x_n)$ is A-statistically bounded with rate of $O\mu(b_n)$ provided that there is a positive number M such that $lim_j \Sigma_{n:|x_n|\ge Mb_n} a_{jn} = 0$. In this case we write $x_n = st_A - O_\mu(b_n)$, as $n \to \infty$.

Theorem 10. *Let $A = (a_{jn})$ be a nonegative regular summability matrix and $(b_n(x))$, be a non-increasing sequence. If the sequence of positive linear operators $B_{n,q}$ is defined by (1) and*

$$\omega(f;\delta_{n,x}) = st_A - o(b_n(x)) \text{ with } \delta_{n,x} = \sqrt{B_{n,q}((t-x)^2;x)}$$

then

$$B_{n,q}(f;x) - f(x) = st_A - o(b_n(x)),$$

where $\omega(f, \delta_n(x))$ is the usual modulus of continuity. Similar results hold when small "o" is replaced by large "O".

Proof. Since $B_{n,q}(e_0, x) = e_0(x)$ and from Cauchy-Schwarz inequality for linear positive operators we have

$$|B_{n,q}(f;x) - f(x)| \le \left[B_{n,q}(e_0, x) + \frac{1}{\delta_{n,x}}(B_{n,q}((e_1 - x^2), x)^{1/2}\right]\omega(f, \delta_{n,x}).$$

Choosing $\delta_{n,x} = \sqrt{(B_{n,q}((e_1 - x^2), x))}$, we get

$$|B_{n,q}(f, x) - f(x)| \le 2\omega(f, \delta_{n,x}).$$

This implies that

$$\frac{1}{b_n(x)} \sum_{n:|B_{n,q}(f,x)-f(x)|\ge\epsilon} a_{jn} \le \frac{1}{b_n(x)} \sum_{n:2\omega(f;\delta_{n,x})\ge\epsilon/2} a_{jn}.$$

Hence the proof follows. $\square$

Replacing "o" by "o_μ" one can get the following result.

Theorem 11. *Let $A = (a_{jn})$,$(b_n(x))$ and $\beta_{n,q}$ be the same as in Theorem 10. Assume that the operators $B_{n,q}$ satisfy the condition*

$$\omega(f; \delta_{n,x}) = st_A - o_\mu(b_n(x)) \text{ with } \delta_{n,x} = \sqrt{B_{n,q}((e_1 - x)^2; x)},$$

Then, for all functions $f \in C_2[0, \infty)$

$$B_{n,q}(f; x) - f(x) = st_A - o_\mu(b_n(x)).$$

Similar conclusions hold when small "o_μ" is replaced by large "O_μ".

3.4 Better estimates

It is well known that the classical Bernstein polynomials preserve constant as well as linear functions. To make the convergence faster, King [30] proposed an approach to modify the Bernstein polynomials, so that the sequence preserves test functions e_0 and e_2, where $e_i(t) = t^i$, $i = 0, 1, 2$. As the operator $B_{n,q}(f; x)$ defined in (1) reproduces only constant functions, this motivated us to propose the modification of this operator, so that it can preserve constant as well as linear functions.

For $f \in C_B[0, \infty)$, let us consider the following K-functional:

$$K_2(f, \delta) = \inf_{g \in W^2}\{||f - g|| + \delta||g''||\}, \tag{16}$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0, \infty) : g'' \in C_B[0, \infty)\}$. From [31], there exists an absolute constant $C > 0$ such that

$$K_2(f, \delta) \le C\omega_2(f, \sqrt{\delta}), \tag{17}$$

where

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \le \sqrt{\delta}} \sup_{x \in [0,\infty)} |f(x + 2h) - 2f(x + h) + f(x)|$$

is the second order modulus of smoothness of f. The modification of the operators given in (1) is defined as

$$\widetilde{B}_{n,q}(f; x) = \sum_{k=0}^{\infty} S_{n,k}^q(r_n^q(x)) \int_0^{\infty/A} b_{n,k}^q(t)f(t)d_qt$$

where $r_n^q(x) = \left(x - \frac{1}{q[n-1]_q}\right)\frac{q^2[n-1]_q}{[n]_q}$, for $x \in I_n = \left[\frac{1}{q[n-1]_q}, \infty\right)$ and $n > 1$.

