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On the proximal point algorithm and demimetric mappings in CAT(0) spaces

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Abstract: In this paper, we introduce and study the class of demimetric mappings in CAT(0) spaces. We then propose a modified proximal point algorithm for approximating a common solution of a finite family of minimization problems and fixed point problems in CAT(0) spaces. Furthermore, we establish strong convergence of the proposed algorithm to a common solution of a finite family of minimization problems and fixed point problems for a finite family of demimetric mappings in complete CAT(0) spaces. A numerical example which illustrates the applicability of our proposed algorithm is also given. Our results improve and extend some recent results in the literature.

Keywords: demimetric mappings, minimization problem, CAT(0) spaces, fixed point problem

MSC: 47H06, 47H09, 47J05, 47J25

1 Introduction

Let D be a nonempty subset of a metric space (X, d). A point $x \in X$ is called a fixed point of a nonlinear mapping $T : D \to X$, if $x = Tx$. We denote by $F(T)$ the set of fixed points of T. The mapping T is said to be:

(i) *nonexpansive*, if for all $x, y \in D$,
$$d(Tx, Ty) \leq d(x, y),$$
(ii) *quasi-nonexpansive*, if $F(T) \neq \emptyset$ and for $y \in F(T),\ x \in D$, we have
$$d(Tx, y) \leq d(x, y),$$
(iii) *k-strictly pseudocontractive*, if there exists $k \in [0, 1)$, such that
$$d^2(Tx, Ty) \leq d^2(x, y) + k\big[d(x, Tx) + d(x, Ty)\big]^2 \text{ for all } x, y \in D,$$
(iv) *k-demicontractive*, if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$, such that
$$d^2(Tx, y) \leq d^2(x, y) + kd^2(Tx, x)\ \forall\, x \in D,\ y \in F(T),$$
(v) *generalized hybrid*, if there exist $\alpha, \beta \in \mathbb{R}$, such that
$$\alpha d^2(Tx, Ty) + (1 - \alpha)d^2(x, Ty) \leq \beta d^2(Tx, y) + (1 - \beta)d^2(x, y) \text{ for all } x, y \in D.$$

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Clearly, the class of nonexpansive mappings (with nonempty fixed points set) is contained in the class of quasi-nonexpansive mappings, while the class of demicontractive mappings contains both the classes of nonexpansive and quasi-nonexpansive mappings. Moreover, there are several examples in the literature which show that the above inclusions are proper (see for example, [1] and the references therein).

Takahashi [2] (see also [3]) recently introduced a new class of nonlinear mappings in a Hilbert space, namely the class of demimetric mappings, which is defined as follows:
Let H be a real Hilbert space and D be a nonempty, closed and convex subset of H. A mapping $T : D \to H$ is called k-demimetric, if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$, such that for any $x \in D$ and $y \in F(T)$, we have

$$\langle x - y, x - Tx \rangle \geq \frac{1-k}{2} \|x - Tx\|^2. \tag{1.1}$$

The class of k-demimetric mappings with $k \in (-\infty, 1)$ is a wide class of mappings known to cover the class of k-demicontractive mappings with $k \in [0, 1)$, generalized hybrid mappings, the metric projections and the resolvents of maximal monotone operators in Hilbert spaces (see [3–5]). We note that the class of k–demimetric and k–demicontractive mappings are both quasi-generalizations of the class of k–strictly pseudocontractive mappings.

The approximation of fixed points of the above nonlinear mappings have been studied extensively by various authors in the settings of both Hilbert and Banach spaces (see [6–12]). The study has now been extended to nonlinear spaces, precisely, CAT(0) spaces. The pioneer work in fixed point theory in CAT(0) spaces was th work of Kirk [13]. After that Dhompongsa and Panyanak [14], Khan and Abass [15], Chan *et al.* [16], among others, continued to obtain interesting results on fixed point theory in CAT(0) spaces. Recently, Berg and Nikolaev [17] introduced an inner product-like notion in CAT(0) spaces called the quasilinearization mapping, which is defined as follows:
Let a pair $(a, b) \in X \times X$, denoted by $\overrightarrow{ab}$, be called a vector. The quasilinearization map $\langle ., . \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)), \quad \text{for all } a, b, c, d \in X. \tag{1.2}$$

It is not difficult to see that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, for all $a, b, c, d, e \in X$. Furthermore, a geodesic space X is said to satisfy the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d),$$

for all $a, b, c, d \in X$. It is well known that a geodesically connected space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality [14].

Using the inner product-like notion, Liu and Chang [18] introduced the following class of demicontractive-type mappings in CAT(0) spaces:
Let X be a CAT(0) space and D be a nonempty subset of X. A mapping $T : D \to X$ is called demicontractive in the sense of [18], if $F(T) \neq \emptyset$ and there exists a constant $k \in (0, 1)$, such that

$$\langle \overrightarrow{Txy}, \overrightarrow{xy} \rangle \leq d^2(x, y) - kd^2(x, Tx), \quad \text{for all } x \in D, y \in F(T). \tag{1.3}$$

Equivalently, $T : D \to X$ is called demicontractive in the sense of [18], if $F(T) \neq \emptyset$ and there exists a constant $k \in (0, 1)$, such that

$$d^2(Tx, y) \leq d^2(x, y) + (1 - 2k)d^2(x, Tx), \quad \text{for all } x \in D, y \in F(T). \tag{1.4}$$

Let X be a CAT(0) space. A mapping $h : X \to (-\infty, \infty]$ is said to be

(i) convex if
$$h(\lambda x \oplus (1 - \lambda)y) \leq \lambda h(x) + (1 - \lambda)h(y) \ \text{ for all } x, y \in X, \ \lambda \in (0, 1),$$

(ii) proper, if $D := \{x \in X : h(x) < +\infty\}$is nonempty, where D denotes the domain of h,

(iii) lower semi-continuous at a point $x \in D$ if

$$h(x) \leq \liminf_{n\to\infty} h(x_n), \tag{1.5}$$

for each sequence $\{x_n\}$ in D, such that $\lim_{n\to\infty} x_n = x$,

(iv) lower semi-continuous on D if it is lower semi-continuous at any point in D.

The Moreau-Yosida resolvent of a proper convex and lower semi-continuous function h for any $\lambda > 0$, is defined as follows:

$$J_{\lambda h}x = arg\min_{u\in X}\left[h(u) + \frac{1}{2\lambda}d^2(u,x)\right]$$

for all $x \in X$. Jost [19] showed that the mapping $J_{\lambda h}$ is well-defined and nonexpansive for all $\lambda > 0$.

The minimization problem deals with finding minimizers of a convex functional, that is, the problem of finding a point $x \in X$, such that

$$h(x) = \min_{u\in X} h(u). \tag{1.6}$$

The set of solutions (minimizers) that satisfy (1.6) is denoted by $arg\min_{u\in X} h(u)$. We note from [19] that $F(J_{\lambda h}) = arg\min_{u\in X} h(u)$.

