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Noncyclic Meir-Keeler contractions and best proximity pair theorems

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Abstract: In this article, we consider the class of noncyclic Meir-Keeler contractions and study the existence and convergence of best proximity pairs for such mappings in the setting of complete CAT(0) spaces. We also discuss asymptotic pointwise noncyclic Meir-Keeler contractions in the framework of uniformly convex Banach spaces and generalize a main result of Chen [Chen C. M., A note on asymptotic pointwise weaker Meir-Keeler type contractions, Appl. Math. Lett., 2012, 25, 1267-1269]. Examples are given to support our main results.

Keywords: best proximity pair, CAT(0) space, uniformly convex Banach space, projection mapping, noncyclic Meir-Keeler contraction

MSC: 47H10, 47H09, 46B20

1 Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be *Meir-Keeler contraction* provided that for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon \quad \text{for all} \quad x, y \in X. \tag{1.1}$$

Another interesting extension of the *Banach contraction principle* was established in [1].

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a Meir-Keeler contraction mapping. Then T has a unique fixed point and the Picard iteration sequence $\{T^n x_0\}$ converges to the fixed of T for any $x_0 \in X$.*

Recently, the class of Meir-Keeler contractions was generalized in [2] as follows.

Definition 1.2. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called a cyclic Meir-Keeler contraction if T is cyclic on $A \cup B$, that is, $T(A) \subseteq B$, $T(B) \subseteq A$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d^*(x, y) < \varepsilon + \delta \Rightarrow d^*(Tx, Ty) < \varepsilon \quad \textit{for all} \quad (x, y) \in A \times B, \tag{1.2}$$

where $d^*(a, b) = d(a, b) - \operatorname{dist}(A, B)$ for any $(a, b) \in A \times B$.

The following result is a generalization of Theorem 1.1 in uniformly convex Banach spaces.

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Theorem 1.3. [2] *Let A and B be nonempty and closed subsets of a uniformly convex Banach space X, such that A is convex. Suppose T is a cyclic Meir-Keeler contraction on $A \cup B$. Then there exists a unique point $z \in A$ for which $d(z, Tz) = \operatorname{dist}(A, B)$. Moreover, for any $x_0 \in A$ the iterate sequence $\{T^{2n}x_0\}$ converges to z.*

We mention that Theorem 1.3 is valid in metric spaces when the pair (A, B) satisfies the property UC (see Theorem 3 of [3] for more details).

Let (A, B) be a nonempty pair in a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be *noncyclic* provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. If $A \cap B = \emptyset$, then it is interesting to study the existence of *best proximity pairs* for the non-self mapping T, that is, a point $(p, q) \in A \times B$, such that

$$p = Tp, \quad q = Tq \quad \text{and} \quad d(p, q) = \operatorname{dist}(A, B).$$

In this case, the existence of a best proximity pair for the noncyclic mapping T is equivalent to the existence of a solution of the following minimization problem:

$$\text{Find} \quad \min_{x \in A} d(x, Tx), \ \min_{y \in B} d(y, Ty) \ \text{and} \ \min_{(x,y) \in A \times B} d(x, y). \tag{1.3}$$

Existence of best proximity pairs for noncyclic mappings was first studied in [4] (see also [5] for a different approach to the same problem) . It was proved that if (A, B) is a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X, and $T : A \cup B \to A \cup B$ is a noncyclic mapping for which $\|Tx - Ty\| \le \|x - y\|$ for all $(x, y) \in A \times B$, then T has at least one best proximity pair (see Theorem 2.1 and Proposition 2.1 of [4]).

In this paper, we study the noncyclic version of the mappings considered in Theorem 1.3, in order to prove the existence and convergence of best proximity pairs using the *metric projection operators* in the setting of complete CAT(0) spaces. We also extend one of the main theorems of [6] to noncyclic mappings and establish a new best proximity pair theorem in uniformly convex Banach spaces. Finally, in the last section of this article, we show that under some sufficient conditions the class of noncyclic relatively u-continuous mappings is continuous on their domains and so the existence of best proximity pairs for such mappings can be obtained easily from the Schauder's fixed point result.

2 Preliminaries

In this section we recall some fundamental concepts which will be used in our coming discussion.

