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Pointwise approximation of $2\pi/r$-periodic functions by matrix operators of their Fourier series with r-differences of the entries

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Abstract: We extend the results of Xh. Z. Krasniqi [Acta Comment. Univ. Tartu. Math., 2013, 17, 89-101] and the authors [Acta Comment. Univ. Tartu. Math., 2009, 13, 11-24] to the case of $2\pi/r$-periodic functions. Moreover, as a measure of approximation r-differences of the entries are used.

Keywords: rate of approximation, summability of Fourier series

MSC: 42A24

1 Introduction

Let $L^p_{2\pi/r}$ $(1 \le p < \infty)$ be the class of all $2\pi/r$–periodic real–valued functions, integrable in the Lebesgue sense with p–th power over $Q_r = [-\pi/r, \pi/r]$ with the norm

$$\|f(\cdot)\|_{L^p_{2\pi/r}} = \left(\int_{Q_r} | f(t) |^p \, dt \right)^{1/p},$$

where $r \in \mathbb{N}$. It is clear that $L^p_{2\pi/r} \subset L^p_{2\pi/1} = L^p_{2\pi}$ and

$$\|f(\cdot)\|_{L^p_{2\pi}} = r^{1/p} \|f(\cdot)\|_{L^p_{2\pi/r}}$$

for $f \in L^p_{2\pi/r}$.

We will consider, for $f \in L^1_{2\pi}$, the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

with the partial sums $S_k f$ and the conjugate Fourier series

$$\widetilde{S}f(x) := \sum_{\nu=1}^{\infty} (a_\nu(f) \sin \nu x - b_\nu(f) \cos \nu x)$$

with the partial sums $\widetilde{S}_k f$. We also know that if $f \in L^1_{2\pi}$, then

$$\widetilde{f}(x) := -\frac{1}{\pi} \int_0^{\pi} \psi_x(t) \frac{1}{2} \cot \frac{t}{2} \, dt,$$

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with

$$\psi_x(t) := f(x+t) - f(x-t),$$

exists for almost all x [1, Theorem (3.1)IV].

Let $A := (a_{n,k})$ be an infinite matrix of real numbers such that

$$a_{n,k} \geq 0 \text{ when } k, n = 0, 1, 2, \ldots, \lim_{n\to\infty} a_{n,k} = 0 \text{ and } \sum_{k=0}^{\infty} a_{n,k} = 1,$$

and let $A^\circ := (a^\circ_{n,k})$, where

$$a^\circ_{n,k} = a_{n,k} \text{ when } k \leq n \text{ and } a^\circ_{n,k} = 0 \text{ when } k > n.$$

We will use the notations

$$A_{n,r} = \sum_{k=0}^{\infty} \left|a_{n,k} - a_{n,k+r}\right|, \quad A^\circ_{n,r} = \sum_{k=0}^{n} \left|a^\circ_{n,k} - a^\circ_{n,k+r}\right|$$

for $r \in \mathbb{N}$ and

$$\begin{pmatrix} T_{n,A}f(x) \\ \widetilde{T}_{n,A}f(x) \end{pmatrix} := \sum_{k=0}^{\infty} a_{n,k} \begin{pmatrix} S_k f(x) \\ \widetilde{S}_k f(x) \end{pmatrix} \quad (n = 0, 1, 2, \ldots),$$

for the A–transformation of $S_k f$ or $\widetilde{S}f$, respectively.

In this paper, we will study the upper bounds of $\left|T_{n,A}f(x) - f(x)\right|$ and $\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right|$ by the functions of modulus of continuity type. These are nondecreasing continuous functions having the following properties: $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$. We will also consider functions from the following subclasses $L^p_{2\pi/r}(\omega)_\beta$, $L^p_{2\pi/r}(\widetilde{\omega})_\beta$ of $L^p_{2\pi/r}$:

$$\begin{aligned} L^p_{2\pi/r}(\omega)_\beta \quad &: \quad = \left\{ f \in L^p_{2\pi/r} : \omega_\beta f(\delta)_{L^p_{2\pi/r}} = O(\omega(\delta)) \text{ when } \delta \in [0, 2\pi] \text{ and } \beta \geq 0 \right\}, \\ L^p_{2\pi/r}(\widetilde{\omega})_\beta \quad &: \quad = \left\{ f \in L^p_{2\pi/r} : \widetilde{\omega}_\beta f(\delta)_{L^p_{2\pi/r}} = O(\widetilde{\omega}(\delta)) \text{ when } \delta \in [0, 2\pi] \text{ and } \beta \geq 0 \right\}, \end{aligned}$$

where $r \in \mathbb{N}$, ω and $\widetilde{\omega}$ are functions of modulus of continuity type. Moreover,

$$\omega_\beta f(\delta)_{L^p_{2\pi/r}} := \sup_{0\leq|t|\leq\delta} \left\{ \left|\sin\frac{rt}{2}\right|^\beta \|\varphi_\cdot(t)\|_{L^p_{2\pi/r}} \right\}$$

with $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$,

$$\widetilde{\omega}_\beta f(\delta)_{L^p_{2\pi/r}} = \sup_{0\leq|t|\leq\delta} \left\{ \left|\sin\frac{rt}{2}\right|^\beta \|\psi_\cdot(t)\|_{L^p_{2\pi/r}} \right\}.$$

It is easy to see that $\widetilde{\omega}_0 f(\cdot)_{L^p_{2\pi/r}} = \widetilde{\omega} f(\cdot)_{L^p_{2\pi/r}}$ and $\omega_0 f(\cdot)_{L^p_{2\pi/r}} = \omega f(\cdot)_{L^p_{2\pi/r}}$ are the classical moduli of continuity. It is clear that

$$\widetilde{\omega}_\beta f(\delta)_{L^p_{2\pi/r}} \leq \widetilde{\omega}_\alpha f(\delta)_{L^p_{2\pi/r}} \text{ and } \omega_\beta f(\delta)_{L^p_{2\pi/r}} \leq \omega_\alpha f(\delta)_{L^p_{2\pi/r}},$$

and consequently

$$L^p_{2\pi/r}(\widetilde{\omega})_\alpha \subseteq L^p_{2\pi/r}(\widetilde{\omega})_\beta \text{ and } L^p_{2\pi/r}(\omega)_\alpha \subseteq L^p_{2\pi/r}(\omega)_\beta$$

for $\beta \geq \alpha \geq 0$

The above deviations were estimated in the paper [2] and generalized in [3] as follows:

Theorem A. [3, Theorem 10, p. 97] *Let $f \in L^p_{2\pi}(\omega)_\beta$ with $1 < p < \infty$* and *$0 \leq \beta < 1 - \frac{1}{p}$, and let ω satisfy the conditions*

$$\left\{ \int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\gamma}|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p}\frac{t}{2} dt \right\}^{1/p} = O_x\left((n+1)^\gamma\right) \tag{1.1}$$

with $0 < \gamma < \beta + \frac{1}{p}$,

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x \left((n+1)^{-1/p} \right) \tag{1.2}$$

and

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{\omega(t)}{t \sin^\beta \frac{t}{2}} \right)^q dt \right\}^{1/q} = O \left((n+1)^{\beta+1/p} \omega \left(\frac{\pi}{n+1} \right) \right), \tag{1.3}$$

where $q = p(p-1)^{-1}$. *Then*

$$\left| T_{n,A^\circ} f(x) - f(x) \right| = O_x \left((n+1)^{\beta + \frac{1}{p} + 1} A^\circ_{n,1} \omega \left(\frac{\pi}{n+1} \right) \right).$$

Theorem B. [3, Theorem 9, p. 97] *Let* $f \in L^p_{2\pi}(\widetilde{\omega})_\beta$ *with* $1 < p < \infty$ and $0 \le \beta < 1 - \frac{1}{p}$, *where* $\widetilde{\omega}$ *satisfies the conditions*

$$\left\{ \int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\gamma} |\psi_x(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x \left((n+1)^\gamma \right) \tag{1.4}$$

with $0 < \gamma < \beta + \frac{1}{p}$,

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x \left((n+1)^{-1/p} \right) \tag{1.5}$$

and

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{\widetilde{\omega}(t)}{t \sin^\beta \frac{t}{2}} \right)^q dt \right\}^{1/q} = O \left((n+1)^{\beta+1/p} \widetilde{\omega} \left(\frac{\pi}{n+1} \right) \right), \tag{1.6}$$

where $q = p(p-1)^{-1}$. *Then*

$$\left| \widetilde{T}_{n,A^\circ} f(x) - \widetilde{f}(x) \right| = O_x \left((n+1)^{\beta + \frac{1}{p} + 1} A^\circ_{n,1} \widetilde{\omega} \left(\frac{\pi}{n+1} \right) \right).$$

In our theorems we generalize the above results by considering functions from $L^p_{2\pi/r}$ and using $A_{n,r}$ with $r \in \mathbb{N}$ instead of $A^\circ_{n,1}$.

Hereafter $\sum_{k=a}^{b} = 0$ when $a > b$.

2 Statement of the results

In this section we will present the estimates of the quantities $\left| T_{n,A} f(x) - f(x) \right|$ and $\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}(x) \right|$. Additionally, we will formulate some remarks and corollaries.

Theorem 1. *Suppose that* $f \in L^p_{2\pi/r}$, $1 < p < \infty$, $r \in \mathbb{N}$, $\beta \ge 0$ *and a function of the modulus of continuity type* ω *satisfies the conditions:*

$$\left\{ \int_0^{\frac{\pi}{r(n+1)}} \left(\frac{\omega(t)}{t \left| \sin \frac{rt}{2} \right|^\beta} \right)^q dt \right\}^{\frac{1}{q}} = O \left((n+1)^{\beta+1/p} \omega \left(\frac{\pi}{n+1} \right) \right), \tag{2.1}$$

where $q = p(p-1)^{-1}$,

$$\left\{ \int_0^{\frac{\pi}{r(n+1)}} \left(\frac{|\varphi_x(t)| \left| \sin \frac{rt}{2} \right|^\beta}{\omega(t)} \right)^p dt \right\}^{\frac{1}{p}} = O_x \left((n+1)^{-1/p} \right) \tag{2.2}$$

and

$$\left\{ \int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \left(\frac{|\varphi_x(t)| \left|\sin \frac{rt}{2}\right|^{\beta}}{\omega(t)\, t^{\gamma}} \right)^{p} dt \right\}^{\frac{1}{p}} = O_x\left((n+1)^{\gamma}\right), \tag{2.3}$$

with $0 < \gamma < \beta + \frac{1}{p}$. *If a matrix A is such that*

$$\left[\sum_{l=0}^{n} \sum_{k=l}^{r+l-1} a_{n,k} \right]^{-1} = O(1) \tag{2.4}$$

is true for $r \in \mathbb{N}$, *then*

$$\left| T_{n,A} f(x) - f(x) \right| = O_x\left((n+1)^{\beta + \frac{1}{p} + 1} A_{n,r}\, \omega\left(\frac{\pi}{n+1}\right)\right).$$

Theorem 2. *Suppose that* $f \in L^{p}_{2\pi/r}$, $1 < p < \infty$, $r \in \mathbb{N}$, $\beta \geq 0$ *and a function of the modulus of continuity type* $\widetilde{\omega}$ *satisfies the conditions:*

$$\left\{ \int_{0}^{\frac{\pi}{r(n+1)}} \left(\frac{\widetilde{\omega}(t)}{t \left|\sin \frac{rt}{2}\right|^{\beta}} \right)^{q} dt \right\}^{\frac{1}{q}} = O\left((n+1)^{\beta + 1/p}\, \widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right), \tag{2.5}$$

where $q = p(p-1)^{-1}$,

$$\left\{ \int_{0}^{\frac{\pi}{r(n+1)}} \left(\frac{|\psi_x(t)| \left|\sin \frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}(t)} \right)^{p} dt \right\}^{1/p} = O_x(1). \tag{2.6}$$

and

$$\left\{ \int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \left(\frac{|\psi_x(t)| \left|\sin \frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}(t)\, t^{\gamma}} \right)^{p} dt \right\}^{\frac{1}{p}} = O_x\left((n+1)^{\gamma}\right), \tag{2.7}$$

with $0 < \gamma < \beta + \frac{1}{p}$. *If a matrix A is such that* (2.4) *holds, then*

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}(x) \right| = O_x\left((n+1)^{\beta + \frac{1}{p} + 1} A_{n,r}\, \widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

Remark 1. *If we consider the following more natural condition*

$$\left\{ \int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \left(\frac{|\varphi_x(t)| \left|\sin \frac{rt}{2}\right|^{\beta}}{\omega(t)\, t^{\gamma}} \right)^{p} dt \right\}^{\frac{1}{p}} = O_x\left((n+1)^{\gamma - \frac{1}{p}}\right),$$

for $\gamma \in \left(\frac{1}{p}, \frac{1}{p} + \beta\right)$ *where* $\beta > 0$, *instead of (2.3) and for* $\widetilde{\omega}$ *and* ψ *analogously, then our estimates take the form*

$$\left| T_{n,A} f(x) - f(x) \right| = O_x\left((n+1)^{\beta + 1} A_{n,r}\, \omega\left(\frac{\pi}{n+1}\right)\right),$$

and

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}(x) \right| = O_x\left((n+1)^{\beta + 1} A_{n,r}\, \widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

