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Research Article

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Rate of Convergence for Ibragimov-Gadjiev-Durrmeyer Operators

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Abstract: The present paper deals with the rate of convergence of the general class of Durrmeyer operators, which are generalization of Ibragimov-Gadjiev operators. The special cases of the operators include some well known operators as particular cases viz. Szász-Mirakyan-Durrmeyer operators, Baskakov-Durrmeyer operators. Here we estimate the rate of convergence of Ibragimov-Gadjiev-Durrmeyer operators for functions having derivatives of bounded variation.

Keywords: Ibragimov-Gadjiev-Durrmeyer operators, rate of convergence, bounded variation.

MSC: 41A25, 41A36

1 Introduction

With the idea of general definition unifying several definitions of Durrmeyer operators, in [1] we have recently introduced a general class of Durrmeyer operators by modifying the Ibragimov-Gadjiev operators introduced in [2]. For the details of these new operators and some approximation properties see [3]. Let us recall these new Durrmeyer operators.

Let $(\varphi_n(t))_{n\in\mathbb{N}}$ and $(\psi_n(t))_{n\in\mathbb{N}}$ be the sequences of functions in $C\left(\mathbb{R}^+\right)$, which is the space of continuous function on $\mathbb{R}^+ := [0,\infty)$ such that $\varphi_n(0)=0$, $\psi_n(t)>0$, for all t and $\lim_{n\to\infty} 1/n^2\psi_n(0)=0$. Also let $(\alpha_n)_{n\in\mathbb{N}}$ denote a sequence of positive numbers satisfying the conditions

$$\lim_{n\to\infty}\frac{\alpha_n}{n}=1 \text{ and } \lim_{n\to\infty}\alpha_n\psi_n(0)=l_1,\ l_1\geq 0.$$

The Ibragimov-Gadjiev-Durrmeyer operators are defined by

$$\begin{aligned} M_n(f;x) &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!} \\ &\quad\times\int_0^{\infty}f(y)\,K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}dy, \end{aligned} \tag{1}$$

where

$$K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))=\left.\frac{\partial^{\nu}}{\partial u^{\nu}}K_n(x,t,u)\right|_{u=\alpha_n\psi_n(t),\,t=0},$$

$x,t\in\mathbb{R}^+$ and $-\infty<u<\infty$, is a sequence of functions of three variables x,t,u such that K_n is entire analytic function with respect to variable u for each $x,t\in\mathbb{R}^+$ and for each $n\in\mathbb{N}$. On the other hand, K_n must meet several conditions under which the $\{M_n\}_{n\in\mathbb{N}}$ represents a method to approximate a function f. These conditions are:

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(1) Every function of this sequence is an entire function with respect to u for fixed $x, t \in \mathbb{R}^+$ and $K_n(x,0,0) = 1$ for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$,

(2) $\left[(-1)^{\nu} \frac{\partial^{\nu}}{\partial u^{\nu}} K_n(x,t,u) \Big|_{u=u_1,t=0} \right] \geq 0$ for $\nu = 0,1,\ldots$, any fixed u_1 and $x \in \mathbb{R}^+$,
(This notation means that the derivative with respect to u is taken ν times, then one set $u = u_1$ and $t = 0$.)

(3) $\frac{\partial^{\nu}}{\partial u^{\nu}} K_n(x,t,u) \Big|_{u=u_1,t=0} = -nx \left[\frac{\partial^{\nu-1}}{\partial u^{\nu-1}} K_{m+n}(x,t,u) \Big|_{u=u_1,t=0} \right]$ for all $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$, $\nu = 0,1,\ldots$, m is a number such that $m+n = 0$ or a natural number.

(4) $K_n(0,0,u) = 1$ for any $u \in \mathbb{R}$, and

$$\lim_{x\to\infty} x^p K_n^{(\nu)}(x,0,u_1) = 0,$$

for any $p \in \mathbb{N}$ and fixed $u = u_1$.

(5) For any fixed t and u, the function $K_n(x,t,u)$ is continuously differentiable with respect to variable $x \in \mathbb{R}^+$ and satisfying the equality

$$\frac{d}{dx} K_n(x,0,u_1) = -nu_1 K_{m+n}(x,0,u_1)$$

for fixed $u = u_1$.