The Proximal Point Algorithm (PPA) is a vital tool for solving problem (1.6). PPA was first introduced for Hilbert spaces by Martinet [20] in 1970 and Rockafellar [21] in 1976. After that several authors have also used PPA to obtain convergence results in Hilbert and Banach spaces (see [22]-[28]). The PPA in CAT(0) spaces started with the work of Bačák [29] in 2013. He introduced the following PPA for solving (1.6) in a CAT(0) space:

$$x_{n+1} = arg\min_{u\in X}\left[h(u) \oplus \frac{1}{2\lambda_n}d^2(y,x_n)\right], \tag{1.7}$$

for $n \in \mathbb{N}$, where $\lambda_n > 0$, such that $\sum_{n=1}^{\infty}\lambda_n = \infty$. Bačák [29] obtained a Δ-convergence result of (1.7) to a minimizer of h. In 2015, Chlomajiak *et al.* [30] considered the following iterative algorithm for finding a minimizer of a proper convex and lower semicontinuous function and common fixed points of two nonexpansive mappings in complete CAT(0) spaces:

$$\begin{cases} z_n = arg\min_{u\in X}\left[h(u) \oplus \frac{1}{2\lambda_n}d^2(u,x_n)\right], \\ y_n = \beta_n x_n \oplus (1-\beta_n)T_1 z_n, \\ x_{n+1} = \alpha_n T_1 x_n \oplus (1-\alpha_n)T_2 y_n \ \text{ for all } n \geq 1, \end{cases} \tag{1.8}$$

where $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \geq 1$ and $\lambda_n \geq \lambda > 0$ for all $n \geq 1$. They showed that the sequence $\{x_n\}$ Δ-converges to an element of $\Gamma := arg\min_{u\in X} h(u) \cap F(T_1) \cap F(T_2)$, provided Γ is nonempty.

Very recently, Lerkchaiyaphum and Phuengrattana [31] proposed the following modified PPA in CAT(0) spaces for finding a common minimizer of a finite family of proper convex and lower semicontinuous functions, and a common fixed point of a finite family of nonexpansive mappings in a CAT(0) space. More precisely, they proved the following theorem:

Theorem 1.1. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{h_i\}_{i=1}^N$ be a finite family of proper, convex and lower semicontinuous functions of D into $(-\infty,\infty]$ and $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \cap_{i=1}^N \arg\min_{u\in D} h_i(u) \cap \cap_{i=1}^N F(T_i)$ is nonempty. For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by*

$$\begin{cases} y_n^{(i)} = arg\min_{u\in X}\left[h_i(u) \oplus \frac{1}{2\lambda_n^{(i)}}d^2(u,x_n)\right], \\ z_n = \beta_n^{(0)}x_n \oplus \beta_n^{(1)}y_n^{(1)} \oplus \beta_n^{(2)}y_n^{(2)} \oplus \cdots \oplus \beta_n^{(N)}y_n^{(N)}, \\ w_n = \gamma_n^{(0)}z_n \oplus \gamma_n^{(1)}T_1 z_n^{(1)} \oplus \gamma_n^{(2)}T_2 z_n^{(2)} \oplus \cdots \oplus \gamma_n^{(N)}T_N z_n^{(N)}, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)w_n \ \textit{for all } n \geq 1, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ *are sequences in* $[0, 1]$, *such that* $0 < a \leq \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$, $\sum_{i=0}^{N} \beta_n^{(i)} = 1$ *and* $\sum_{i=0}^{N} \gamma_n^{(i)} = 1$ *for all* $n \geq 1$, *and* $\{\lambda_n^{(i)}\}$ *is a sequence such that* $\lambda_n^{(i)} \leq \lambda^{(i)} > 0$ *for all* $n \geq 1, i = 1, 2, \cdots, N$. *Then,* $\{x_n\}$ *Δ–converges to an element of* $\mathcal{F}$.

Inspired by the works of Takahashi [3], Lerkchaiyaphum and Phuengrattana [31], we introduce the class of *k-demimetric* mappings in the framework of CAT(0) spaces and prove a strong convergence theorem for a common solution of a finite family of minimization problems and fixed point problems involving this class of mappings in complete CAT(0) spaces. Our results improve and extend the work of Takahashi [3], Chlomajiak *et al.* [30], Lerkchaiyaphum and Phuengrattana [31].

2 Preliminaries

Let (X, d) be a metric space, $x, y \in X$ and $I = [0, d(x, y)]$. A curve c (or simply a geodesic path) joining x to y is an isometry $c : I \to X$, such that $c(0) = x$, $c(d(x, y)) = y$ and $d(c(t), c(t') = |t - t'|$ for all $t, t' \in I$. The image of a geodesic path is called the geodesic segment, which is denoted by $[x, y]$ whenever it is unique. We say a metric space X is a geodesic space if for every pair of points $x, y \in X$, there is a minimal geodesic from x to y. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three vertices (points in X) with unparameterized geodesic segments between each pair of vertices. For any geodesic triangle there is comparison (Alexandrov) triangle $\bar{\Delta} \subset \mathbb{R}^2$, such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$, for $i, j \in \{1, 2, 3\}$.

A geodesic space X is a CAT(0) space if the distance between an arbitrary pair of points on a geodesic triangle Δ does not exceed the distance between its corresponding pair of points on its comparison triangle $\bar{\Delta}$. If Δ and $\bar{\Delta}$ are geodesic and comparison triangles in X respectively, then Δ is said to satisfy the CAT(0) inequality for all points x, y of Δ and $\bar{x}, \bar{y}$ of $\bar{\Delta}$ if

$$d(x, y) = d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{2.1}$$

Let x, y, z be points in X and y_0 be the midpoint of the segment $[y, z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2} d^2(x, y) + \frac{1}{2} d^2(x, z) - \frac{1}{4} d(y, z). \tag{2.2}$$

For more properties of CAT(0) spaces, see [32–34] and the references therein.

Let $\{x_n\}$ be a bounded sequence in X and $r(., \{x_n\}) : X \to [0, \infty)$ be a continuous mapping defined by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) := \inf\{r(x, \{x_n\}) : x \in X\}$ while the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$. It is known that in a Hadamard space X, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ (see [35, 36]).

Definition 2.1. Let D be a nonempty closed and convex subset of a complete CAT(0) space X. A mapping $T : D \to D$ is said to be Δ-demiclosed, if for any bounded sequence $\{x_n\}$ in X, such that $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x = Tx$.

Definition 2.2. Let D be a nonempty closed and convex subset of a CAT(0) space X. The metric projection is a mapping $P_D : X \to D$ which assigns to each $x \in X$, the unique point $P_D x$ in D, such that

$$d(x, P_D x) = \inf\{d(x, y) : y \in D\}.$$

Recall that a mapping T is *firmly nonexpansive* (see [37]), if

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \text{ for all } x, y \in X. \tag{2.3}$$

It follows from the Cauchy-Schwartz inequality that firmly nonexpansive mappings are nonexpansive. Metric projection mapping is an example of a firmly nonexpansive mapping (see [37, Corollary 3.8]). The notion of firmly nonexpansive mappings was first introduced in nonlinear settings in [38]. We also remark here that (2.3) corresponds to property (P_2) (Definition 2.7) of [39].

We give some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and Δ-convergence by " $\to$ " and " $\rightharpoonup$ ", respectively.

Lemma 2.3. *[14] Let X be a CAT(0) space, then for each $x, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x, y]$, such that*

$$d(z,x) = (1-t)d(x,y) \text{ and } d(z,y) = td(x,y). \tag{2.4}$$

In this case, we write $z = tx \oplus (1-t)y$.

Lemma 2.4. *[19] Let (X, d) be a complete CAT(0) space and $h : X \to (-\infty, \infty]$ be proper, convex and lower semi-continuous. Then the following identity holds:*

$$J_{\lambda h}x = J_{\mu h}\left(\frac{\lambda-\mu}{\lambda}J_{\lambda h}x \oplus \frac{\mu}{\lambda}x\right),$$

for all $x \in X$ and $\lambda \geq \mu > 0$.

Lemma 2.5. *[14, 40] Let X be a CAT(0) space. Then for all $x, y, z \in X$ and all $t \in [0,1]$, we have*

1. $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$,
2. $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$,
3. $d^2(z, tx \oplus (1-t)y) \leq t^2d^2(z,x) + (1-t)^2d^2(z,y) + 2t(1-t)\langle\overrightarrow{zx}, \overrightarrow{zy}\rangle$.