These considerations are natural because in the case of norm approximation the new conditions, as well as the old ones, always hold for $f \in L^{p}_{2\pi/r}(\omega)_{\beta}$ *or* $f \in L^{p}_{2\pi/r}(\widetilde{\omega})_{\beta}$ *with* $\|\varphi_{\cdot}(t)\|_{L^{p}_{2\pi/r}}$ *instead of* $|\varphi_x(t)|$ *or with* $\|\psi_{\cdot}(t)\|_{L^{p}_{2\pi/r}}$ *instead of* $|\psi_x(t)|$, *respectively. Therefore, if* $f \in L^{p}_{2\pi/r}(\omega)_{\beta}$ *or* $f \in L^{p}_{2\pi/r}(\widetilde{\omega})_{\beta}$, *then*

$$\left\| T_{n,A} f(\cdot) - f(\cdot) \right\|_{L^{p}_{2\pi/r}} = O\left((n+1)^{\beta + 1} A_{n,r}\, \omega\left(\frac{\pi}{n+1}\right)\right)$$

or

$$\left\| \widetilde{T}_{n,A} f(\cdot) - \widetilde{f}(\cdot) \right\|_{L^{p}_{2\pi/r}} = O\left((n+1)^{\beta + 1} A_{n,r}\, \widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right),$$

respectively.

Remark 2. *For $r = 1$, we can observe that if we use, in the proof of Theorem 1 the estimate $\left|D^{\circ}_{k,1}(t)\right| \le k + \frac{1}{2}$ from Lemma 2, then we additionally need the condition*

$$\sum_{k=0}^{\infty} (k+1)\, a_{n,k} = O(n+1). \tag{2.8}$$

In this case we can apply the weaker condition

$$\left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{t\,|\varphi_x(t)|}{\omega(t)} \right)^p \left| \sin \frac{t}{2} \right|^{\beta p} dt \right\}^{\frac{1}{p}} = O_x\left((n+1)^{-1-1/p} \right)$$

instead of the condition (2.2).

Remark 3. *We note that our extra conditions* (2.4) *and* (2.8) *for a lower triangular infinite matrix A always hold.*

Corollary 1. *Using Remark 3 and the obvious inequality*

$$A_{n,r} \le r A_{n,1} \text{ for } r \in \mathbb{N} \tag{2.9}$$

our results improve and generalize the aforementioned Theorems A and B of Xh. Z. Krasniqi [3] without the assumption $\beta < 1 - \frac{1}{p}$.

Remark 4. *We note that instead of $L^p_{2\pi/r}(\omega)_\beta$ and $L^p_{2\pi/r}\left(\widetilde{\omega}\right)_\beta$ we can consider other subclasses of $L^p_{2\pi/r}$ generated by any function of the modulus of continuity type e.g. ω_x such that*

$$\omega_x(f,\delta) = \sup_{|t|\le\delta} |\varphi_x(t)| \le \omega_x(\delta)$$

or

$$\omega_x(f,\delta) = \frac{1}{\delta}\int_0^{\delta} |\varphi_x(t)|\, dt \le \omega_x(\delta).$$

Likewise in the conjugate case.

3 Auxiliary results

We begin this section by introducing some notation from [4] and [1, Section 5 of Chapter II]. For $r = 1, 2\ldots$ let

$$D^{\circ}_{k,r}(t) = \frac{\sin \frac{(2k+r)t}{2}}{2\sin \frac{rt}{2}},\ \widetilde{D^{\circ}}_{k,r}(t) = \frac{\cos \frac{(2k+r)t}{2}}{2\sin \frac{rt}{2}} \text{ and } \widetilde{D}_{k,r}(t) = \frac{\cos \frac{rt}{2} - \cos \frac{(2k+r)t}{2}}{2\sin \frac{rt}{2}}$$

It is clear by [1] that

$$S_k f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, D^{\circ}_{k,1}(t)\, dt,$$

$$\widetilde{S}_k f(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, \widetilde{D}_{k,1}(t)\, dt$$

and

$$T_{n,A} f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t) \sum_{k=0}^{\infty} a_{n,k} D^{\circ}_{k,1}(t)\, dt,$$

$$\widetilde{T}_{n,A}f(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\sum_{k=0}^{\infty} a_{n,k}\widetilde{D}_{k,1}(t)\,dt.$$

Hence

$$T_{n,A}f(x) - f(x) = \frac{1}{\pi}\int_{0}^{\pi} \varphi_x(t)\sum_{k=0}^{\infty} a_{n,k}D^{\circ}_{k,1}(t)\,dt$$

and

$$\widetilde{T}_{n,A}f(x) - \widetilde{f}(x) = \frac{1}{\pi}\int_{0}^{\pi} \psi_x(t)\sum_{k=0}^{\infty} a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\,dt.$$

Now, we present very useful property of functions of the modulus of continuity type.

Lemma 1. [1] *A function ω of the modulus of continuity type on the interval $[0, 2\pi]$ satisfies the following condition*

$$\delta_2^{-1}\omega(\delta_2) \le 2\delta_1^{-1}\omega(\delta_1) \text{ for } \delta_2 \ge \delta_1 > 0.$$

Next, we present the known estimates.

Lemma 2. [1] *If* $0 < |t| \le \pi$, *then*

$$\left|D^{\circ}_{k,1}(t)\right| \le \frac{\pi}{2|t|}, \quad \left|\widetilde{D^{\circ}}_{k,1}(t)\right| \le \frac{\pi}{2|t|}, \quad \left|\widetilde{D}_{k,1}(t)\right| \le \frac{\pi}{|t|}$$

and if $t \subset \mathbb{R}$, then

$$\left|D^{\circ}_{k,1}(t)\right| \le k + \frac{1}{2}$$

for $k \in N$.