Lemma 5. *For each $x \in I_n$, by a simple computation, we have*
1.
$$\widetilde{B}_{n,q}(1; x) = 1$$

2.
$$\widetilde{B}_{n,q}(t; x) = x$$

3.
$$\widetilde{B}_{n,q}(t^2; x) = \frac{[n-1]_q}{q^2[n-2]_q}x^2 + \frac{-2 + [2]_q^2}{q^3[n-2]_q}x + \frac{1 - [2]_q^2 + q[2]_q}{q^4[n-1]_q[n-2]_q}.$$

Consequently for each $x \in I_n$, we have the following equalities:

$$\widetilde{B}_{n,q}(t - x; x) = 0.$$

$$\widetilde{B}_{n,q}((t - x)^2; x) = \left(\frac{[n-1]_q}{q^2[n-2]_q} - 1\right)x^2 + \left(\frac{-2 + [2]_q^2}{q^3[n-2]_q}\right)x + \frac{1 - [2]_q^2 + q[2]_q}{q^4[n-1]_q[n-2]_q}$$

$$= \xi_n(x), (say). \tag{18}$$

Theorem 12. *Let $f \in C_B[0, \infty)$ and $x \in I_n$. Then, there exists a positive constant C such that*

$$|\widetilde{B}_{n,q}(f;x) - f(x)| \le C\omega_2(f, \sqrt{\xi_n(x)}),$$

where $\xi_n(x)$ is given by (18).

Proof. Let $g \in W^2$, $x \in I_n$ and $t \in [0, \infty)$. By Taylor's expansion we have

$$g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - u)g''(u)du.$$

Applying $\widetilde{B}_{n,q}$ on both sides and using Lemma 4, we get

$$\widetilde{B}_{n,q}(g;x) - g(x) = \widetilde{B}_{n,q}((t - x);x)g'(x) + \widetilde{B}_{n,q}\left(\int_x^t (t - u)g''(u)du;x\right).$$

Obviously, we have

$$\left|\int_x^t (t - u)g''(u)du\right| \le (t - x)^2 ||g''||.$$

Therefore

$$\left|\widetilde{B}_{n,q}(g;x) - g(x)\right| \le \widetilde{B}_{n,q}((t - x)^2;x)||g''|| = \xi_n(x)||g''||.$$

Since $|B_{n,q}(f;x)| \le ||f||$, we get

$$|\widetilde{B}_{n,q}(f;x) - f(x)| \le |\widetilde{B}_{n,q}(f - g;x)| + |(f - g)(x)| + |\widetilde{B}_{n,q}(g;x) - g(x)| \le 2||f - g|| + \xi_n(x)||g''||.$$

Finally, taking the infimum over all $g \in W^2$ on the right side of above inequality and using (16)-(17) we obtain