(6) $\frac{n+\nu m}{1+u_1 mx} \frac{\partial^{\nu}}{\partial u^{\nu}} K_n(x,t,u) \Big|_{u=u_1,t=0} = n \frac{\partial^{\nu}}{\partial u^{\nu}} K_{n+m}(x,t,u) \Big|_{u=u_1,t=0}$ for all $x \in \mathbb{R}^+$, $n \in \mathbb{N}$, $\nu = 0,1,\ldots$ and fixed $u = u_1$.

With the assumptions imposed on K_n, the operators M_n are linear and positive. Furthermore, using Taylor's expansion of $K_n(x,t,u)$ at any point $u_1 \in \mathbb{R}$ with the assumption (1), it follows that

$$\sum_{\nu=0}^{\infty} K_n^{(\nu)}(x,0,\alpha_n\psi_n(0)) \frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!} = 1.$$

Also, the following equality holds:

$$\int_0^{\infty} K_n^{(\nu)}(x,0,u_1)\, dx = (-1)^{\nu} \frac{\nu!}{(n-m)\, u_1^{\nu+1}}.$$

The operators (1) reduce to some well-known Durrmeyer operators in particular cases:

(i) if we choose $K_n(x,t,u) = [1+t+ux]^{-n}$, $\alpha_n = n$, $\psi_n(0) = 1/n$, the operators (1) reduce to Baskakov-Durrmeyer operators,

(ii) if we choose $K_n(x,t,u) = e^{-n(t+ux)}$, $\alpha_n = n$, $\psi_n(0) = 1/n$, the operators (1) reduce to Szász-Durrmeyer operators,

(iii) If $K_n(x,t,u)$ is entire analytic function with respect to u and $K_n(x,t,u) = K_n(t+ux)$, $\alpha_n = n$, $\psi_n(0) = 1/n$, the operators (1) reduce to generalized Baskakov-Durrmeyer operators given in [4].

The simultaneous approximation behaviors of (1) were presented in [3].

The rate of approximation for functions with derivatives of bounded variation is an active topic. Several researchers have studied this problem, just to mention mention the works of Zeng, Tao and Cheng (see [5–7]) who estimated the rate of convergence of modified Szász operators and Lupas Bezier operators for functions having derivatives of bounded variation. We also mention some of the papers devoted to this subject which were written for different operators as [8–12], and references therein. Also for some other results for different Durrmeyer type operators, one can refer to [13]. In this paper, we shall estimate the rate of convergence of the operators M_n for functions having derivatives of bounded variation.

2 Some Lemmas

In this section we give some useful lemmas which are necessary to prove our main results. Throughout the rest of paper, we assume $u_1 = \alpha_n\psi_n(t)$ and $t = 0$.

Lemma 1. *Let $T_{n,r}(x)$, $r \in \mathbb{N}$, be the moment of order r of M_n, that is,*

$$\begin{aligned} T_{n,r}(x) &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty} K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!} \\ &\quad\times\int_0^{\infty}(y-x)^r K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\,dy, \end{aligned}$$

then for $n > m(r+2)$, we have the following recurrence relation

$$u_1[n-m(r+2)]\,T_{n,r+1}(x) = x(1+u_1mx)\left[T'_{n,r}(x)+2rT_{n,r-1}(x)\right]+(1+2xu_1m)(r+1)\,T_{n,r}(x).$$

Also

$$\begin{aligned} T_{n,0}(x) &= 1, \\ T_{n,1}(x) &= \frac{2mx}{(n-2m)}+\frac{1}{(n-2m)\,\alpha_n\psi_n(0)}, \\ T_{n,2}(x) &= x^2\left[\frac{m(2n+6m)\,\alpha_n\psi_n(0)}{(n-2m)(n-3m)\,\alpha_n\psi_n(0)}\right]+x\left[\frac{2n+6m}{(n-2m)(n-3m)\,\alpha_n\psi_n(0)}\right]+\frac{2}{(n-2m)(n-3m)\,\alpha_n^2\psi_n^2(0)}. \end{aligned}$$

For each $x \geq 0$ and $n > 3m$ one can easily obtain from the recurrence formula that

$$T_{n,r}(x) = \mathcal{O}\left(\left(\frac{1}{n\alpha_n\psi_n(0)}\right)^{\left\lfloor\frac{r+1}{2}\right\rfloor}\right)\left(x^r+\cdots+x+1\right),$$

where $\lfloor\cdot\rfloor$ is integral part of $(r+1)/2$.