Lemma 2.6. *[41] Let X be a complete CAT(0) space. For any $t \in [0,1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$, we have*

$$\langle\overrightarrow{u_tx}, \overrightarrow{uy}\rangle \quad \leq t\langle\overrightarrow{ux}, \overrightarrow{uy}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{uy}\rangle.$$

Lemma 2.7. *[42] Let X be a CAT(0) space and $z \in X$. Let $x_1, \cdots, x_N \in X$ and $\gamma_1, \cdots, \gamma_N$ be real numbers in $[0,1]$, such that $\sum_{i=1}^N \gamma_i = 1$. Then the following inequality holds:*

$$\sum_{i=1}^{N} \oplus\gamma_i d^2(x_i, z) \leq \sum_{i=1}^{N}\gamma_i d^2(x_i,z) - \sum_{i,j=1, i\neq j}^{N}\gamma_i\gamma_j d^2(x_i,x_j).$$

Lemma 2.8. *[43] Every bounded sequence in a complete CAT(0) space has a Δ-convergent subsequence.*

Lemma 2.9. *[44] Let X be a complete CAT(0) space, $\{x_n\}$ be a bounded sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty}\langle\overrightarrow{x_nx}, \overrightarrow{yx}\rangle \leq 0$ for all $y \in X$.*

Lemma 2.10. *[45] Let X be a complete CAT(0) space and $T : X \to X$ be a nonexpansive mapping. Then T is Δ-demiclosed.*

Lemma 2.11. *[46] Let X be a complete CAT(0) space and $h : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous mapping. Then, for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda}d^2(J_{\lambda h}x, y) - \frac{1}{2\lambda}d^2(x,y) + \frac{1}{2\lambda}d^2(x, J_{\lambda h}x) + h(J_{\lambda h}x) \leq h(y). \tag{2.5}$$

Lemma 2.12. *[47] Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$ *and* $\{\gamma_n\}$ *satisfy the following conditions:*
(i) $\{\alpha_n\} \subset [0,1]$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$,
(iii) $\gamma_n \geq 0 (n \geq 0)$, $\Sigma_{n=0}^{\infty}\gamma_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.13. *[48] Let* $\{a_n\}$ *be a sequence of real numbers, such that there exists a subsequence* $\{n_j\}$ *of* $\{n\}$ *with* $a_{n_j} < a_{n_j+1}$ *for all* $j \in \mathbb{N}$. *Then there exists a nondecreasing sequence* $\{m_k\} \subset \mathbb{N}$, *such that* $m_k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.

3 Main results

We first give the definition of a k-demimetric mapping in a CAT(0) space. We begin with the following facts which led to our definition.

If T is a k-demicontractive mapping with $k \in [0, 1)$, then

$$d^2(Tx, y) \leq d^2(x, y) + kd^2(x, Tx) \text{ for all } x \in X, y \in F(T). \tag{3.1}$$

Also, by definition of quasilinearization mapping (see (1.2)), we obtain that

$$2\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle = d^2(x, y) + d^2(Tx, x) - d^2(Tx, y).$$

That is,

$$d^2(Tx, y) = d^2(x, y) + d^2(Tx, x) - 2\langle \overrightarrow{xTx}, \overrightarrow{xy} \rangle,$$

which implies from (3.1) that

$$\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle \geq \frac{1-k}{2} d^2(x, Tx). \tag{3.2}$$

Motivated by (3.2) above, we define the demimetric mapping in a CAT(0) space as follows:

Definition 3.1. Let X be a CAT(0) space and D be a nonempty closed and convex subset of X. A mapping $T : D \to X$ is said to be *k-demimetric* if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$, such that

$$\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle \geq \frac{1-k}{2} d^2(x, Tx) \text{ for all } x \in X, y \in F(T). \tag{3.3}$$

Clearly, the class of k-demimetric mappings with $k \in (-\infty, 1)$ contains the class of k-demicontractive mappings with $k \in [0, 1)$.

Remark 3.2. If T is a generalized hybrid mapping with $F(T) \neq \emptyset$, then for $x \in D$ and $y \in F(T)$ we obtain that

$$\alpha d^2(Tx, y) + (1-\alpha)d^2(x, y) \leq \beta d^2(Tx, y) + (1-\beta)d^2(x, y),$$

which implies that

$$d^2(Tx, y) \leq d^2(x, y). \tag{3.4}$$

Now, from (3.4) and the definition of quasilinearization, we obtain that

$$2\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle = d^2(x, Tx) + d^2(x, y) - d^2(y, Tx) \geq d^2(x, Tx) + d^2(x, y) - d^2(x, y),$$

which implies that

$$\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle \geq \frac{1-0}{2} d^2(x, Tx). \tag{3.5}$$

Also, T is firmly nonexpansive if

$$d^2(Tx, Ty) \leq \langle \overrightarrow{xy}, \overrightarrow{TxTy} \rangle \text{ for all } x, y \in D.$$

If $F(T) \neq \emptyset$, then for all $x \in D$ and $y \in F(T)$, we have that

$$d^2(Tx, y) \leq \langle \overrightarrow{xy}, \overrightarrow{Txy} \rangle.$$

Therefore, the following implications hold:

$$\begin{aligned}
&\langle \overrightarrow{Txy}, \overrightarrow{Txy} \rangle \leq \langle \overrightarrow{Txy}, \overrightarrow{xy} \rangle \\
&\Rightarrow \langle \overrightarrow{yTx}, \overrightarrow{yTx} \rangle + \langle \overrightarrow{yTx}, \overrightarrow{xy} \rangle \leq 0 \\
&\Rightarrow \langle \overrightarrow{yTx}, \overrightarrow{yx} \rangle + \langle yTx, \overrightarrow{xTx} \rangle + \langle \overrightarrow{yTx}, \overrightarrow{xy} \rangle \leq 0 \\
&\Rightarrow \langle \overrightarrow{Txy}, \overrightarrow{xy} \rangle + \langle \overrightarrow{yTx}, \overrightarrow{xy} \rangle + \langle \overrightarrow{yx}, \overrightarrow{xTx} \rangle + \langle \overrightarrow{xTx}, \overrightarrow{xTx} \rangle \leq 0 \\
&\Rightarrow \langle \overrightarrow{TxTx}, \overrightarrow{xy} \rangle + d^2(x, Tx) \leq \langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle, \\
&\Rightarrow \langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle + \langle \overrightarrow{xy}, \overrightarrow{TxTx} \rangle \geq d^2(x, Tx),
\end{aligned}$$

which implies that

$$\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle \geq \frac{1-(-1)}{2} d^2(x, Tx). \tag{3.6}$$

Thus, (3.6) and (3.5) show that generalized hybrid mappings with nonempty fixed point sets and firmly nonexpansive mappings with nonempty fixed point sets are 0 and -1 demimetric mappings respectively. Since metric projection mappings are an example of firmly nonexpansive mappings, then they are demimetric mappings.

Example 3.3. Let $T : [0, 1] \to [0, 1]$ be defined by $Tx = x - x^j,\ j \geq 1$. Then T is k-demimetric with $k = -1$.

Proof. Clearly, $F(T) = \{0\}$. Now, for all $x \in [0, 1]$ and $j \geq 1$, we obtain that

$$\begin{aligned}
\langle x - 0, x - Tx \rangle &= \langle x, x^j \rangle \\
&= \frac{1}{2}\left[|x|^2 + |x^j|^2 - |x - x^j|^2\right] \\
&= \frac{1}{2}\left[|x|^2 + |x^j|^2 - |x|^2 + 2|x||x^j|^2 - |x^j|^2\right] \\
&\geq \frac{1}{2}\left[2|x^j||x^j|\right] = |x^j|^2.
\end{aligned}$$

That is,

$$\langle x - 0, x - Tx \rangle \quad \geq \quad \frac{1-(-1)}{2}|x^j|^2.$$

Hence, we have that $\langle x - 0, x - Tx \rangle \geq \frac{1-(-1)}{2}|x - Tx|^2$. □

We now study some fixed point properties of k-demimetric mappings in CAT(0) spaces.