Lemma 3. [4, 5] *Let $m, n, r \in \mathbb{N}$, $l \in \mathbb{Z}$ and $(a_n) \subset \mathbb{C}$. If $t \neq \frac{2l\pi}{r}$, then*

$$\begin{aligned}
\sum_{k=n}^{m} a_k \sin kt &= -\sum_{k=n}^{m}(a_k - a_{k+r})\widetilde{D^{\circ}}_{k,r}(t) + \sum_{k=m+1}^{m+r} a_k\widetilde{D^{\circ}}_{k,-r}(t) - \sum_{k=n}^{n+r-1} a_k\widetilde{D^{\circ}}_{k,-r}(t),\\
\sum_{k=n}^{m} a_k \cos kt &= \sum_{k=n}^{m}(a_k - a_{k+r})D^{\circ}_{k,r}(t) - \sum_{k=m+1}^{m+r} a_k D^{\circ}_{k,-r}(t) + \sum_{k=n}^{n+r-1} a_k D^{\circ}_{k,-r}(t)
\end{aligned}$$

for every $m \ge n$.

We additionally need two estimates as a consequence of Lemma 3.

Lemma 4. *Let $r \in \mathbb{N}$, $l \in \mathbb{Z}$ and $(a_{n,k})_{k=0}^{\infty} \subset [0, \infty)$ such that $\lim_{k\to\infty} a_{n.k} = 0$. If $t \neq \frac{2l\pi}{r}$, then*

$$\left|\sum_{k=0}^{\infty} a_{n,k}D^{\circ}_{k,1}(t)\right| \le \frac{1}{2\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}\left(A_{n,r} + \sum_{k=0}^{r-1} a_{n,k}\right) \le \frac{1}{\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}A_{n,r}.$$

Proof. By Lemma 3,

$$\begin{aligned}
\sum_{k=0}^{\infty} a_{n,k}D^{\circ}_{k,1}(t) &= \sum_{k=0}^{\infty} a_{n,k}\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\\
&= \frac{1}{2\sin\frac{t}{2}}\left(\sum_{k=0}^{\infty} a_{n,k}\sin kt\cos\frac{t}{2} + \sum_{k=0}^{\infty} a_{n,k}\cos kt\sin\frac{t}{2}\right)\\
&= \frac{\cos\frac{t}{2}}{2\sin\frac{t}{2}}\left(-\sum_{k=0}^{\infty}\left(a_{n,k} - a_{n,k+r}\right)\widetilde{D^{\circ}}_{k,r}(t) - \sum_{k=0}^{r-1} a_{n,k}\widetilde{D^{\circ}}_{k,-r}(t)\right)
\end{aligned}$$

$$+\frac{1}{2}\left(\sum_{k=0}^{\infty}\left(a_{n,k}-a_{n,k+r}\right)\ D_{k,r}^{\circ}\left(t\right)+\sum_{k=0}^{r-1}a_{n,k}D_{k,-r}^{\circ}\left(t\right)\right)$$

and our lemma is proved. □

Lemma 5. *Let $r\in\mathbb{N}$, $l\in\mathbb{Z}$ and $(a_{n,k})_{k=0}^{\infty}\subset[0,\infty)$ such that $\lim_{k\to\infty}a_{n.k}=0$. If $t\neq\frac{2l\pi}{r}$, then*

$$\left|\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}\left(t\right)\right|\leq\frac{1}{2\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}\left(A_{n,r}+\sum_{k=0}^{r-1}a_{n,k}\right)\leq\frac{1}{\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}A_{n,r}.$$

Proof. By Lemma 3,

$$\begin{aligned}\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}\left(t\right) &= \sum_{k=0}^{\infty}a_{n,k}\frac{\cos\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\\ &= \frac{1}{2\sin\frac{t}{2}}\left(\sum_{k=0}^{\infty}a_{n,k}\cos kt\cos\frac{t}{2}-\sum_{k=0}^{\infty}a_{n,k}\sin kt\sin\frac{t}{2}\right)\\ &= \frac{\cos\frac{t}{2}}{2\sin\frac{t}{2}}\left(\sum_{k=0}^{\infty}\left(a_{n,k}-a_{n,k+r}\right)D_{k,r}^{\circ}\left(t\right)+\sum_{k=0}^{r-1}a_{n,k}D_{k,-r}^{\circ}\left(t\right)\right)\\ &\quad-\frac{1}{2}\left(-\sum_{k=0}^{\infty}\left(a_{n,k}-a_{n,k+r}\right)\widetilde{D^{\circ}}_{k,r}\left(t\right)-\sum_{k=0}^{r-1}a_{n,k}\widetilde{D^{\circ}}_{k,-r}\left(t\right)\right)\end{aligned}$$

and thus our proof is complete. □

We also need some special conditions which which follow from that which was mentioned earlier.

Lemma 6. *Let $p\in(1,\infty)$. If condition (2.1) holds with $q=p\left(p-1\right)^{-1}$, $r\in\mathbb{N}-\{1\}$, $\beta\geq0$ and some function ω of the modulus of continuity type, then*

$$\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{\omega\left(t\right)}{t\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right\}^{1/q}=O\left(\left(n+1\right)^{\beta+1/p}\omega\left(\frac{\pi}{n+1}\right)\right),$$

where $m\in\left\{0,\ldots,\left[\frac{r}{2}\right]-1\right\}$.

Proof. By substituting $t=\frac{2(m+1)\pi}{r}-u$ we obtain

$$\begin{aligned}\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{\omega\left(t\right)}{t\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right\}^{1/q} &= \left\{\int_{0}^{\frac{\pi}{r(n+1)}}\left(\frac{\omega\left(\frac{2(m+1)\pi}{r}-u\right)}{\left(\frac{2(m+1)\pi}{r}-u\right)\left|\sin\frac{r}{2}\left(\frac{2(m+1)\pi}{r}-u\right)\right|^{\beta}}\right)^{q}du\right\}^{1/q}\\ &= \left\{\int_{0}^{\frac{\pi}{r(n+1)}}\left(\frac{\omega\left(\frac{2(m+1)\pi}{r}-u\right)}{\left(\frac{2(m+1)\pi}{r}-u\right)\left|\sin\frac{ru}{2}\right|^{\beta}}\right)^{q}du\right\}^{1/q}.\end{aligned}$$

Since $\frac{2(m+1)\pi}{r}-u\geq u$ by Lemma 1,

$$\left\{\int_{0}^{\frac{\pi}{r(n+1)}}\left(\frac{\omega\left(\frac{2(m+1)\pi}{r}-u\right)}{\left(\frac{2(m+1)\pi}{r}-u\right)\left|\sin\frac{ru}{2}\right|^{\beta}}\right)^{q}du\right\}^{1/q}\leq\left\{\int_{0}^{\frac{\pi}{r(n+1)}}\left(2\frac{\omega\left(u\right)}{u\left|\sin\frac{ru}{2}\right|^{\beta}}\right)^{q}du\right\}^{1/q}.$$

Hence, by (2.1) our estimate follows. □

Lemma 7. *Let $p \in (1,\infty)$. If condition (2.1) holds with $q = p(p-1)^{-1}$, $r \in \mathbb{N}$, $\beta \geq 0$ and some function ω of the modulus of continuity type, then*

$$\left\{ \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \left(\frac{\omega(t)}{t\left|\sin\frac{rt}{2}\right|^{\beta}} \right)^{q} dt \right\}^{1/q} = O\left((n+1)^{\beta+1/p}\,\omega\left(\frac{\pi}{n+1}\right)\right),$$

where $m \in \left\{0,\ldots,\left[\frac{r}{2}\right]\right\}$ and $\beta < 1 - \frac{1}{p}$.