Proof. Using the identity

$$\frac{d}{dx}\,K_n(x,t,u)\Big|_{\substack{u=u_1\\t=0}} = -nu_1\,K_{m+n}(x,t,u)\Big|_{\substack{u=u_1\\t=0}}$$

we have

$$\begin{aligned} x(1+u_1mx)\,T'_{n,r}(x) &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty} x(1+u_1mx)\,\frac{d}{dx}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!} \\ &\quad\times\int_0^{\infty}(y-x)^r K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\,dy \\ &\quad -x(1+u_1mx)\,r(n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty} K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!} \\ &\quad\times\int_0^{\infty}(y-x)^{r-1} K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\,dy \\ &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}(\nu-xu_1n)\,K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!} \\ &\quad\times\int_0^{\infty}(y-x)^r K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\,dy - rx(1+u_1mx)\,T_{n,r-1}(x). \end{aligned}$$

Also using the identity given in the assumption (6), we get

$$\begin{aligned}
x(1+u_1mx)\left[T'_{n,r}(x)+rT_{n,r-1}(x)\right] &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\\
&\quad\times\left[\int_0^\infty(\nu-yu_1n)\,K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))(y-x)^r\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\right.\\
&\quad\left.+u_1n\int_0^\infty K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))(y-x)^{r+1}\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\right]\\
&= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\\
&\quad\times\left[\int_0^\infty y(1+yu_1m)\frac{d}{dy}K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))(y-x)^r\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\right.\\
&\quad\left.+u_1n\int_0^\infty K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))(y-x)^{r+1}\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\right]
\end{aligned}$$

$$\begin{aligned}
&= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\\
&\quad\times\left[\int_0^\infty u_1m\frac{d}{dy}K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))(y-x)^{r+2}\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\right.\\
&\quad+(1+2xu_1m)\int_0^\infty\frac{d}{dy}K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))(y-x)^{r+1}\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\\
&\quad+x(1+xu_1m)\int_0^\infty\frac{d}{dy}K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))(y-x)^{r}\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\\
&\quad\left.+u_1n\int_0^\infty K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))(y-x)^{r+1}\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\right]\\
&= -u_1m(r+2)\,T_{n,r+1}(x)-(1+2xu_1m)(r+1)\,T_{n,r}(x)\\
&\quad -x(1+xu_1m)\,rT_{n,r-1}(x)+u_1nT_{n,r+1}(x).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
[u_1n-u_1m(r+2)]\,T_{n,r+1}(x) &= x(1+u_1mx)\left[T'_{n,r}(x)+rT_{n,r-1}(x)\right]\\
&\quad+(1+2xu_1m)(r+1)\,T_{n,r}(x)+x(1+xu_1m)\,rT_{n,r-1}(x)\\
&= x(1+u_1mx)\left[T'_{n,r}(x)+2rT_{n,r-1}(x)\right]+(1+2xu_1m)(r+1)\,T_{n,r}(x).
\end{aligned}$$

The moments can be obtained easily by using the above recurrence relation, keeping in mind the fact that

$$\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}=1$$

and

$$\int_0^\infty K_n^{(\nu)}(x,0,u_1)\,dx=(-1)^\nu\frac{\nu!}{(n-m)\,u_1^{\nu+1}},$$

we omit the details. □

Remark 1. Let $x\in(0,\infty)$ and $C>2$; then for n sufficiently large, Lemma 1 yields that

$$\frac{2mu_1x^2+2x}{(n-3m)\,u_1}\le T_{n,2}(x)\le\frac{Cx(1+mu_1x)}{(n-3m)\,u_1}.$$

Remark 2. Let $x\in(0,\infty)$ and $C>2$. For n sufficiently large, using Cauchy-Schwarz inequality, it follows from Remark 1 that,

$$(n-m)\,\alpha_n\varphi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\int_0^\infty|y-x|\,K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy$$
$$\le\ [T_{n,2}(x)]^{1/2}\le\sqrt{\frac{Cx(1+mu_1x)}{(n-3m)\,u_1}}.$$