Proposition 3.4. *Let X be a complete CAT(0) space and $T : X \to X$ be a k-demimetric mapping with $k \in (-\infty, 1)$, such that $F(T)$ is nonempty. Then $F(T)$ is closed and convex.*

Proof. We first show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$, such that $\{x_n\}$ converges to x^*. Then from Definition 3.5, we have that

$$\langle \overrightarrow{x^* x_n}, \overrightarrow{x^* Tx^*} \rangle \geq \frac{1-k}{2} d^2(x^*, Tx^*),$$

which implies by the Cauchy-Schwarz inequality that

$$d(x^*, x_n) d(x^*, Tx^*) \geq \frac{1-k}{2} d^2(x^*, Tx^*). \tag{3.7}$$

Taking limits on both sides of (3.7), we obtain that $\frac{1-k}{2} d^2(x^*, Tx^*) \leq 0$. By the condition on k, we obtain that $d(x^*, Tx^*) = 0$. Thus, $x^* \in F(T)$. Therefore, $F(T)$ is closed.

Next, we show that $F(T)$ is convex. For this, let $x, y \in F(T)$. Then it suffices to show that $(tx \oplus (1-t)y) \in F(T)$, for $t \in [0, 1]$. Set $z = tx \oplus (1-t)y$, $t \in [0, 1]$. Then by Definition 3.1, we obtain from Lemma 2.6 that

$$\begin{aligned} d^2(z, Tz) &= \langle \overrightarrow{zTz}, \overrightarrow{zTz} \rangle \\ &= \langle \overrightarrow{(tx \oplus (1-t)y)Tz}, \overrightarrow{zTz} \rangle \\ &\leq t \langle \overrightarrow{xTz}, \overrightarrow{zTz} \rangle + (1-t) \langle \overrightarrow{yTz}, \overrightarrow{zTz} \rangle \\ &= t \left[\langle \overrightarrow{xz}, \overrightarrow{zTz} \rangle + \langle \overrightarrow{zTz}, \overrightarrow{zTz} \rangle \right] + (1-t) \left[\langle \overrightarrow{yz}, \overrightarrow{zTz} \rangle + \langle \overrightarrow{zTz}, \overrightarrow{zTz} \rangle \right] \\ &\leq \frac{t(k-1)}{2} d^2(z, Tz) + t d^2(z, Tz) + \frac{(1-t)(k-1)}{2} d^2(z, Tz) + (1-t) d^2(z, Tz) \\ &= \frac{k-1}{2} d^2(z, Tz) + d^2(z, Tz), \end{aligned}$$

which implies that $\frac{k-1}{2} d^2(z, Tz) \geq 0$. By the condition on k, we obtain that $d^2(z, Tz) \leq 0$. Hence, $z = Tz$ and this yields the desired conclusion. □

The following Lemma is a cardinal property of all kinds of mappings derived from strictly pseudocontractions. The Lemma first appeared in the setting of Hilbert spaces [[49], Theorem 2]. We state the lemma for k–demimetric mappings in a CAT(0) space setting and give the proof for completeness.

Lemma 3.5. *Let X be a CAT(0) space and $T : X \to X$ be a k-demimetric mapping with $k \in (-\infty, \lambda]$ and $\lambda \in (0, 1)$, such that $F(T)$ is nonempty. Suppose that $T_\lambda x = \lambda x \oplus (1-\lambda)Tx$. Then T_λ is quasi-nonexpansive and $F(T_\lambda) = F(T)$.*

Proof. Let $x \in X$ and $z \in F(T)$. Then, from Definition 3.1 and Lemma 2.6 we obtain that

$$\begin{aligned} \langle \overrightarrow{zx}, \overrightarrow{xT_\lambda x} \rangle &= \langle \overrightarrow{zx}, \overrightarrow{x(\lambda x \oplus (1-\lambda)Tx)} \rangle \\ &= \langle \overrightarrow{(\lambda x \oplus (1-\lambda)Tx)x}, \overrightarrow{xz} \rangle \\ &\leq \lambda \langle \overrightarrow{xx}, \overrightarrow{xz} \rangle + (1-\lambda) \langle \overrightarrow{Txx}, \overrightarrow{xz} \rangle \\ &= (1-\lambda) \langle \overrightarrow{xz}, \overrightarrow{Txx} \rangle \\ &\leq \frac{(1-\lambda)^2(k-1)}{2(1-\lambda)} d^2(x, Tx). \end{aligned} \tag{3.8}$$

Now, from Lemma 2.3, we obtain that $d^2(x, T_\lambda x) = (1-\lambda)^2 d^2(x, Tx)$. Substituting this in (3.8), we obtain

$$\langle \overrightarrow{zx}, \overrightarrow{xT_\lambda x} \rangle \leq \frac{k-1}{2(1-\lambda)} d^2(x, T_\lambda x),$$

which implies that

$$\begin{aligned} \langle \overrightarrow{xz}, \overrightarrow{xT_\lambda x} \rangle &\geq \frac{1-k}{2(1-\lambda)} d^2(x, T_\lambda x) \\ &\geq \frac{1}{2} d^2(x, T_\lambda x). \end{aligned}$$

Thus, by (1.2), we obtain that

$$d^2(x, T_\lambda x) + d^2(z, x) - d^2(z, T_\lambda x) \ge d^2(x, T_\lambda x).$$

That is,

$$d^2(z, T_\lambda x) \le d^2(z, x).$$

Hence, T_λ is quasi-nonexpansive.

We next show that $F(T_\lambda) = F(T)$. Let $x \in F(T_\lambda)$, then $x = T_\lambda x$. So,

$$\begin{aligned} d(x, Tx) &= d(\lambda x \oplus (1-\lambda)Tx, Tx) \\ &\le \lambda d(x, Tx), \end{aligned}$$

which implies that $(1-\lambda)d(x, Tx) \le 0$. Since $\lambda < 1$, we obtain that $d(x, Tx) \le 0$. Therefore, $x \in F(T)$, and thus $F(T_\lambda) \subseteq F(T)$.

Conversely, let $x \in F(T)$, then $x = Tx$. By Lemma 2.5, we obtain

$$\begin{aligned} d(x, T_\lambda x) &= d(Tx, \lambda x \oplus (1-\lambda)Tx) \\ &\le \lambda d(Tx, x) + (1-\lambda)d(Tx, Tx) \\ &= 0, \end{aligned}$$

which implies that $d(x, T_\lambda x) = 0$. Thus, $x \in F(T_\lambda)$ and therefore $F(T) \subseteq F(T_\lambda)$. Hence, we obtain the desired result. □

Theorem 3.6. *Let D be a nonempty closed and convex subset of a complete CAT(0) space X, and $h_i : X \to (-\infty, \infty]$, $i = 1, \cdots, N$ be a finite family of proper, convex and lower semi-continuous functions. Let $T_i : D \to D$, $i = 1, \cdots, N$ be a finite family of k_i-demimetric mappings with $k_i \in (-\infty, \lambda]$ and $\lambda \in (0, 1)$. Suppose that $\Gamma = (\cap_{i=1}^N argmin_{u\in X} h_i(u)) \cap (\cap_{i=1}^N F(T_i))$ is nonempty and $\{x_n\}$ is a sequence generated for arbitrary $x_1, u \in X$ by*

$$\begin{cases} v_n = (1-t_n)x_n \oplus t_n u, \\ y_n = J_{r_n h_1} \circ J_{r_n h_2} \circ \cdots \circ J_{r_n h_N} v_n, \\ z_n = P_D\big(\beta_n^{(0)} v_n \oplus \beta_n^{(1)} y_n \oplus \cdots \oplus \beta_n^{(N)} y_n\big), \\ w_n = \gamma_n^{(0)} z_n \oplus \gamma_n^{(1)} T_{1\lambda} z_n \oplus \gamma_n^{(1)} T_{2\lambda} z_n \cdots \oplus \gamma_n^{(N)} T_{N\lambda} z_n, \\ x_{n+1} = \alpha_n v_n \oplus (1-\alpha_n) w_n \ \ \text{for all } n \ge 1, \end{cases} \tag{3.9}$$

where $T_{i\lambda} x = \lambda x \oplus (1-\lambda)T_i x$, such that $T_{i\lambda}$ are Δ-demiclosed for each $i = 1, 2, \ldots, N$. Suppose that $\{t_n\}$, $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ are sequences in [0,1], such that the following conditions are satisfied:

$C1:$ $0 < a \le \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \le b < 1$, $\sum_{i=0}^N \beta_n^{(i)} = 1$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \ge 1$,
$C2:$ $\lim\limits_{n\to\infty} t_n = 0$, $\sum_{n=1}^\infty t_n = \infty$,
$C3:$ $\{r_n\}$ is a sequence of real numbers, such that $r_n \ge r > 0$ for all $n \ge 1$.

Then, the sequence $\{x_n\}$ converges strongly to a point in Γ.

Proof. Let $p \in \Gamma$, from Lemma 3.5, we obtain that $p = T_{i\lambda} p$. Also, we have that $p = J_{r_n h_i} p$, $i = 1, 2, \cdots, N$. Thus, we obtain from (3.9), Lemma 2.7 and Lemma 3.5 that

$$\begin{aligned} d(w_n, p) &= d(\gamma_n^{(0)} z_n \oplus \gamma_n^{(1)} T_{1\lambda} z_n \oplus \cdots \oplus \gamma_n^{(N)} T_{N\lambda} z_n, p) \\ &\le \gamma_n^{(0)} d(z_n, p) + \sum_{i=1}^N \gamma_n^{(i)} d(T_{i\lambda} z_n, p) \\ &\le \gamma_n^{(0)} d(z_n, p) + \sum_{i=1}^N \gamma_n^{(i)} d(z_n, p) \\ &= d(z_n, p). \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\begin{aligned} d(z_n, p) &\le d(\beta_n^{(0)} v_n \oplus \beta_n^{(1)} y_n \oplus \cdots \oplus \beta_n^{(N)} y_n, p) \\ &\le \beta_n^{(0)} d(v_n, p) + \sum_{i=1}^{N} \beta_n^{(i)} d(y_n, p) \\ &\le \beta_n^{(0)} d(v_n, p) + \sum_{i=1}^{N} \beta_n^{(i)} d(J_{r_n h_1} \circ J_{r_n h_2} \cdots \circ J_{r_n h_N} v_n, p) \\ &\le \beta_n^{(0)} d(v_n, p) + \sum_{i=1}^{N} \beta_n^{(i)} d(v_n, p) \\ &= d(v_n, p). \end{aligned} \tag{3.11}$$

From (3.9), (3.10) and (3.11), we have that

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n v_n \oplus (1-\alpha_n) w_n, p) \\ &\le \alpha_n d(v_n, p) + (1-\alpha_n) d(w_n, p) \\ &\le \alpha_n d(v_n, p) + (1-\alpha_n) d(z_n, p) \\ &\le \alpha_n d(v_n, p) + (1-\alpha_n) d(v_n, p) \\ &= d(v_n, p) \qquad (3.12) \\ &= d((1-t_n) x_n \oplus t_n, p) \\ &\le (1-t_n) d(x_n, p) + t_n d(u, p) \\ &\le \max\{d(x_n, p), d(u, p)\}, \end{aligned}$$

which implies by induction that

$$d(x_{n+1}, p) \le \max\{d(x_1, p), d(u, p)\}, \quad \text{for all } n \ge 1.$$

Hence $d(x_n, p)$ is bounded, and so are $\{v_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{y_n\}$.

Now from (3.9), (3.10), (3.11), Lemma 2.5 and Lemma 2.7, we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n v_n \oplus (1-\alpha_n) w_n, p) \\ &\le \alpha_n d^2(v_n, p) + (1-\alpha_n) d^2(w_n, p) - \alpha(1-\alpha_n) d^2(v_n, w_n) \\ &\le \alpha_n d^2(v_n, p) + (1-\alpha_n)[\gamma_n^{(0)} d^2(z_n, p) + \sum_{i=1}^{N} \gamma_n^{(i)} d^2(T_{i\lambda} z_n, p) - \sum_{i=1}^{N} \gamma_n^{(0)} \gamma_n^{(i)} d^2(z_n, T_{i\lambda} z_n) \\ &\quad - \sum_{i=1, i\neq j}^{N} \gamma_n^{(i)} \gamma_n^{(j)} d^2(T_{i\lambda} z_n, T_{j_\lambda} z_n)] - \alpha_n(1-\alpha_n) d^2(v_n, w_n) \\ &\le \alpha_n d^2(v_n, p) + (1-\alpha_n)[\gamma_n^{(0)} d^2(z_n, p) + \sum_{i=1}^{N} \gamma_n^{(i)} d^2(z_n, p) - \sum_{i=1}^{N} \gamma_n^{(0)} \gamma_n^{(i)} d^2(z_n, T_{i\lambda} z_n) \\ &\quad - \sum_{i=1, i\neq j}^{N} \gamma_n^{(i)} \gamma_n^{(j)} d^2(T_{i\lambda_i} z_n, T_{j_\lambda} z_n)] - \alpha_n(1-\alpha_n) d^2(v_n, w_n) \\ &\le \alpha_n d^2(v_n, p) + (1-\alpha_n)[d^2(z_n, p) - \sum_{i=1}^{N} \gamma_n^{(0)} \gamma_n^{(i)} d^2(z_n, T_{i\lambda} z_n)] - \alpha_n(1-\alpha_n) d^2(v_n, w_n) \\ &\le \alpha_n d^2(v_n, p) + (1-\alpha_n)[\beta_n^{(0)} d^2(v_n, p) + \sum_{i=1}^{N} \beta_n^{(i)} d^2(y_n, p) - \sum_{i=1}^{N} \beta_n^{(0)} \beta_n^{(i)} d^2(v_n, y_n) \\ &\quad - \sum_{i=1, i\neq j}^{N} \beta_n^{(i)} \beta_n^{(j)} d^2(y_n^{(i)}, y_n^{(j)})] - \alpha_n(1-\alpha_n) \sum_{i=1}^{N} \gamma_n^{(0)} \gamma_n^{(i)} d^2(z_n, T_{i\lambda} z_n) - \alpha_n(1-\alpha_n) d^2(v_n, w_n) \end{aligned}$$

$$
\begin{aligned}
& d^2(v_n,p)-(1-\alpha_n)\sum_{i=1}^{N}\beta_n^{(0)}\beta_n^{(i)}d^2(v_n,y_n)-(1-\alpha_n)\sum_{i=1,i\neq j}^{N}\beta_n^{(i)}\beta_n^{(j)}d^2(y_n^{(i)},y_n^{(j)})\\
& -(1-\alpha_n)\sum_{i=1}^{N}\gamma_n^{(0)}\gamma_n^{(i)}d^2(z_n,T_{i\lambda}z_n)-\alpha_n(1-\alpha_n)d^2(v_n,w_n)\\
\le & (1-t_n)d^2(x_n,p)+t_nd^2(u,p)-t_n(1-t_n)d^2(u,x_n)-(1-\alpha_n)\sum_{i=1}^{N}\beta_n^{(0)}\beta_n^{(i)}d^2(v_n,y_n)\\
& -(1-\alpha_n)\sum_{i=1}^{N}\gamma_n^{(0)}\gamma_n^{(i)}d^2(z_n,T_{i\lambda}z_n)-\alpha_n(1-\alpha_n)d^2(v_n,w_n) \qquad (3.13)\\
\le & (1-t_n)d^2(x_n,p)+t_nd^2(u,p)-t_n(1-t_n)d^2(u,x_n)-\alpha_n(1-\alpha_n)d^2(v_n,w_n).
\end{aligned}
$$