Definition 2.1. *A Banach space X is said to be uniformly convex if there exists a strictly increasing function $\delta : (0, 2] \to [0, 1]$, such that the following implication holds for all $x, y, p \in X$, $R > 0$ and $r \in [0, 2R]$:*

$$\begin{cases} \|x - p\| \le R \\ \|y - p\| \le R \\ \|x - y\| \ge r \end{cases} \Rightarrow \|\frac{x + y}{2} - p\| \le (1 - \delta(\frac{r}{R}))R$$

It is well known that Hilbert spaces and l^p spaces $(1 < p < \infty)$ are uniformly convex Banach spaces.

Given (A, B), a pair of nonempty subsets of a normed linear space X, then its proximal pair is the pair (A_0, B_0) given by

$$A_0 = \{x \in A \colon \|x - y'\| = \operatorname{dist}(A, B) \text{ for some } y' \in B\},$$

$$B_0 = \{y \in B \colon \|x' - y\| = \operatorname{dist}(A, B) \text{ for some } x' \in A\}.$$

Proximal pairs may be empty but, in particular, if A and B are nonempty weakly compact and convex, then (A_0, B_0) is a nonempty weakly compact convex pair in X. We will say that the pair (A, B) is *proximinal* provided that $A_0 = A$ and $B_0 = B$.

A metric space (X, d) is said to be a *(uniquely) geodesic space* if every two points x and y of X are joined by a *(unique) geodesic*, i.e, a map $c : [0, l] \subseteq \mathbb{R} \to X$, such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all

$t, t' \in [0, l]$. A subset A of a geodesic space X is said to be *convex* if the image of any geodesic that joins each pair of points x and y of A (geodesic segment $[x, y]$) is contained in A. A point z in X belongs to a geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$, such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$. We write $z = (1 - t)x \oplus ty$ for simplicity. Notice that this point may not be unique. Any Banach space is for instance a geodesic space with usual segments as geodesic segments.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\triangle(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$, such that $d_{\mathbb{E}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

CAT(0) : Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

For details about CAT(0) spaces, we refer to [7, 8].

The next lemma plays an important role in our results.

Lemma 2.2. *([9]) Let (X, d) be a* CAT(0) *space and let (A, B) be a nonempty and closed pair of subsets in X. Suppose B is bounded. Then (A_0, B_0) is a nonempty, bounded and closed pair. Moreover, if (A, B) is convex, then (A_0, B_0) is also convex.*

Let (X, d) be a metric space and C be a nonempty subset of X. The metric projection operator $\mathcal{P}_C : X \to 2^C$ is defined as

$$\mathcal{P}_C(x) := \{y \in C : d(x, y) = \operatorname{dist}(\{x\}, C)\},$$

where 2^C denotes the set of all subsets of X. It is well known that if C is a nonempty, closed and convex subset of a complete CAT(0) space X, then the metric projection $\mathcal{P}_C$ is single-valued from X onto C (see [7] for more details).

Now, suppose (A, B) is a nonempty, closed and convex pair in a complete CAT(0) space X, such that B is bounded. By Lemma 2.2 (A_0, B_0) is also nonempty, closed and convex.

Consider the mapping $\mathcal{P} : A_0 \cup B_0 \to A_0 \cup B_0$ as below

$$\mathcal{P}(x) := \begin{cases} \mathcal{P}_{B_0}(x), & \text{if } x \in A_0, \\ \mathcal{P}_{A_0}(x), & \text{if } x \in B_0. \end{cases}$$

Then $\mathcal{P}$ is a single-valued cyclic mapping on $A_0 \cup B_0$, that is, $T(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$. Furthermore, for each $x \in A_0 \cup B_0$ we have $d(x, \mathcal{P}x) = \operatorname{dist}(A, B)$ (see [10, 11] for more information).

We finish this section by recalling the following geometric notion.

Definition 2.3. [3] *Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy property UC if for any $\{x_n\}$ and $\{z_n\}$ sequences in A and $\{y_n\}$ sequence in B, such that* $\lim_{n\to\infty} d(x_n, y_n) = \operatorname{dist}(A, B)$ *and* $\lim_{n\to\infty} d(z_n, y_n) = \operatorname{dist}(A, B)$, *we have* $\lim_{n\to\infty} d(x_n, z_n) = 0$.