Lemma 2. *Let $x\in(0,\infty)$ and $C>2$, then for n sufficiently large, we have*

$$\begin{aligned}\lambda_n(x,\omega) &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\int_0^\omega K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\\ &\le \frac{Cx(1+mu_1x)}{(n-3m)(x-\omega)^2u_1},\ 0\le\omega<x.\end{aligned}$$

$$\begin{aligned}1-\lambda_n(x,z) &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\\ &\quad\times\int_z^\omega K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\\ &\le \frac{Cx(1+mu_1x)}{(n-3m)(z-x)^2u_1},\ x\le z<\infty.\end{aligned}$$

Proof. The proof of the above lemma follows easily by using Remark 1. For instance, for the first inequality for n sufficiently large and $0\le\omega<x$, we have

$$\begin{aligned}\lambda_n(x) &= (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\\ &\quad\times\int_0^\omega K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\\ &\le (n-m)\,\alpha_n\varphi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\\ &\quad\times\int_0^\omega\frac{(y-x)^2}{(\omega-x)^2}K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy\\ &\le \frac{T_{n,2}(x)}{(\omega-x)^2}\le\frac{C(x+mu_1x)}{(n-3m)(x-\omega)^2u_1}.\end{aligned}$$

The proof of the second inequality follows along the similar lines. □

3 Main Results

By $DB_q(0,\infty)$, (where q is some positive integer) we mean the class of absolutely continuous functions f defined on $(0,\infty)$ satisfying the following conditions:

(*i*) $f(t)=O(t^q),\ t\to\infty$

(*ii*) the function f has the first derivative on the interval $(0,\infty)$ which coincide a.e. with a function which is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that for all functions $f\in DB_q(0,\infty)$ we can have the representation

$$f(x)=f(c)+\int_c^x\psi(t)dt,\ x\ge c>0.$$

The main theorem of this paper is stated as:

Theorem 3. *Let $f \in DB_q(0,\infty)$, $q > 0$ and $x \in (0,\infty)$. Then for $C > 2$ and n sufficiently large, we have*

$$\begin{aligned}
|M_n(f;x) - f(x)| &\leq \frac{Cx(1+mu_1x)}{(n-3m)u_1}\left(\sum_{k=1}^{\lfloor\sqrt{n}\rfloor}\bigvee_{x-x/k}^{x+x/k}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x)\right) \\
&+ \frac{C(1+mu_1x)}{(n-3m)u_1x}\left(\left|f(2x) - f(x) - xf'\left(x^+\right)\right| + |f(x)|\right) + O\left(n^{-q}\right) \\
&+ \left|f'(x^+)\right|\sqrt{\frac{Cx(1+mu_1x)}{(n-3m)u_1}} + \frac{\left|f'\left(x^+\right) - f'\left(x^-\right)\right|}{2}\sqrt{\frac{C(x+mu_1x)}{(n-3m)u_1}} \\
&+ \frac{\left[f'\left(x^+\right) + f'\left(x^-\right)\right]}{2}\frac{(2mx\alpha_n\psi_n(0)+1)}{(n-2m)\alpha_n\psi_n(0)},
\end{aligned}$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, and f_x is defined by

$$f_x(t) = \begin{cases} f(t) - f(x^-), & 0 \leq t < x; \\ 0, & t = x; \\ f(t) - f(x^+), & x < t < \infty. \end{cases}$$

Proof. Using the mean value theorem, we can write

$$\begin{aligned}
|M_n(f;x) - f(x)| &\leq (n-m)\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!} \\
&\times\int_0^\infty |f(y) - f(x)|\,K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}dy \\
&= \int_0^\infty\left|\int_x^y (n-m)\alpha_n\varphi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\right. \\
&\times\left.\left(\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\right)^2 f'(u)\,K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,du\right|dy
\end{aligned}$$