From (3.5) and condition C2, we obtain that

$$d(v_n,x_n)\le t_nd(u,x_n)\to 0, \quad as\ n\ \to\infty. \tag{3.14}$$

Now we divide the rest of the proof into two cases:

Case 1: Assume that $\{d^2(x_n,p)\}$ is a monotonically non-increasing sequence. Clearly, $\{d^2(x_n,p)\}$ is convergent and

$$d^2(x_n,p)-d^2(x_{n+1},p)\to 0, \quad as\ n\ \to\infty.$$

So from (3.13), we have

$$
\begin{aligned}
\alpha_n(1-\alpha)d^2(v_n,w_n) &\le (1-t_n)d^2(x_n,p)+t_nd^2(u,p)-d^2(x_{n+1},p)\\
&= t_n[d^2(u,p)-d^2(x_n,p)]+d^2(x_n,p)-d^2(x_{n+1},p),
\end{aligned}
$$

which implies by condition C2 that

$$\lim_{n\to\infty} d(v_n,w_n)=0. \tag{3.15}$$

Similarly,

$$(1-\alpha_n)\sum_{i=1}^{N}\gamma_n^{(0)}\gamma_n^{(i)}d^2(z_n,T_\lambda z_n)\le t_n[d^2(u,p)-d(x_n,p)]+d^2(x_n,p)-d^2(x_{n+1},p)\to 0, \quad as\ n\ \to\infty.$$

Hence, by condition C2, we obtain that

$$(1-\alpha_n)\sum_{i=1}^{N}\gamma_n^{(0)}\gamma_n^{(i)}d^2(z_n,T_{i\lambda}z_n)\to 0,$$

and thus,

$$\lim_{n\to\infty} d(z_n,T_{i\lambda}z_n)=0,\ i=1,2,\cdots,N. \tag{3.16}$$

In a similar way, from (3.11) we obtain that

$$\lim_{n\to\infty} d(v_n,y_n)=\lim_{n\to\infty} d(J_{r_nh_1}\circ\cdots\circ J_{r_nh_N}v_n,v_n)=0. \tag{3.17}$$

Let $c_n^{(i)}=J_{r_nh_i}c_n^{(i+1)}$, $i=1,2,\cdots,N$, where $c_n^{(N+1)}=v_n$ for all $n\ge 1$. Then, $c_n^{(1)}=y_n$. By Lemma 2.11, we obtain

$$\frac{1}{2r_n}d^2(c_n^{(i)},p)-\frac{1}{2r_n}d^2(c_n^{(i+1)},p)+\frac{1}{2r_n}d^2(c_n^{(i+1)},c_n^{(i)})+h(c_n^{(i)})\le h(p).$$

Since $h(p)\le h(c_n^{(i)})$, we obtain

$$d^2(c_n^{(i)},c_n^{(i+1)})\le d^2(c_n^{(i+1)},p)-d^2(c_n^{(i)},p). \tag{3.18}$$

Now, taking the sum from $i = 1$ to $i = N$ in (3.18), from (3.17) we obtain that

$$\begin{aligned}\sum_{i=1}^{N} d^2(c_n^{(i)}, c_n^{(i+1)}) &\leq d^2(c_n^{(N+1)}, p) - d^2(c_n^{(1)}, p)\\ &= d^2(v_n, p) - d^2(y_n, p)\\ &\leq d^2(y_n, v_n) + 2d(y_n, v_n)d(v_n, p) \to 0, \text{ as } n \to \infty,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(c_n^{(i)}, c_n^{(i+1)}) = 0, \ i = 1, 2, \cdots, N. \tag{3.19}$$

Thus, for each $i = 1, 2, \cdots, N$, we obtain by applying the triangle inequality that $\lim_{n\to 0} d(c_n^{(i)}, c_n^{(N+1)}) = 0$. That is,

$$\lim_{n\to\infty} d(c_n^{(i)}, v_n) = 0, \ i = 1, 2, \cdots, N. \tag{3.20}$$

Since $r_n \geq r > 0$ for all $n \geq 1$, from Lemma 2.5, Lemma 2.4, (3.19) and the nonexpansivity of J_{rh_i}, $i = 1, 2, \cdots, N$ we obtain that

$$\begin{aligned}d(c_n^{(i+1)}, J_{rh_i}c_n^{(i+1)}) &\leq d(c_n^{(i+1)}, J_{r_nh_i}c_n^{(i+1)}) + d(J_{r_nh_i}c_n^{(i+1)}, J_{rh_i}c_n^{(i+1)})\\ &= d(c_n^{(i+1)}, c_n^{(i)}) + d\left(J_{rh_i}\left(\frac{r_n - r}{r_n}J_{r_nh_i}c_n^{(i+1)} \oplus \frac{r}{r_n}c^{(i+1)}\right), J_{rh_i}c_n^{(i+1)}\right)\\ &\leq d(c_n^{(i+1)}, c_n^{(i)}) + d\left(\frac{r_n - r}{r_n}J_{r_nh_i}c_n^{(i+1)} \oplus \frac{r}{r_n}c_n^{(i+1)}, \ c_n^{(i+1)}\right)\\ &\leq \left(2 - \frac{r}{r_n}\right) d(c_n^{(i+1)}, c_n^{(i)}) \to 0, \text{ as } n \to \infty, \ i = 1, 2, \cdots, N.\end{aligned} \tag{3.21}$$

By (3.19), (3.20) and (3.21), we obtain that

$$\begin{aligned}d(J_{rh_i}v_n, v_n) &\leq d(J_{rh_i}v_n, J_{rh_i}c_n^{(i+1)}) + d(J_{rh_i}c_n^{(i+1)}, c_n^{(i+1)}) + d(c_n^{(i+1)}, c_n^{(i)}) + d(c_n^{(i)}, v_n)\\ &\leq d(v_n, c_n^{(i)}) + d(c_n^{(i)}, c_n^{(i+1)}) + d(J_{rh_i}c_n^{(i+1)}, c_n^{(i+1)}) + d(c_n^{(i+1)}, c_n^{(i)}) + d(c_n^{(i)}, v_n)\\ &= 2d(v_n, c_n^{(i)}) + 2d(c_n^{(i)}, c_n^{(i+1)}) + d(J_{rh_i}c_n^{(i+1)}, c_n^{(i+1)}) \to 0, \text{ as } n \to \infty.\end{aligned}$$

That is,

$$\lim_{n\to\infty} d(J_{rh_i}v_n, v_n) = 0, \ i = 1, 2, \cdots, N. \tag{3.22}$$

Let $a_n = \beta_n^{(0)}v_n \oplus \beta_n^{(1)}y_n \oplus \beta_n^{(2)}y_n \cdots \oplus \beta_n^{(N)}y_n$. Then,

$$\begin{aligned}d(a_n, x_n) &= \beta_n^{(0)}d(v_n, x_n) + \sum_{i=1}^{N}\beta_n^{(i)}d(y_n, x_n)\\ &\leq \beta_n^{(0)}d(v_n, x_n) + \sum_{i=1}^{N}\beta_n^{(i)}d(y_n, v_n) + \sum_{i=1}^{N}\beta_n^{(i)}d(v_n, x_n),\end{aligned}$$

which implies from (3.14) and (3.17) that

$$\lim_{n\to\infty} d(a_n, x_n) \quad = \quad 0. \tag{3.23}$$

We know that P_D is firmly nonexpansive. Thus, from (3.10), (3.11) and (3.15) we obtain that

$$\begin{aligned}d^2(z_n, a_n) &\leq d^2(a_n, p) - d^2(z_n, p)\\ &\leq d^2(v_n, p) - d^2(z_n, p)\\ &\leq d^2(v_n, p) - d^2(w_n, p)\\ &\leq d^2(v_n, w_n) + 2d(v_n, w_n)d(w_n, p) \to 0, \ as \ n \to \infty.\end{aligned} \tag{3.24}$$