Notice that property UC is not symmetric, that is, it is not true that if (A, B) has property UC then so does (B, A). It was proved in [12, 13] that if (A, B) is a nonempty and closed pair in a uniformly convex Banach space (or CAT(0)), space such that A is convex, then (A, B) has property UC.

Lemma 2.4. [3] *Let A and B be two nonempty subsets of a metric space (X, d). Assume that (A, B) satisfies property UC. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B, respectively, such that either of the following holds:*

$$\lim_{m\to\infty} \sup_{n\geq m} d(x_m, y_n) = d(A, B) \text{ or } \lim_{n\to\infty} \sup_{m\geq n} d(x_m, y_n) = d(A, B).$$

Then $\{x_n\}$ is a Cauchy sequence.

3 Existence and convergence of best proximity pairs

3.1 Noncyclic Meir-Keeler contractions

Here we present the noncyclic version of Definition 1.2 in order to study the existence and convergence of best proximity pairs.

Definition 3.1. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called a noncyclic Meir-Keeler contraction if T is noncyclic and satisfies condition (2) of Definition 1.2.*

The next lemma will be used in our coming discussion.

Lemma 3.2. *(see also [14] for the same result in uniformly convex Banach spaces) Let (A, B) be a nonempty, closed and convex pair in a complete* CAT(0) *space X, such that B is bounded. Then the projection mapping $\mathcal{P} : A_0 \cup B_0 \to A_0 \cup B_0$ is continuous.*

Proof. Let $\{x_n\}$ be a sequence in A_0, such that $x_n \to x \in A_0$. Then

$$d(\mathcal{P}x_n, x) \leq d(\mathcal{P}x_n, x_n) + d(x_n, x) \to \operatorname{dist}(A, B),$$

which implies that $d(\mathcal{P}x_n, x) \to \operatorname{dist}(A, B)$. Thereby, $\mathcal{P}x_n \to \mathcal{P}x$ and so $\mathcal{P}|_{A_0}$ is continuous. Equivalently, we can show that $\mathcal{P}|_{B_0}$ is continuous. □

We now prove the main result of this section.

Theorem 3.3. *Let (A, B) be a nonempty, closed and convex pair in a complete* CAT(0) *space X and $T : A \cup B \to A \cup B$ a noncyclic Meir-Keeler contraction mapping. For an arbitrary element $x_0 \in A_0$ define*

$$\begin{cases} x_n = T^n x_0, \\ y_n = \mathcal{P}x_n, \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$. If either A or B is bounded, then the sequence $\{(x_n, y_n)\} \subseteq A_0 \times B_0$ converges to a best proximity pair of the mapping T.

Proof. Notice that from Lemma 2.2 the pair (A_0, B_0) is nonempty, closed and convex and so it has property UC. Put $\delta_n := d^\star(x_n, y_{n+1})$. We claim that $\delta_n \to 0$. Note that if $\delta_k = 0$ for some $k \in \mathbb{N}$, then

$$\delta_{k+1} = d^\star(x_{k+1}, y_{k+2}) \leq d^\star(x_k, y_{k+1}) = 0,$$

which implies that $\delta_n = 0$ for all $n \geq k$. Besides, if $\delta_n > 0$ for all $n \in \mathbb{N}$, then from Proposition 1 of [2] there exists a nondecreasing and continuous L-function φ for which

$$\delta_{n+1} = d^\star(x_{n+1}, y_{n+2}) < \varphi(d^\star(x_n, y_{n+1})) = \varphi(\delta_n).$$