Also, using the identity

$$f'(u) = \frac{f'\left(x^+\right) + f'\left(x^-\right)}{2} + (f')_x(u) + \frac{f'\left(x^+\right) - f'\left(x^-\right)}{2}sgn(u-x) + \left[f'(x) - \frac{f'\left(x^+\right) + f'\left(x^-\right)}{2}\right]\chi_x(u),$$

where

$$\chi_x(u) = \begin{cases} 1, & u = x; \\ 0, & u \neq x. \end{cases}$$

Obviously, we have

$$\begin{aligned}
&(n-m)\alpha_n\psi_n(0)\int_0^\infty\left(\int_x^y\left(f'(x) - \frac{f'\left(x^+\right) + f'\left(x^-\right)}{2}\right)\chi_x(u)du\right) \\
&\times\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\right)^2 K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,dy = 0.
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
|M_n(f;x)-f(x)| &\leq (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\\
&\quad\times\int_0^{\infty}|f(y)-f(x)|\,K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}dy\\
&= (n-m)\,\alpha_n\psi_n(0)\int_0^{\infty}\left|\int_x^y\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2\right.\\
&\quad\left.\times K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\left(\frac{f'(x^+)+f'(x^-)}{2}+(f')_x(u)\right)du\right|dy\\
&\quad+(n-m)\,\alpha_n\psi_n(0)\int_0^{\infty}\left|\int_x^y\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2\right.\\
&\quad\left.\times K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\left(\frac{f'(x^+)-f'(x^-)}{2}sgn(u-x)\right)du\right|dy.
\end{aligned}
$$

Also, it can be verified that

$$
\begin{aligned}
&(n-m)\,\alpha_n\psi_n(0)\int_0^{\infty}\left|\int_x^y\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2\right.\\
&\left.\times K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\left(\frac{f'(x^+)-f'(x^-)}{2}sgn(u-x)\right)du\right|dy\\
\leq\ &\frac{\left|f'(x^+)-f'(x^-)\right|}{2}[T_{n,2}(x)]^{1/2}
\end{aligned}
\tag{2}
$$

and

$$
\begin{aligned}
&(n-m)\,\alpha_n\psi_n(0)\int_0^{\infty}\left|\int_x^y\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2\right.\\
&\left.\times K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\left(\frac{f'(x^+)+f'(x^-)}{2}\right)du\right|dy\\
\leq\ &\frac{f'(x^+)+f'(x^-)}{2}T_{n,1}(x).
\end{aligned}
\tag{3}
$$

Combining (2)-(3), we have

$$
\begin{aligned}
|M_n(f;x)-f(x)| &\leq \left|(n-m)\,\alpha_n\psi_n(0)\int_x^{\infty}\left(\int_x^y(f')_x(u)du\right)\right.\\
&\quad\times\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,dy\\
&\quad+(n-m)\,\alpha_n\psi_n(0)\int_0^{x}\left(\int_x^y(f')_x(u)du\right)\\
&\quad\left.\times\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,dy\right|\\
&\quad+\frac{\left|f'(x^+)-f'(x^-)\right|}{2}[T_{n,2}(x)]^{1/2}+\frac{\left[f'(x^+)+f'(x^-)\right]}{2}T_{n,1}(x)\\
&= |A_n(f,x)+B_n(f,x)|+\frac{\left|f'(x^+)-f'(x^-)\right|}{2}[T_{n,2}(x)]^{1/2}+\frac{\left[f'(x^+)+f'(x^-)\right]}{2}T_{n,1}(x).
\end{aligned}
$$

Collecting the above relations, we obtain the estimates

$$\left|M_n(f;x)-f(x)\right| \leq \left|A_n(f,x)+B_n(f,x)+C_n(f,x)\right| + \frac{\left|f'(x^+)-f'(x^-)\right|}{2}\left[T_{n,2}(x)\right]^{1/2} + \frac{\left[f'(x^+)+f'(x^-)\right]}{2}T_{n,1}(x) \tag{4}$$

with the denotations

$$\begin{aligned} A_n(f,x) &= (n-m)\,\alpha_n\psi_n(0)\int_0^x\left(\int_x^y (f')_x(u)du\right) \\ &\quad\times\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2 K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,dy, \\ B_n(f,x) &= (n-m)\,\alpha_n\psi_n(0)\int_x^{2x}\left(\int_x^y (f')_x(u)du\right) \\ &\quad\times\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2 K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,dy, \\ C_n(f,x) &= (n-m)\,\alpha_n\psi_n(0)\int_{2x}^{\infty}\left(\int_x^y (f')_x(u)du\right) \\ &\quad\times\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2 K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,dy. \end{aligned}$$