$$|\widetilde{B}_{n,q}(f;x) - f(x)| \le C\omega_2(f, \sqrt{\xi_n(x)},$$

which completes the proof of the theorem.   □

References

[1]   Lupas A., A q-analogue of the Bernstein operator, In: Seminar on Numerical and Statistical Calculus (Cluj-Napoca, 1987), Univ. 'Babeş-Bolyai', Cluj-Napoca, 1987, 9, 85-92
[2]   Phillips G. M., Bernstein polynomials based on the q-integers, The heritage of P. L. Chebyshev: a Festschrift in honour of the 70th birthday of Prof. T. J. Rivlin. Ann. Numer. Math., 1997, 4(1-4), 511-518
[3]   Aral A., A generalization of Szász-Mirakyan operators based on q-integers, Math. Comput. Model., 2008, 47(9), 1052-1062
[4]   Aral A., Gupta V., The q-derivative and applications to q-Szász Mirakyan operators, Calcolo, 2006, 43(3), 151-170
[5]   Gupta V., Mahmudov N. I., Approximation properties of the q-Szász-Mirakyan-beta Operators, Ind. J. Indus. Appl. Math., 2012, 3(2), 41-53
[6]   Gupta V., Aral A., Convergence of the q analogue of Szász-Beta operators, Appl. Math. Comput., 2010, 216(2), 374-380
[7]   Örkcü M., Doğru O., q-Szász-Mirakjan Kantorovich type operators preserving some test functions, Appl. Math. Lett., 2011, 24(9), 1588-1593
[8]   Gupta V., Srivastava G. S., Sahai A., On simultaneous approximation by Szász-beta operators, Soochow J. Math., 1995, 21(1), 1-11
[9]   Yüksel I., Dinlemez U., Voronovskaja type approximation theorem for q-Szász-beta operators, Appl. Math. Comp., 2014, 235, 555-559
[10]   Aral A., Gupta V., Agarwal R. P., Applications of q-Calculus in Operator Theory, Berlin: Springer, 2013
[11]   Gupta V., Agarwal R. P., Convergence estimates in approximation theory, Berlin: Springer, 2014

[12] Kac V., Cheung P., Quantum calculus, Springer, New York, 2002

[13] Özarslan M. A., Duman O., Local approximation behaviour of modified SMK operators, Miskolc Math. Notes, 2010, 11(1), 87-99

[14] Erençin A., Durrmeyer type modification of generalized Baskakov operators, Appl. Math. Comput., 2003, 218(8), 4384-4390

[15] Agratini O., Doğru O., Weighted approximation by q-Szász-King type operators, Taiwanese J. Math., 2010, 14(4), 1283

[16] Erençin A., Bascanbaz-Tunca G., Approximation properties of a class of linear positive operators in weighted spaces, Cr. Acad. Bulg. Sci., 2010, 63(10), 1397-1404

[17] Agrawal P. N., Karsli H., and Goyal M. , Szász-Baskakov type operators based on q-integers, J. Inequal Appl. 2014, 2014, 1-18.

[18] Lopez-Moreno A. J., Weighted simultaneous approximation with Baskakov type operators, Acta Math. Hung., 2004, 104(1-2), 143-151

[19] Lenze B., On Lipschitz type maximal functions and their smoothness spaces, Indag. Math., 1988, 50(1), 53-63

[20] Kolk E., Matrix summability of statistically convergent sequences, Analysis, 1993, 13(1-2), 77-83

[21] Ersan S., Doğru O., Statistical approximation properties of q-Bleimann, Butzer and Hahn operators, Math. Comput. Modelling, 2009, 49(7), 1595-1606

[22] Gadjiev A. D., Orhan C., Some approximation theorems via statistical convergence, J. Math., 2002, 32(1)

[23] Gupta V., Radu C., Statistical approximation properties of q-Baskakov-Kantorovich operators, Cent. Eur. J. Math., 2009, 7(4), 809-818

[24] Ispir N., Gupta V., A-Statistical approximation by the generalized Kantorovich-Bernstein type rational operators, Southeast Asian Bull. Math., 2008, 32, 87-97

[25] Örkcü M., Doğru O., Statistical approximation of a kind of Kantorovich type q-Szász-Mirakjan operators, Nonlinear Anal., 2012, 75(5), 2874-2882

[26] Örkcü M., Doğru O., Weighted statistical approximation by Kantorovich type q-Szász-Mirakjan operators, Appl. Math. Comput., 2011, 217(20), 7913-7919

[27] Ünal Z., Özarslan M. A., Duman O., Approximation properties of real and complex Post-Widder operators based on q-integers, Miskolc Math Notes, 2012, 13, 581-603

[28] Duman O., Orhan C., Statistical approximation by positive linear operators, Stud. Math., 2004, 161(2), 187-197

[29] Duman O., Khan M. K., Orhan C., A-statistical convergence of approximating operators, Math. Inequal. Appl., 2003, 6(4), 689-699

[30] King J. P., Positive linear operators which preserve x^2, Acta Math. Hung., 2003, 99(3), 203-208

[31] DeVore R.A., Lorentz G.G., Constructive Approximation, Springer, Berlin 2013