Now by Lemma 2 of [2], we conclude that $\lim_{n\to\infty} \delta_n = 0$. Thus, $d(x_n, y_{n+1}) \to \operatorname{dist}(A, B)$ and $d(x_{n+1}, y_{n+1}) = \operatorname{dist}(A, B)$ for all $n \in \mathbb{N}$. In view of the fact that (A, B) has property UC, $d(x_n, x_{n+1}) \to 0$. Moreover, by the fact that $d(x_n, y_n) = \operatorname{dist}(A, B) = d(x_{n+1}, y_{n+1})$ and using Lemma 4.3 of [11], we obtain $d(y_n, y_{n+1}) = d(x_n, x_{n+1}) \to 0$. Fix $\varepsilon > 0$ and choose $r \in (0, \varepsilon)$, such that $\varphi(\varepsilon + r) \leq \varepsilon$. Then there exists $N \in \mathbb{N}$, such that

$$\max\{d(x_{m+1}, x_m), d(y_{m+1}, y_m)\} < r, \quad d^\star(x_m, y_{m+1}) < \varepsilon \quad \text{for all} \quad m \geq N.$$

Consider $m \geq N$. We prove that

$$d^\star(x_m, y_{n+1}) < r + \varepsilon \quad \text{for all} \quad n \geq m. \tag{3.2}$$

Clearly, (5) holds for $n = m$. Assume that (5) is satisfied for some $n \geq m$. Then

$$d^\star(x_{m+1}, y_{n+2}) \leq \varphi(d^\star(x_m, y_{n+1})) \leq \varphi(r + \varepsilon) \leq \varepsilon,$$

and so,

$$d^\star(x_m, y_{n+2}) \leq d(x_m, x_{m+1}) + d^\star(x_{m+1}, y_{n+2}) < r + \varepsilon,$$

that is, (5) holds. Therefore,

$$\lim_{m\to\infty} \sup_{n\geq m} d(x_m, y_{n+1}) = \operatorname{dist}(A, B).$$

It now follows from Lemma 2.4 that $\{x_n\}$ and so $\{y_n\}$ are Cauchy sequences. Suppose that $x_n \to p \in A_0$. Since $\mathcal{P}|_{A_0}$ is continuous $y_n = \mathcal{P}x_n \to \mathcal{P}p := q$. It is worth noticing that T and $\mathcal{P}$ are commuting on $A_0 \cup B_0$. Indeed, if $x \in A_0$, then

$$d(Tx, \mathcal{P}Tx) = \operatorname{dist}(A, B), \quad d(Tx, T\mathcal{P}x) \leq d(x, \mathcal{P}x) = \operatorname{dist}(A, B).$$

This implies that $\mathcal{P}Tx = T\mathcal{P}x$. Similarly, if $x \in B_0$, then the result follows. Thereby,

$$d(Tx_n, \mathcal{P}Tp) = d(Tx_n, T\mathcal{P}p) \leq d(x_n, q) \to \operatorname{dist}(A, B).$$

Hence, $Tx_n \to Tp$. Because of the fact that $d(x_n, Tx_n) \to 0$, the point p is a fixed point of T in A_0. On the other hand,

$$Tq = T\mathcal{P}p = \mathcal{P}Tp = \mathcal{P}p = q.$$

Thus, (p, q) is a best proximity pair of the mapping T and $(x_n, y_n) \to (p, q) \in A_0 \times B_0$. □

Let us illustrate Theorem 3.3 with the following examples.

Example 3.1. Let $X = \mathbb{R}^2$ and d be the *river metric* on X defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2, \\ |x_1 - x_2| + |y_1| + |y_2|, & \text{if } x_1 \neq x_2. \end{cases}$$

It is well known that $(\mathbb{R}^2, d)$ is a complete CAT(0) space (see [15]). Suppose $A = \{(0, x) : 0 \leq x \leq \frac{1}{2}\}$ and $B = \{(1, y) : y \geq 0\}$. Thus, (A, B) is a closed and convex pair and that A is bounded. Clearly, $\operatorname{dist}(A, B) = 1$. Define the noncyclic mapping $T : A \cup B \to A \cup B$ with $T(0, x) = (0, x^2)$ and $T(1, y) = (1, \frac{y}{2})$. We have

$$d(T(0, x), T(1, y)) = d((0, x^2), (1, \frac{y}{2})) = 1 + x^2 + \frac{y}{2},$$

$$d((0, x), (1, y)) = 1 + x + y.$$

This implies that the mapping T is noncyclic Meir-Keeler contraction. Therefore, all of the conditions of Theorem 3.3 hold and T has a best proximity pair which is the point $((0, 0), (1, 0))$.

Here, we present an example to show that the convergence result of Theorem 3.3 may not be concluded if the geodesic space X is not CAT(0).