Applying Remark 2 and Lemma 1, we have

$$\begin{aligned} \left|M_n(f;x)-f(x)\right| &\leq \left|A_n(f,x)\right|+\left|B_n(f,x)\right|+\left|C_n(f,x)\right| \\ &\quad+\frac{\left|f'(x^+)-f'(x^-)\right|}{2}\sqrt{\frac{C(x+mu_1x)}{(n-3m)\,u_1}} \\ &\quad+\frac{\left[f'(x^+)+f'(x^-)\right]}{2}\frac{(2mx\alpha_n\varphi_n(0)+1)}{(n-2m)\,\alpha_n\psi_n(0)}. \end{aligned}$$

In order to complete the proof, it is sufficient to estimate the terms $A_{n,r}(f,x)$, $B_{n,r}(f,x)$ and $C_{n,r}(f,x)$. Applying integration by parts and Lemma 2 with $\omega = x - x/\sqrt{n}$, we have

$$\begin{aligned} \left|A_n(f,x)\right| &= \left|\int_0^x\int_x^y (f')_x(u)du\,d_y\lambda_n(x,y)\right| \\ &\leq \left(\int_0^{x-x/\sqrt{n}}+\int_{x-x/\sqrt{n}}^x\right)\left|(f')_x(y)\right|\left|\lambda_n(x,y)\right|dy \\ &\leq \frac{C(x+mu_1x)}{(n-3m)\,u_1}\int_0^{x-x/\sqrt{n}}\bigvee_y^x\left((f')_x\right)\frac{1}{(x-y)^2}dy+\int_{x-x/\sqrt{n}}^x\bigvee_y^x\left((f')_x\right)dy \\ &\leq \frac{C(x+mu_1x)}{(n-3m)\,u_1}\int_0^{x-x/\sqrt{n}}(x-y)^{-2}\bigvee_y^x\left((f')_x\right)dy+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^x\left((f')_x\right). \end{aligned}$$

By the substitution of $u = \frac{x}{t-x}$, we obtain

$$\begin{aligned} \int_0^{x-x/\sqrt{n}}(x-y)^{-2}\bigvee_y^x\left((f')_x\right)dy &= x^{-1}\int_1^{\sqrt{n}}\bigvee_{x-x/u}^x\left((f')_x\right)du \\ &\leq x^{-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor}\int_k^{k+1}\bigvee_{x-x/u}^x\left((f')_x\right)du \\ &\leq x^{-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor}\bigvee_{x-x/k}^x\left((f')_x\right). \end{aligned}$$

Thus we have

$$|A_n(f,x)| \le \frac{Cx(1+mu_1x)}{(n-3m)u_1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor}\bigvee_{x-x/k}^{x}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}((f')_x). \tag{5}$$

On the other hand we have

$$\begin{aligned}
|B_n(f,x)| &= \left|-\int_x^{2x}\int_x^y (f')_x(u)du d_y(1-\lambda_n(x,y))\right| \\
&\le \left|\int_x^{2x}(f')_x(u)du\right||1-\lambda_n(x,2x)| + \int_x^{2x}|(f')_x(y)||1-\lambda_n(x,y)|dy \\
&\le \frac{C(1+mu_1x)}{(n-3m)u_1x}\left|f(2x)-f(x)-xf'(x^+)\right| + \int_x^{x+x/\sqrt{n}}\bigvee_x^y((f')_x)dy \\
&\quad + \frac{C(1+mu_1x)}{(n-3m)u_1}\int_{x+x/\sqrt{n}}^{2x}(y-x)^{-2}\bigvee_x^y((f')_x)dy.
\end{aligned}$$

By the substitution of $u=\frac{x}{t-x}$, we obtain

$$\begin{aligned}
\int_{x+x/\sqrt{n}}^{2x}(y-x)^{-2}\bigvee_x^y((f')_x)dy &= x^{-1}\int_1^{\sqrt{n}}\bigvee_{x-x/u}^{x}((f')_x)du \\
&\le x^{-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor}\int_k^{k+1}\bigvee_x^{x+x/u}((f')_x)du \\
&\le x^{-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor}\bigvee_x^{x+x/u}((f')_x).
\end{aligned}$$