Proof. By substituting $t = \frac{2m\pi}{r} + u$, analogously to the above proof, we obtain

$$\begin{aligned}
\left\{ \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \left(\frac{\omega(t)}{t\left|\sin\frac{rt}{2}\right|^{\beta}} \right)^{q} dt \right\}^{1/q}
&= \left\{ \int_{0}^{\frac{\pi}{r(n+1)}} \left(\frac{\omega\left(\frac{2m\pi}{r}+u\right)}{\left(\frac{2m\pi}{r}+u\right)\left|\sin\frac{r}{2}\left(\frac{2m\pi}{r}+u\right)\right|^{\beta}} \right)^{q} du \right\}^{1/q} \\
&= \left\{ \int_{0}^{\frac{\pi}{r(n+1)}} \left(\frac{\omega\left(\frac{2m\pi}{r}+u\right)}{\left(\frac{2m\pi}{r}+u\right)\left|\sin\frac{ru}{2}\right|^{\beta}} \right)^{q} du \right\}^{1/q} \\
&\leq \left\{ \int_{0}^{\frac{\pi}{r(n+1)}} \left(2\frac{\omega(u)}{u\left|\sin\frac{ru}{2}\right|^{\beta}} \right)^{q} du \right\}^{1/q} = O_x\left((n+1)^{\beta+1/p}\,\omega\left(\frac{\pi}{n+1}\right)\right)
\end{aligned}$$

and we have the desired estimate. □

Lemma 8. *Suppose that $f \in L^{p}_{2\pi/r}$, where $1 \leq p < \infty$ and $r \in \mathbb{N} - \{1\}$. If condition (2.2) holds with some function ω of the modulus of continuity type and $\beta \geq 0$, then*

$$\left\{ \int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}} \left(\frac{|\varphi_x(t)|}{\omega(t)} \right)^{p} \left|\sin\frac{rt}{2}\right|^{\beta p} dt \right\}^{\frac{1}{p}} = O_x\left((n+1)^{-1/p}\right),$$

where $m \in \left\{0,\ldots,\left[\frac{r}{2}\right]-1\right\}$.

Proof. By substituting $t = \frac{2(m+1)\pi}{r} - u$, we obtain

$$\begin{aligned}
&\left\{ \int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}} \left(\frac{|\varphi_x(t)|}{\omega(t)} \right)^{p} \left|\sin\frac{rt}{2}\right|^{\beta p} dt \right\}^{1/p} \\
&= \left\{ \int_{0}^{\frac{\pi}{r(n+1)}} \left(\frac{\left|\varphi_x\left(\frac{2(m+1)\pi}{r}-u\right)\right|}{\omega\left(\frac{2(m+1)\pi}{r}-u\right)} \left|\sin\frac{r}{2}\left(\frac{2(m+1)\pi}{r}-u\right)\right|^{\beta} \right)^{p} du \right\}^{1/p} \\
&= \left\{ \int_{0}^{\frac{\pi}{r(n+1)}} \left(\frac{|\varphi_x(u)|}{\omega\left(\frac{2(m+1)\pi}{r}-u\right)} \left|\sin\frac{ru}{2}\right|^{\beta} \right)^{p} du \right\}^{1/p}.
\end{aligned}$$

Since $\frac{2(m+1)\pi}{r} - u \geq u$, we have

$$\left\{ \int_{0}^{\frac{\pi}{r(n+1)}} \left(\frac{|\varphi_x(u)|}{\omega\left(\frac{2(m+1)\pi}{r}-u\right)} \left|\sin\frac{ru}{2}\right|^{\beta} \right)^{p} du \right\}^{1/p} \leq \left\{ \int_{0}^{\frac{\pi}{r(n+1)}} \left(\frac{|\varphi_x(u)|}{\omega(u)} \left|\sin\frac{ru}{2}\right|^{\beta} \right)^{p} du \right\}^{1/p}.$$

Hence, by (2.2) our estimate follows. □

Lemma 9. *Suppose that $f \in L^p_{2\pi/r}$, where $1 \le p < \infty$ and $r \in \mathbb{N}$. If condition (2.2) holds with some function ω of the modulus of continuity type and $\beta \ge 0$, then*

$$\left\{\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \left(\frac{|\varphi_x(t)|}{\omega(t)}\right)^p \left|\sin\frac{rt}{2}\right|^{\beta p} dt\right\}^{\frac{1}{p}} = O_x\left((n+1)^{-1/p}\right),$$

where $m \in \left\{0, \dots, \left[\frac{r}{2}\right]\right\}$.

Proof. By substituting $t = \frac{2m\pi}{r} + u$, analogously to the above proof, we obtain

$$\begin{aligned}\left\{\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \left(\frac{|\varphi_x(t)|}{\omega(t)}\right)^p \left|\sin\frac{rt}{2}\right|^{\beta p} dt\right\}^{1/p} &= \left\{\int_0^{\frac{\pi}{r(n+1)}} \left(\frac{\left|\varphi_x\left(\frac{2m\pi}{r}+u\right)\right|}{\omega\left(\frac{2m\pi}{r}+u\right)} \left|\sin\frac{r}{2}\left(\frac{2m\pi}{r}+u\right)\right|^{\beta}\right)^p du\right\}^{1/p}\\ &= \left\{\int_0^{\frac{\pi}{r(n+1)}} \left(\frac{|\varphi_x(u)|}{\omega\left(\frac{2m\pi}{r}+u\right)} \left|\sin\frac{ru}{2}\right|^{\beta}\right)^p du\right\}^{1/p}\\ &\le \left\{\int_0^{\frac{\pi}{r(n+1)}} \left(\frac{|\varphi_x(u)|}{\omega(u)} \left|\sin\frac{ru}{2}\right|^{\beta}\right)^p du\right\}^{1/p} = O_x\left((n+1)^{-1/p}\right)\end{aligned}$$

and we have the desired estimate. □

Lemma 10. *Suppose that $f \in L^p_{2\pi/r}$, where $1 \le p < \infty$ and $r \in \mathbb{N} - \{1\}$. If condition (2.3) holds with some function ω of the modulus of continuity type and $\beta, \gamma \ge 0$, then*

$$\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}} \left(\frac{|\varphi_x(t)|\left|\sin\frac{rt}{2}\right|^{\beta}}{\omega(t)\left(\frac{2(m+1)\pi}{r}-t\right)^{\gamma}}\right)^p dt\right\}^{1/p} = O_x\left((n+1)^{\gamma}\right),$$

where $m \in \left\{0, \dots, \left[\frac{r}{2}\right]-1\right\}$.