Example 3.2. Consider the Banach space $X = \mathbb{R}^2$ with the supremum norm. In this case X is a geodesic metric space which is not uniquely geodesic and so is not a CAT(0) space. Suppose $A = \{(t, 1) : 0 \leq t \leq 1\}$ and $B = \{(s, 0) : 0 \leq s \leq 1\}$. Then (A, B) is a compact and convex pair and $\operatorname{dist}(A, B) = 1$. Moreover, $A = A_0$ and $B = B_0$. Define the mapping $T : A \cup B \to A \cup B$ with

$$T(t, 1) = \begin{cases} (1, 1) & \text{if } t \in \mathbb{Q} \cap [0, 1], \\ (\frac{t}{2}, 1) & \text{if } t \in \mathbb{Q}^c \cap [0, 1], \end{cases} \quad \text{and} \quad T(s, 0) = (0, 0).$$

For $\mathbf{x} = (t, 1) \in A$ and $\mathbf{y} = (s, 0) \in B$ we have the following cases:

Case 1. If $t \in \mathbb{Q} \cap [0, 1]$, then

$$\|T\mathbf{x} - T\mathbf{y}\| = \|(1, 1)\| = 1 = \operatorname{dist}(A, B).$$

Case 2. If $t \in \mathbb{Q}^c \cap [0, 1]$, then

$$\|T\mathbf{x} - T\mathbf{y}\| = \|(\frac{t}{2}, 1)\| = 1 = \text{dist}(A, B).$$

Therefore, T is a noncyclic Meir-Keeler contraction mapping. We note that $((1, 1), (0, 0))$ is a best proximity pair for the mapping T. It is worth noting that if $\mathbf{x}_0 = (t_0, 1) \in A_0$ with $t_0 \in \mathbb{Q}^c \cap [0, 1]$, then

$$\mathbf{x}_n = T^n\mathbf{x}_0 = (\frac{t_0}{2^n}, 1) \to (0, 1), \quad \mathbf{y}_n = \mathcal{P}\mathbf{x}_n = (\frac{t_0}{2^n}e_1, 0) \to (0, 0),$$

whereas, the point $((0, 1), (0, 0))$ is not a best proximity pair for the mapping T.

3.2 Asymptotic pointwise noncyclic Meir-Keeler contractions

In this section, we establish a best proximity pair theorem for a generalized class of noncyclic Meir-Keeler contractions in uniformly convex Banach spaces. We refer to [16] for a cyclic version of these conclusions in order to study the existence of best proximity points. To this end, we recall some notions of [6].

Definition 3.4. [6] *The function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a weaker Meir-Keeler-type function, if for each $\eta > 0$, there exists $\delta > \eta$, such that for $t \in \mathbb{R}^+$ with $\eta \le t < \delta$, there exists $n_0 \in \mathbb{N}$, such that $\psi^{n_0}(t) < \eta$.*

The notion of asymptotic pointwise weaker Meir-Keeler-type ψ-contractions was introduced in [6] as follows.

Definition 3.5. [6] *Let A be a nonempty subset of a normed linear space X and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a weaker Meir-Keeler-type function. A mapping $T : A \to A$ is said to be an asymptotic pointwise weaker Meir-Keeler-type ψ-contraction if for each $i \in \mathbb{N}$ and for each $x, y \in A$,*

$$\|T^i x - T^i y\| \le \psi^i(\|x\|)\|x - y\|.$$

The next theorem is the main result of [6].

Theorem 3.6. [6] *Let A be nonempty weakly compact convex subset of a Banach space X and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a weaker Meir-Keeler-type function, such that for each $t \in \mathbb{R}^+$, $\{\psi^i(t)\}_{i\in\mathbb{N}}$ is nonincreasing. Suppose that $T : A \to A$ is an asymptotic pointwise weaker Meir-Keeler-type ψ-contraction. Then T has a unique fixed point $z \in A$, and for each $x \in A$, the sequence of Picard iterates, $\{T^n(x)\}$ converges in norm to z.*

The following lemma will be useful in the main result of this section.