Thus we have

$$|B_n(f,x)| \le \frac{C(1+mu_1x)}{(n-3m)u_1x}\left|f(2x)-f(x)-xf'(x^+)\right| + \frac{C(1+mu_1x)}{(n-3m)u_1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor}\bigvee_x^{x+x/u}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_x^{x+x/\sqrt{n}}((f')_x). \tag{6}$$

Finally, applying Hölder's inequality, we proceed as follows for the estimation of $C_n(f,x)$:

$$\begin{aligned}
|C_n(f,x)| &= \Bigg|(n-m)\alpha_n\psi_n(0)\int_{2x}^{\infty}\left(\int_x^y(f')_x(u)du\right) \\
&\quad\times\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\right)^2dy\Bigg| \\
&\le \Bigg|(n-m)\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\right)^2 \\
&\quad\times\int_{2x}^{\infty}(f(y)-f(x))K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))dy\Bigg| \\
&\quad+\left|f'(x^+)\right|\Bigg|(n-m)\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0)) \\
&\quad\times\left(\frac{[-\alpha_n\psi_n(0)]^\nu}{(\nu)!}\right)^2\int_{2x}^{\infty}(y-x)K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))dy\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq (n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2\\
&\times\int_{2x}^{\infty}K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,C_1y^{2q}dy\\
&+\frac{|f(x)|}{x^2}(n-m)\,\alpha_n\varphi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\\
&\times\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2\int_{2x}^{\infty}K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,(y-x)^2\,dy\\
&+\left|f'(x^+)\right|(n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\\
&\times\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2\int_{2x}^{\infty}|y-x|\,K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,dy.
\end{aligned}\tag{7}
$$

To estimate the integral

$$
(n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2\int_{2x}^{\infty}K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,C_1y^{2q}dy,
$$

we proceed as follows:

Obviously, $y\geq 2x$ implies that $t\leq 2(y-x)$ and it follows from Lemma 1, that

$$
\begin{aligned}
&(n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2\int_{2x}^{\infty}K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,C_1y^{2q}dy\\
\leq\;& C_12^q(n-m)\,\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2\\
&\times\int_{2x}^{\infty}K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,(y-x)^{2q}\,dy\\
=\;& C_12^qT_{n,2q}(x)=O\left(n^{-q}\right)\ \ (n\to\infty).
\end{aligned}
$$

Applying Schwarz inequality and Remark 1, third term in right hand side of (7) can be estimated as follows:

$$
\begin{aligned}
&\left|f'(x^+)\right|(n-m)\,\alpha_n\psi_n(0)\int_{2x}^{\infty}\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,|y-x|\,dy\\
\leq\;&\left|f'(x^+)\right|(n-m)\,\alpha_n\psi_n(0)\int_{0}^{\infty}\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\\
&\times\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,|y-x|\,dy\\
\leq\;&\left|f'(x^+)\right|\left((n-m)\,\alpha_n\psi_n(0)\int_{0}^{\infty}\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\right.\\
&\left.\times\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,(y-x)^2\,dy\right)^{1/2}\\
&\times\left((n-m)\,\alpha_n\psi_n(0)\int_{0}^{\infty}\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(\frac{[-\alpha_n\psi_n(0)]^{\nu}}{(\nu)!}\right)^2K_n^{(\nu)}(y,0,\alpha_n\psi_n(0))\,dy\right)^{1/2}\\
=\;&\left|f'(x^+)\right|\sqrt{\frac{Cx\,(1+mu_1x)}{(n-3m)\,u_1}}.
\end{aligned}
$$

So, we have

$$
|C_n(f,x)|\leq O\left(n^{-q}\right)+\frac{C\,(1+mu_1x)}{(n-3m)\,u_1x}+\left|f'(x^+)\right|\sqrt{\frac{Cx\,(1+mu_1x)}{(n-3m)\,u_1}}.\tag{8}
$$

Therefore combining (4), (5), (6) and (8) we get desired result. □

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