Proof. By substituting $t = \frac{2(m+1)\pi}{r} - u$, we obtain

$$\begin{aligned}&\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}} \left(\frac{|\varphi_x(t)|\left|\sin\frac{rt}{2}\right|^{\beta}}{\omega(t)\left(\frac{2(m+1)\pi}{r}-t\right)^{\gamma}}\right)^p dt\right\}^{1/p}\\ &= \left\{\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \left(\frac{\left|\varphi_x\left(\frac{2(m+1)\pi}{r}-u\right)\right|}{\omega\left(\frac{2(m+1)\pi}{r}-u\right)u^{\gamma}} \left|\sin\frac{r}{2}\left(\frac{2(m+1)\pi}{r}-u\right)\right|^{\beta}\right)^p du\right\}^{1/p}\\ &= \left\{\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \left(\frac{|\varphi_x(u)|}{\omega\left(\frac{2(m+1)\pi}{r}-u\right)u^{\gamma}} \left|\sin\frac{ru}{2}\right|^{\beta}\right)^p du\right\}^{1/p}.\end{aligned}$$

Since $\frac{2(m+1)\pi}{r} - u \ge u$, we have

$$\left\{\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \left(\frac{|\varphi_x(u)|}{\omega\left(\frac{2(m+1)\pi}{r}-u\right)u^{\gamma}} \left|\sin\frac{ru}{2}\right|^{\beta}\right)^p du\right\}^{1/p} \le \left\{\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \left(\frac{|\varphi_x(u)|}{\omega(u)u^{\gamma}} \left|\sin\frac{ru}{2}\right|^{\beta}\right)^p du\right\}^{1/p}.$$

Hence, by (2.3) our estimate follows. □

Lemma 11. *Suppose that $f \in L^p_{2\pi/r}$, where $1 \le p < \infty$ and $r \in \mathbb{N}$. If condition (2.3) holds with some function ω of the modulus of continuity type and $\beta, \gamma \ge 0$, then*

$$\left\{ \int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} \left(\frac{|\varphi_x(t)| \left|\sin\frac{rt}{2}\right|^\beta}{\omega(t)\left(t-\frac{2m\pi}{r}\right)^\gamma} \right)^p dt \right\}^{1/p} = O_x\left((n+1)^\gamma\right),$$

where $m \in \left\{0, \ldots, \left[\frac{r}{2}\right]\right\}$.

Proof. By substituting $t = \frac{2m\pi}{r} + u$, analogously to the above proof, we obtain

$$\begin{aligned}
\left\{ \int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} \left(\frac{|\varphi_x(t)| \left|\sin\frac{rt}{2}\right|^\beta}{\omega(t)\left(t-\frac{2m\pi}{r}\right)^\gamma} \right)^p dt \right\}^{1/p} &= \left\{ \int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \left(\frac{\left|\varphi_x\left(\frac{2m\pi}{r}+u\right)\right|}{\omega\left(\frac{2m\pi}{r}+u\right)u^\gamma} \left|\sin\frac{r}{2}\left(\frac{2m\pi}{r}+u\right)\right|^\beta \right)^p du \right\}^{1/p} \\
&= \left\{ \int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \left(\frac{|\varphi_x(u)|}{\omega\left(\frac{2m\pi}{r}+u\right)u^\gamma} \left|\sin\frac{ru}{2}\right|^\beta \right)^p du \right\}^{1/p} \\
&\le \left\{ \int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}} \left(\frac{|\varphi_x(u)|}{\omega(u)u^\gamma} \left|\sin\frac{ru}{2}\right|^\beta \right)^p du \right\}^{1/p} \\
&= O_x\left((n+1)^\gamma\right)
\end{aligned}$$

and we have the desired estimate. □

4 Proofs of the results

4.1 Proof of Theorem 1

It is clear that

$$\begin{aligned}
T_{n,A}f(x) - f(x) &= \frac{1}{\pi}\sum_{m=0}^{[r/2]} \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D^\circ_{k,1}(t)\,dt + \frac{1}{\pi}\sum_{m=0}^{[r/2]-1} \int_{\frac{2m\pi}{r}+\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D^\circ_{k,1}(t)\,dt \\
&= I_1(x) + I_2(x)
\end{aligned}$$

for an odd r, and

$$\begin{aligned}
T_{n,A}f(x) - f(x) &= \frac{1}{\pi}\sum_{m=0}^{[r/2]-1} \left(\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} + \int_{\frac{2m\pi}{r}+\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}} \right) \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D^\circ_{k,1}(t)\,dt \\
&= I_1'(x) + I_2(x)
\end{aligned}$$

for an even r, whence

$$\left|T_{n,A}f(x) - f(x)\right| \le |I_1(x)| + \left|I_1'(x)\right| + |I_2(x)|.$$

By Lemmas 2 and 4,

$$|I_1(x)| \le \frac{1}{\pi}\sum_{m=0}^{[r/2]} \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} |\varphi_x(t)| \left| \sum_{k=0}^{\infty} a_{n,k} D^\circ_{k,1}(t) \right| dt$$

$$= \frac{1}{\pi}\sum_{m=0}^{[r/2]}\left(\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} + \int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\right)|\varphi_x(t)|\left|\sum_{k=0}^{\infty} a_{n,k}D^{\circ}_{k,1}(t)\right| dt$$

$$\leq \frac{1}{2}\sum_{m=0}^{[r/2]}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\frac{|\varphi_x(t)|}{t}dt + \frac{1}{\pi}A_{n,r}\sum_{m=0}^{[r/2]}\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\frac{|\varphi_x(t)|}{\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}dt.$$