Lemma 3.7. *Let (A, B) be a nonempty, weakly compact and convex pair of subsets of a Banach space X and let $T : A \cup B \to A \cup B$ be a noncyclic mapping. Suppose that $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a weaker Meir-Keeler-type function, such that for each $t \in \mathbb{R}$, $\{\psi^i(t)\}_{i\in\mathbb{N}}$ is nonincreasing, and for each $(x, y) \in A \times B$*

$$\|T^i x - T^i y\|^* \le \psi^i(\|x\|)\|x - y\|^* \text{ for all } y \in B, \tag{3.3}$$

$$\|T^i x - T^i y\|^* \le \psi^i(\|y\|)\|x - y\|^* \text{ for all } x \in A. \tag{3.4}$$

Then for any $(x, y) \in A \times B$ there exits a point $(v, w) \in A \times B$, such that

$$\limsup_i \|T^i x - w\| = \text{dist}(A, B) = \limsup_i \|v - T^i y\|.$$

Proof. For some fixed element $x \in A$ define $f : B \to [0, \infty)$ with

$$f(y) = \limsup_i \|T^i x - y\|^* \quad \text{for all} \quad y \in B.$$

By the fact that B is weakly compact and convex, f attains its minimum at a point $w \in B$. We have

$$\begin{aligned} f(T^j y) = & \ \limsup_i ||T^i x - T^j y||^* \\ = & \ \limsup_i ||T^{i+j} x - T^j y||^* \\ = & \ \limsup_i ||T^j(T^i x) - T^j y||^* \\ \le & \ \limsup_i \psi^j(||y||)||T^i x - y||^* \\ = & \ \psi^j(||y||) f(y) \end{aligned}$$

for all $y \in B$. Since $w \in B$ is a minimum of f, we obtain

$$f(w) \le f(T^j w) \le \psi^j(||w||) f(w) \quad \text{for all} \quad j \in \mathbb{N}. \tag{3.5}$$

From Theorem 3 of [6], we conclude that $\lim_j \psi^j ||w|| = 0$. It now follows from (8) that $f(w) = 0$ and so $f(T^j w) = 0$ for all $j \in \mathbb{N}$. Therefore,

$$\limsup_i ||T^i x - w|| = \operatorname{dist}(A, B) = \limsup_i ||T^i x - T^j w|| \quad \text{for all} \quad j \in \mathbb{N}. \tag{3.6}$$

By a similar argument for $y \in B$ we can find an element $v \in A$, such that

$$\limsup_i ||v - T^i y|| = \operatorname{dist}(A, B) = \limsup_i ||T^j v - T^i y|| \quad \text{for all} \quad j \in \mathbb{N}, \tag{3.7}$$

and hence the lemma. □

Here, we present the following new fixed point result.

Theorem 3.8. *Let (A, B) be a nonempty, weakly compact and convex pair in a Banach space X and $T : A \cup B \to A \cup B$ be a noncyclic mapping. Assume that $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a weaker Meir-Keeler-type function, such that for each $t \in \mathbb{R}$, $\{\psi^i(t)\}_{i \in \mathbb{N}}$ is nonincreasing, and for each $(x, y) \in A \times B$*

$$||T^i x - T^i y|| \le \psi^i(||x||)||x - y|| \quad \textit{for all} \quad y \in B,$$

$$||T^i x - T^i y|| \le \psi^i(||y||)||x - y|| \quad \textit{for all} \quad x \in A.$$

Then $A \cap B$ is nonempty and T has a unique fixed point in $A \cap B$. Furthermore, for each $x \in A \cap B$, if $x_n = T^n x$, then $\{x_n\}$ converges to the fixed point of T.

Proof. Let $x \in A$. It follows from Lemma 3.7 that there exists $w \in B$, such that

$$\limsup_i \|T^i x - w\| = \limsup_i \|T^i x - Tw\| = 0,$$

which implies that $w \in B$ is a fixed point of the mapping $T|_B$. Again, by Lemma 3.7 there exists an element $v \in A$, such that

$$\|v - w\| = \limsup_i \|v - T^i w\| = 0,$$

which yields that $v = w$. So $A \cap B$ is nonempty. Now it is sufficient to note that $A \cap B$ is also a weakly compact and convex subset of X and T maps $A \cap B$ into itself. Hence, the result follows from Theorem 3.6. □

In what follows we provide some sufficient conditions to ensure the existence, as well as convergence, of best proximity pairs for weaker Meir-Keeler noncyclic contractions.