From (3.23) and (3.24), we obtain that

$$\lim_{n\to\infty} d(z_n, x_n) \quad = \quad 0. \tag{3.25}$$

Using a similar method as in [50], [51] and [52], and the fact that $\{x_n\}$ is bounded, it follows from Lemma 2.8 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $\Delta - \lim_{k\to\infty} x_{n_k} = z$. It follows from (3.25) that there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$, such that $\Delta - \lim_{k\to\infty} z_{n_k} = z$. By a similar argument, we have that $\Delta - \lim_{k\to\infty} v_{n_k} = z$. Since T_{i_λ} is Δ-demiclosed for each $i = 1, 2, \cdots, N$, it follows from (3.16) and Lemma 3.5 that $z \in \cap_{i=1}^{N} F(T_{i_\lambda}) = \cap_{i=1}^{N} F(T_i)$. Also, since J_{rh_i} is nonexpansive, for each $i = 1, 2, \cdots, N$, we obtain from (3.22) and Lemma 2.10 that $z \in \cap_{i=1}^{N} F(J_{rh_i}) = \left(\cap_{i=1}^{N} argmin_{y\in X} h_i(y)\right)$. Hence, $z \in \Gamma$.

Furthermore, for an arbitrary $u \in X$, by Lemma 2.9 we have that

$$\limsup_{n\to\infty} \langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle \le 0, \tag{3.26}$$

which implies by condition C1 that

$$\limsup_{n\to\infty} \left(t_n d^2(z, u) + 2(1 - t_n)\langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle \right) \le 0. \tag{3.27}$$

We now show that $\{x_n\}$ converges strongly to z. By (3.12) and Lemma 2.5, we obtain

$$\begin{aligned} d^2(x_{n+1}, z) &\le d^2(v_n, z) \\ &\le (1 - t_n)^2 d^2(z, x_n) + t_n^2 d^2(z, u) + 2t_n(1 - t_n)\langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle \\ &\le (1 - t_n) d^2(z, x_n) + t_n \left(t_n d^2(z, u) + 2(1 - t_n)\langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle \right). \end{aligned} \tag{3.28}$$

Hence, by (3.27) and Lemma 2.12, we conclude that $\{x_n\}$ converges strongly to z.

Case 2: Suppose that $\{d^2(x_n, p)\}$ is not monotonically non-increasing. Then, there exists a subsequence $\{d^2(p, x_{n_i})\}$ of $\{d^2(p, x_n)\}$, such that $d^2(p, x_{n_i}) < d^2(p, x_{n_i+1})$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.13, there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$, such that $m_k \to \infty$, and

$$d^2(p, x_{m_k}) \le d^2(p, x_{m_k+1}) \text{ and } d^2(p, x_k) \le d^2(p, x_{m_k+1}) \text{ for all } k \in \mathbb{N}. \tag{3.29}$$

Thus, by (3.12), (3.29) and Lemma 2.5, we obtain

$$\begin{aligned} 0 &\le \lim_{k\to\infty} \left(d^2(p, x_{m_k+1}) - d^2(p, x_{m_k}) \right) \\ &\le \limsup_{n\to\infty} \left(d^2(p, x_{n+1}) - d^2(p, x_n) \right) \\ &\le \limsup_{n\to\infty} \left(d^2(p, z_n) - d^2(p, x_n) \right) \\ &\le \limsup_{n\to\infty} \left((1 - t_n) d^2(p, x_n) + t_n d^2(p, u) - d^2(p, x_n) \right) \\ &= \limsup_{n\to\infty} \left[t_n \left(d^2(p, u) - d^2(p, x_n) \right) \right] = 0, \end{aligned}$$

which implies that

$$\lim_{k\to\infty} \left(d^2(p, x_{m_k+1}) - d^2(p, x_{m_k}) \right) = 0. \tag{3.30}$$

Following the arguments as in **Case 1**, we can show that

$$\lim_{k\to\infty} \left(t_{m_k} d^2(z, u) + 2(1 - t_{m_k})\langle \overrightarrow{zu}, \overrightarrow{zx_{m_k}} \rangle \right) \le 0. \tag{3.31}$$

Also, by (3.28), we have

$$d^2(z, x_{m_k+1}) \le (1 - t_{m_k}) d^2(z, x_{m_k}) + t_{m_k} \left(t_{m_k} d^2(z, u) + 2(1 - t_{m_k})\langle \overrightarrow{zu}, \overrightarrow{zx_{m_k}} \rangle \right).$$

Since $d^2(z, x_{m_k}) \le d^2(z, x_{m_k+1})$, we obtain

$$d^2(z, x_{m_k}) \le \left(t_{m_k} d^2(z, u) + 2(1 - t_{m_k})\langle \overrightarrow{zu}, \overrightarrow{zx_{m_k}} \rangle \right).$$

Thus, by (3.31), we get

$$\lim_{k\to\infty} d^2(z, x_{m_k}) = 0. \tag{3.32}$$

It then follows from (3.29), (3.30) and (3.32) that $\lim_{k\to\infty} d^2(z, x_k) = 0$. Therefore, we conclude from both cases that $\{x_n\}$ converges to $z \in \Gamma$. □

By setting $N = 1$ in Theorem 3.6, we obtain the following result:

Corollary 3.7. *Let D be a nonempty closed and convex subset of a complete CAT(0) space X, and $h : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Let $T : D \to D$ be a k-demimetric mapping with $k \in (-\infty, \lambda]$ and $\lambda \in (0, 1)$. Suppose that $\Gamma = ((argmin_{u\in X} h(u)) \cap F(T))$ is nonempty and for arbitrary $x_1, u \in X$ the sequence $\{x_n\}$ is defined by*

$$\begin{cases} v_n = (1 - t_n)x_n \oplus t_n u, \\ y_n = J_{r_n h} v_n, \\ z_n = P_D\big(\beta_n^{(0)} v_n \oplus \beta_n^{(1)} y_n\big), \\ w_n = \gamma_n^{(0)} z_n \oplus \gamma_n^{(1)} T_\lambda z_n, \\ x_{n+1} = \alpha_n v_n \oplus (1 - \alpha_n) w_n \ \text{ for all } n \ge 1, \end{cases}$$

where T_λ is as defined in Lemma 3.5, such that T_λ is Δ-demiclosed. Suppose that $\{t_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in [0,1], such that the following conditions are satisfied:

$C1: \ 0 < a \le \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \le b < 1, \sum_{i=0}^{1} \beta_n^{(i)} = 1$ *and* $\sum_{i=0}^{1} \gamma_n^{(i)} = 1$ *for all* $n \ge 1$,
$C2: \ \lim_{n\to\infty} t_n = 0, \sum_{n=1}^{\infty} t_n = \infty$,
$C3: \ \{r_n\}$ *is a sequence of real numbers such that* $r_n \ge r > 0$.

Then, the sequence $\{x_n\}$ converges strongly to a point in Γ.