Consider the estimates

$$\left|\sin\frac{t}{2}\right| \geq \frac{|t|}{\pi} \text{ for } t \in [0,\pi],$$

$$\left|\sin\frac{rt}{2}\right| \geq \frac{rt}{\pi} - 2m \text{ for } t \in \left[\frac{2m\pi}{r}, \frac{2m\pi}{r}+\frac{\pi}{r}\right],$$

where $m \in \{0, \dots, [r/2]\}$ with $r \in \mathbb{N} - \{1\}$. By the Hölder inequality with $p > 1$ and $q = \frac{p}{p-1}$ we therefore obtain

$$|I_1(x)| \leq \frac{1}{2}\sum_{m=0}^{[r/2]}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\frac{|\varphi_x(t)|}{t}dt + A_{n,r}\sum_{m=0}^{[r/2]}\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\frac{|\varphi_x(t)|}{t\left(\frac{rt}{\pi}-2m\right)}dt$$

$$\leq \frac{1}{2}\sum_{m=0}^{[r/2]}\left[\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\left(\frac{|\varphi_x(t)|}{\omega(t)}\left|\sin\frac{rt}{2}\right|^{\beta}\right)^p dt\right]^{\frac{1}{p}} \cdot \left[\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\left(\frac{\omega(t)}{t\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^q dt\right]^{\frac{1}{q}}$$

$$+\frac{\pi}{r}A_{n,r}\sum_{m=0}^{[r/2]}\left[\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\left(\frac{|\varphi_x(t)|}{\omega(t)\left(t-\frac{2m\pi}{r}\right)^{\gamma}}\left|\sin\frac{rt}{2}\right|^{\beta}\right)^p dt\right]^{\frac{1}{p}}\left[\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\left(\frac{\omega(t)\left(t-\frac{2m\pi}{r}\right)^{\gamma}}{t\left(t-\frac{2m\pi}{r}\right)\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^q dt\right]^{\frac{1}{q}}.$$

Using (2.2) with Lemma 9, (2.1) with Lemma 7 and (2.3) with Lemma 11 we have

$$\left|I_1(x)\right| = O_x(1)\sum_{m=0}^{[r/2]}(n+1)^{-\frac{1}{p}}(n+1)^{\beta+1/p}\,\omega\left(\frac{\pi}{n+1}\right)$$

$$+O_x(1)A_{n,r}\sum_{m=0}^{[r/2]}(n+1)^{\gamma}\left[\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\left(\frac{\omega(t)\left(t-\frac{2m\pi}{r}\right)^{\gamma}}{t\left(t-\frac{2m\pi}{r}\right)\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^q dt\right]^{\frac{1}{q}}.$$

Since

$$\left[\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\left(\frac{\omega(t)\left(t-\frac{2m\pi}{r}\right)^{\gamma}}{t\left(t-\frac{2m\pi}{r}\right)\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^q dt\right]^{\frac{1}{q}} = \left[\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{\omega\left(t+\frac{2m\pi}{r}\right)t^{\gamma}}{t\left(t+\frac{2m\pi}{r}\right)\left|\sin\frac{rt+2m\pi}{2}\right|^{\beta}}\right)^q dt\right]^{\frac{1}{q}}$$

$$\leq \left[\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{2\omega(t)\,t^{\gamma}}{t^2\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^q dt\right]^{\frac{1}{q}}$$

$$\leq \left[2^q\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{\omega(t)}{t^{2-\gamma}\left|\frac{rt}{\pi}\right|^{\beta}}\right)^q dt\right]^{\frac{1}{q}}$$

$$\leq \frac{\omega\left(\frac{\pi}{r(n+1)}\right)}{\frac{\pi}{r(n+1)}}\left[4^q\left(\frac{\pi}{r}\right)^{\beta q}\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}t^{(\gamma-1-\beta)q}dt\right]^{\frac{1}{q}}$$

$$= O\left((n+1)\,\omega\left(\frac{\pi}{n+1}\right)\right)\left(\frac{\pi}{r(n+1)}\right)^{(\gamma-1-\beta)+\frac{1}{q}}$$

$$= O\left((n+1)^{1-\gamma+\beta+\frac{1}{p}}\,\omega\left(\frac{\pi}{n+1}\right)\right),$$

for $0 < \gamma < \beta + \frac{1}{p}$, we obtain

$$\begin{aligned}
|I_1(x)| =& O_x(1)\sum_{m=0}^{[r/2]}(n+1)^{-\frac{1}{p}}(n+1)^{\beta+\frac{1}{p}}\,\omega\left(\frac{\pi}{n+1}\right)\\
&+O_x(1)A_{n,r}\sum_{m=0}^{[r/2]}(n+1)^{\gamma}(n+1)^{1-\gamma+\beta+\frac{1}{p}}\,\omega\left(\frac{\pi}{n+1}\right)\\
=&O_x\left((n+1)^{\beta}\,\omega\left(\frac{\pi}{n+1}\right)\right)+O_x\left((n+1)^{1+\beta+\frac{1}{p}}A_{n,r}\omega\left(\frac{\pi}{n+1}\right)\right).
\end{aligned}$$

Analogously,

$$\left|I_1'(x)\right| = O_x\left((n+1)^{\beta}\,\omega\left(\frac{\pi}{n+1}\right)\right)+O_x\left((n+1)^{1+\beta+\frac{1}{p}}A_{n,r}\omega\left(\frac{\pi}{n+1}\right)\right).$$

Similarly,

$$\begin{aligned}
|I_2(x)| &\leq \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}}|\varphi_x(t)|\left|\sum_{k=0}^{\infty}a_{n,k}D_{k,1}^{\circ}(t)\right|dt\\
&= \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\left(\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}+\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\right)|\varphi_x(t)|\left|\sum_{k=0}^{\infty}a_{n,k}D_{k,1}^{\circ}(t)\right|dt\\
&\leq \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\frac{|\varphi_x(t)|}{\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}A_{n,r}dt+\frac{1}{2}\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\frac{|\varphi_x(t)|}{t}dt.
\end{aligned}$$

Consider again the estimates

$$\left|\sin\frac{t}{2}\right| \geq \frac{|t|}{\pi} \text{ for } t\in[0,\pi],$$

$$\left|\sin\frac{rt}{2}\right| \geq 2(m+1)-\frac{rt}{\pi} \text{ for } t\in\left[\frac{2(m+1)\pi}{r}-\frac{\pi}{r},\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}\right],$$

where $m\in\{0,\dots,[r/2]-1\}$ with $r\in\mathbb{N}-\{1\}$. By the Hölder inequality with $p>1$ and $q=\frac{p}{p-1}$ we get

$$\begin{aligned}
|I_2(x)| &\leq A_{n,r}\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\frac{|\varphi_x(t)|}{\frac{rt}{\pi}\left(\frac{2(m+1)\pi}{r}-t\right)}dt+\frac{1}{2}\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\frac{|\varphi_x(t)|}{t}dt\\
&\leq \frac{\pi}{r}A_{n,r}\sum_{m=0}^{[r/2]-1}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\left(\frac{|\varphi_x(t)|}{\omega(t)\left(\frac{2(m+1)\pi}{r}-t\right)^{\gamma}}\left|\sin\frac{rt}{2}\right|^{\beta}\right)^{p}dt\right]^{\frac{1}{p}}\\
&\cdot\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\left(\frac{\omega(t)\left(\frac{2(m+1)\pi}{r}-t\right)^{\gamma}}{t\left(\frac{2(m+1)\pi}{r}-t\right)\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}\\
&+\frac{1}{2}\sum_{m=0}^{[r/2]-1}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{|\varphi_x(t)|}{\omega(t)}\left|\sin\frac{rt}{2}\right|^{\beta}\right)^{p}dt\right]^{\frac{1}{p}}\cdot\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{\omega(t)}{t\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}.
\end{aligned}$$