Theorem 3.9. *Let (A, B) be a nonempty, closed and convex pair in a uniformly convex Banach space X, such that either A, or B is bounded. Suppose $T : A \cup B \to A \cup B$ is a noncyclic mapping satisfying (6) and (7). If T is weakly continuous on A_0, then T has a best proximity pair, and for any $x_0 \in A_0$ the iteration sequence $\{(x_n, y_n)\} \subseteq A_0 \times B_0$ defined in (4) converges to a best proximity pair of the mapping T.*

Proof. Assume that B is bounded. Then the pair (A_0, B_0) is nonempty, closed and convex. Let $x_0 \in A_0$. From relation (9) of Lemma 3.7 there exists a point $w \in B_0$ for which

$$\limsup_i \|T^i x_0 - w\| = \mathrm{dist}(A, B) = \limsup_i \|T^i x_0 - Tw\|.$$

Since X is uniformly convex and B_0 is convex we must have $w = Tw$. On the other hand,

$$\lim_{i\to\infty} \sup_{j\geq i} \|T^i x_0 - T^j w\| = \lim_{i\to\infty} \|T^i x_0 - w\| = \mathrm{dist}(A, B).$$

Now from Lemma 2.4 we obtain that the sequence $\{x_i\}$ is Cauchy in A_0 and so there exists a point $v \in A_0$, such that $x_i \to v$. Thus, $\|v - w\| = \mathrm{dist}(A, B)$. Since $T|_{A_0}$ is weakly continuous, $Tx_i \rightharpoonup Tv$, where "$\rightharpoonup$" denotes the weak convergence. It follows from the lower semi-continuity of the norm that

$$\|Tv - w\| \leq \liminf_{i\to\infty} \|Tx_i - w\| = \mathrm{dist}(A, B).$$

Strict convexity of X yields that $Tv = v$. By the fact that the projection mapping $\mathcal{P}$ is continuous on A_0 we conclude that

$$y_i = \mathcal{P}x_i \to \mathcal{P}v.$$

Therefore,

$$\|v - T\mathcal{P}(v)\| = \|Tv - T\mathcal{P}(v)\| = \|Tv - \mathcal{P}(Tv)\| = \mathrm{dist}(A, B),$$

and so, $T\mathcal{P}(v) = \mathcal{P}(v)$. Hence, $(v, \mathcal{P}v)$ is a best proximity pair for the mapping T and the sequence (x_i, y_i) converges to $(v, \mathcal{P}v)$. □

4 Continuity of noncyclic relatively u-continuous mappings in CAT(0) spaces

In this section, we consider another class of noncyclic mappings, called noncyclic relatively u-continuous mappings, which was studied in [11, 17] (see also [18] for the cyclic version of such mappings).

Definition 4.1. *Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be a noncyclic relatively u-continuous mapping if T is noncyclic on $A \cup B$ and satisfies the following condition:*

$$\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{if} \quad d^\star(x, y) < \delta, \quad \text{then} \quad d^\star(Tx, Ty) < \varepsilon$$

for all $(x, y) \in A \times B$.

It is worth noticing that this class of mappings contains the class of noncyclic Meir-Keeler contractions as a subclass.

Example 4.1. Consider the space $X = \mathbb{R}^2$ with the river metric d defined in Example 3.1. Suppose $A = \{(x, 0) : 0 \leq x \leq 1\}$ and $B = \{(y, 1) : 0 \leq y \leq 1\}$. Obviously, $\mathrm{dist}(A, B) = 1$. Let $T : A \cup B \to A \cup B$ be defined with

$$T(x, 0) = (\sqrt{x}, 0), \quad T(y, 1) = (\sqrt{y}, 1),$$

where $x, y \in [0, 1]$. Let $\mathbf{x} := (x, 0) \in A$ and $\mathbf{y} := (y, 1) \in B$. Then

$$d^\star(T\mathbf{x}, T\mathbf{y}) = \sqrt{x} + \sqrt{y}, \quad d^\star(\mathbf{x}, \mathbf{y}) = x + y.$$