By setting T_i to be a k-demicontractive mapping for each $i = 1, 2, \ldots, N$ in Theorem 3.6, we obtain the following result:

Corollary 3.8. *Let D be a nonempty closed and convex subset of a complete CAT(0) space X, and $h_i : X \to (-\infty, \infty]$, $i = 1, \cdots, N$ be a finite family of proper convex and lower semi-continuous functions. Let $T_i : X \to X$, $i = 1, \cdots, N$ be a finite family of k_i-demicontactive mappings with $k_i \in (-\infty, \lambda]$ and $\lambda \in (0, 1)$. Suppose that $\Gamma = (\cap_{i=1}^N argmin_{u\in X} h_i(u)) \cap (\cap_{i=1}^N F(T_i))$ is nonempty and $\{x_n\}$ is a sequence generated for arbitrary $x_1, u \in X$ by*

$$\begin{cases} v_n = (1 - t_n)x_n \oplus t_n u, \\ y_n = J_{r_n h_1} \circ J_{r_n h_2} \circ \cdots \circ J_{r_n h_N} v_n, \\ z_n = P_D\big(\beta_n^{(0)} v_n \oplus \beta_n^{(1)} y_n \oplus \cdots \oplus \beta_n^{(N)} y_n\big), \\ w_n = \gamma_n^{(0)} z_n \oplus \gamma_n^{(1)} T_{1\lambda} z_n \oplus \gamma_n^{(1)} T_{2\lambda} z_n \cdots \oplus \gamma_n^{(N)} T_{N\lambda} z_n, \\ x_{n+1} = \alpha_n v_n \oplus (1 - \alpha_n) w_n \ \text{ for all } n \ge 1, \end{cases} \tag{3.33}$$

where $T_{i\lambda} x = \lambda x \oplus (1 - \lambda) T_i x$, such that $T_{i\lambda}$ are Δ-demiclosed for each $i = 1, 2, \ldots, N$. Suppose that $\{t_n\}$, $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ are sequences in [0,1], such that the following conditions are satisfied:

$C1: \ 0 < a \le \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \le b < 1, \sum_{i=0}^{N} \beta_n^{(i)} = 1$ *and* $\sum_{i=0}^{N} \gamma_n^{(i)} = 1$ *for all* $n \ge 1$,
$C2: \ \lim_{n\to\infty} t_n = 0, \sum_{n=1}^{\infty} t_n = \infty$,
$C3: \ \{r_n\}$ *is a sequence of real numbers such that* $r_n \ge r > 0$ *for all* $n \ge 1$.

Then, the sequence $\{x_n\}$ converges strongly to a point in Γ.

4 Numerical example

In this section, we give a numerical example to illustrate Theorem 3.6.
Let $X = \mathbb{R}$, endowed with the usual metric and $D = [0, 1]$. Then,

$$P_D(x) = \begin{cases} 0, & if\ x < 0, \\ x, & if\ x \in D, \\ 1, & if\ x > 1 \end{cases}$$

is a metric projection onto D. For $N = 2$, define $T_i : D \to D$, by $T_i x = x - x^i$, $i = 1, 2$. Then, T is (-1)–demimetric (see Example 3.3). Now, define $h_i : \mathbb{R} \to (-\infty, \infty]$ by $h_i(x) = \frac{1}{2}|B_i(x) - b_i|^2$, where $B_i(x) = 2ix$ and $b_i = 0$, $i = 1, 2$. Since B_i is continuous and linear for each $i = 1, 2$, then we have that h_i is proper, convex and lower semicontinuous mapping. Let $r_n = 1$ for all $n \geq 1$, then

$$\begin{aligned} J_{1h_i}(x) = Proxh_i x &= arg \min_{y \in D} \left(h_i(y) + \frac{1}{2}|y - x|^2\right) \\ &= \left(I + B_i^T B_i\right)^{-1}\left(x + B_i^T b_i\right). \end{aligned}$$

Take $t_n = \frac{1}{2n+1}$, $\beta_n^{(0)} = \frac{n}{4n+1}$, $\beta_n^{(1)} = \frac{n+1}{4n+1}$, $\beta_n^{(2)} = \frac{2n}{4n+1}$, $\gamma_n^{(0)} = \frac{3n}{5n+7}$, $\gamma_n^{(1)} = \frac{n+7}{5n+7}$, $\gamma_n^{(2)} = \frac{n}{5n+7}$ and $\alpha_n = \frac{4n}{6n+1}$, then conditions C1 and C2 of Theorem 3.6 are satisfied. Therefore, for $x_1, u \in \mathbb{R}$, after applying our algorithm (3.9) becomes

$$\begin{cases} v_n = (1 - t_n)x_n + t_n u, \\ y_n = J_1{}^{(1)}\left(J_1{}^{(2)}(v_n)\right), \\ z_n = P_D\left(\beta_n^{(0)} v_n + \beta_n^{(1)} y_n + \beta_n^{(2)} y_n\right), \\ w_n = \gamma_n^{(0)} z_n + \gamma_n^{(1)} T_{1\lambda} z_n + \gamma_n^{(2)} T_{2\lambda} z_n, \\ x_{n+1} = \alpha_n v_n + (1 - \alpha_n) w_n \ \text{ for all } n \geq 1. \end{cases}$$

Case 1: Take $x_1 = 0.5$ and $u = 0.5$.
Case 2: Take $x_1 = 0.5$ and $u = 1$.
Case 3: Take $x_1 = 1$ and $u = 0.5$.

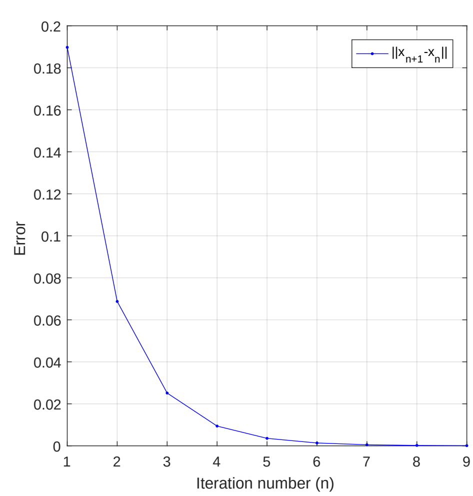

Figure 1: Errors vs number of iterations for **Case 1**.

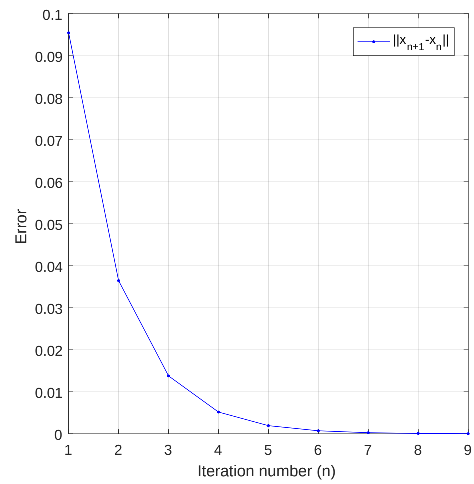

Figure 2: Errors vs number of iterations for **Case 2**.

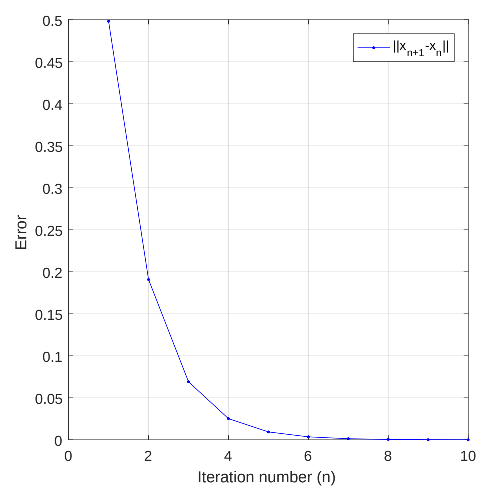

Figure 3: Errors vs number of iterations for **Case 3**.

Declaration

The authors declare that they have no competing interests.

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