Using (2.2) with Lemma 8 , (2.1) with Lemma 6 and (2.3) with Lemma 10 we have

$$\begin{aligned}|I_2(x)| &= O_x(1)\,A_{n,r}\sum_{m=0}^{[r/2]-1}(n+1)^{\gamma}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\left(\frac{\omega(t)\left(\frac{2(m+1)\pi}{r}-t\right)^{\gamma}}{t\left(\frac{2(m+1)\pi}{r}-t\right)\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}\\ &\quad+O_x(1)\sum_{m=0}^{[r/2]-1}(n+1)^{-1/p}(n+1)^{\beta+1/p}\,\omega\left(\frac{\pi}{n+1}\right).\end{aligned}$$

Since

$$\begin{aligned}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\left(\frac{\omega(t)\left(\frac{2(m+1)\pi}{r}-t\right)^{\gamma}}{t\left(\frac{2(m+1)\pi}{r}-t\right)\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}} &= \left[\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{\omega\left(\frac{2(m+1)\pi}{r}-t\right)t^{\gamma}}{t\left(\frac{2(m+1)\pi}{r}-t\right)\left|\sin\frac{-rt+2(m+1)\pi}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}\\ &\le \left[\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{2\omega(t)\,t^{\gamma}}{t^{2}\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}\\ &\le \left[2^{q}\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{\omega(t)\,t^{\gamma}}{t^{2}\left|\frac{rt}{\pi}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}\\ &\le \frac{\omega\left(\frac{\pi}{r(n+1)}\right)}{\frac{\pi}{r(n+1)}}\left[4^{q}\left(\frac{\pi}{r}\right)^{\beta q}\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}t^{(\gamma-1-\beta)q}dt\right]^{\frac{1}{q}}\\ &= O\left((n+1)\,\omega\left(\frac{\pi}{n+1}\right)\right)\left(\frac{\pi}{r(n+1)}\right)^{(\gamma-1-\beta)+1/q}\\ &= O\left((n+1)^{1-\gamma+\beta+\frac{1}{p}}\,\omega\left(\frac{\pi}{n+1}\right)\right),\end{aligned}$$

for $0<\gamma<\beta+\frac{1}{p}$, we obtain

$$\begin{aligned}|I_2(x)| &= O_x(1)\,A_{n,r}\sum_{m=0}^{[r/2]-1}(n+1)^{\gamma}(n+1)^{1-\gamma+\beta+\frac{1}{p}}\,\omega\left(\frac{\pi}{n+1}\right)\\ &\quad+O_x(1)\sum_{m=0}^{[r/2]-1}(n+1)^{-1/p}(n+1)^{\beta+1/p}\,\omega\left(\frac{\pi}{n+1}\right)\\ &= O_x\left((n+1)^{1+\beta+\frac{1}{p}}A_{n,r}\,\omega\left(\frac{\pi}{n+1}\right)\right)+O_x\left((n+1)^{\beta}\,\omega\left(\frac{\pi}{n+1}\right)\right).\end{aligned}$$

Finally, by applying condition (2.4) we have

$$\begin{aligned}\left[(n+1)A_{n,r}\right]^{-1} = \left[\sum_{l=0}^{n}A_{n,r}\right]^{-1} &\le \left[\sum_{l=0}^{n}\sum_{k=l}^{\infty}\left|a_{n,k}-a_{n,k+r}\right|\right]^{-1}\\ \le \left[\sum_{l=0}^{n}\left|\sum_{k=l}^{\infty}\left(a_{n,k}-a_{n,k+r}\right)\right|\right]^{-1} &= \left[\sum_{l=0}^{n}\sum_{k=l}^{r+l-1}a_{n,k}\right]^{-1} = O(1)\end{aligned}$$

and our proof is complete. □

4.2 Proof of Theorem 2

Analogously, as in the proof of Theorem 1, we consider an odd r and an even r. Then,

$$\widetilde{T}_{n,A}f(x)-\widetilde{f}(x) = \frac{1}{\pi}\sum_{m=0}^{[r/2]}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\,dt+\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\,dt$$

$$= \widetilde{I}_1(x) + \widetilde{I}_2(x)$$

for an odd r, and

$$\begin{aligned}\widetilde{T}_{n,A}f(x) - \widetilde{f}(x) &= \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} \psi_x(t)\sum_{k=0}^{\infty} a_{n,k}\widetilde{D^\circ}_{k,1}(t)\,dt + \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}} \psi_x(t)\sum_{k=0}^{\infty} a_{n,k}\widetilde{D^\circ}_{k,1}(t)\,dt \\ &= \widetilde{I}_1'(x) + \widetilde{I}_2(x)\end{aligned}$$

for an even r, whence

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right| \le \left|\widetilde{I}_1(x)\right| + \left|\widetilde{I}_1'(x)\right| + \left|\widetilde{I}_2(x)\right|.$$

Further, we can observe that the quantities $\widetilde{I}_1(x)$, $\widetilde{I}_1'(x)$, $\widetilde{I}_2(x)$ are similar to the quantities $I_1(x)$, $I_1'(x)$, $I_2(x)$ from the proof of Theorem 1 and the estimates of $\widetilde{D^\circ}_{k,1}(t)$ and $\sum_{k=0}^{\infty} a_{n,k}\widetilde{D^\circ}_{k,1}(t)$ are similar to $D^\circ_{k,1}(t)$ and $\sum_{k=0}^{\infty} a_{n,k}D^\circ_{k,1}(t)$. The only difference is that we will use (2.5), (2.6) and (2.7) instead of (2.1), (2.2)and (2.3) and Lemmas 6-11 with $\widetilde{\omega}$ and ψ instead of ω and φ, respectively. Therefore, we obtain the same estimates of these terms, immediately. Thus our proof is complete. □

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