Let $\varepsilon > 0$ be given. Since the function $t \to \sqrt{t}$ is continuous at zero, there exists $\delta > 0$, such that $\sqrt{t} < \frac{\varepsilon}{2}$, whenever $0 < t < \delta$. Now if $d^\star(\mathbf{x}, \mathbf{y}) < \delta$, then $x + y < \delta$ and so

$$d^\star(T\mathbf{x}, T\mathbf{y}) = \sqrt{x} + \sqrt{y} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

that is, T is noncyclic relatively u-continuous. Notice that if $\mathbf{x} = (0, 0)$, $\mathbf{y} = (\frac{1}{4}, 1)$, then

$$d(T\mathbf{x}, T\mathbf{y}) = 1 + \frac{1}{2} > 1 + \frac{1}{4} = d(\mathbf{x}, \mathbf{y}),$$

which implies that T is neither noncyclic Meir-Keeler contraction, nor noncyclic relatively nonexpansive. It is worth noting that T has a best proximity pair which are the points $((0, 0), (0, 1))$ and $((1, 0), (1, 1))$.

In [4] the following interesting theorem was proved to show that every noncyclic relatively nonexpansive mapping is nonexpansive on its domain (see also [9] for the same problem in CAT(0) spaces).

Theorem 4.2. *(Proposition 3.2 of [4]) Let (A, B) be a nonempty, bounded, closed, convex and proximinal pair in a Hilbert space $\mathbb{H}$. If $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping, that is, T is noncyclic and $\|Tx - Ty\| \leq \|x - y\|$ for all $(x, y) \in A \times B$, then T is nonexpansive on $A \cup B$.*

The main purpose of this section is to obtain a similar result to Theorem 4.2 for noncyclic relatively u-continuous mappings in the setting of CAT(0) spaces. In order to do this, we recall the following geometric notion of geodesic spaces.

Definition 4.3. *A geodesic metric space (X, d) is said to satisfy condition* (C) *provided that for any $a, b, u \in X$ and $R, r \geq 0$ with $R \geq \max\{d(u, a), d(u, b)\}$ and $r \leq d(u, \frac{1}{2}a \oplus \frac{1}{2}b)$, we have*

$$\frac{d(a, b)}{2} \leq \sqrt{R^2 - r^2}.$$

For example every CAT(0) space satisfies condition (C) (see [7], p. 177).

We are now ready to state the main conclusion of this section.

Theorem 4.4. *Let (A, B) be a nonempty, closed, convex and proximinal pair in a complete* CAT(0) *space X. If $T : A \cup B \to A \cup B$ is a noncyclic relatively u-continuous mapping, then T is a continuous mapping.*

Proof. Let $x \in A$ be an arbitrary element. Since (A, B) is proximinal, there exists $y \in B$ for which $d(x, y) = \operatorname{dist}(A, B)$. By the u-continuity of T, for any positive integer n there is a $\delta_n > 0$ and a neighborhood of x defined as $U(x, \delta_n) := \{u \in A_0 : d(u, x) < \delta_n\}$, such that $u \in U(x, \delta_n)$ implies

$$d(Tu, Ty) \leq \frac{1}{n} + \operatorname{dist}(A, B).$$

Let $R_n := \frac{1}{n} + \operatorname{dist}(A, B)$ and $r = \operatorname{dist}(A, B)$. Then for $u \in U(x, \delta_n)$ we have

$$r \leq d(Ty, [\frac{1}{2}Tu \oplus \frac{1}{2}Tx]), \quad \max\{d(Ty, Tx), d(Ty, Tu)\} \leq R_n.$$

Since the space X satisfies condition (C), $\frac{d(Tu,Tx)}{2} \leq \sqrt{R_n^2 - r^2}$, and so,

$$d(Tu, Tx) \leq 2\sqrt{(\frac{1}{n})^2 + (\frac{2}{n})\operatorname{dist}(A, B)} := \varepsilon_n.$$

For any $\varepsilon > 0$, choose N sufficiently large, such that $n \geq N$ implies $\varepsilon_n < \varepsilon$. Then for $u \in U(x, \delta_n)$ we have $d(Tu, Tx) < \varepsilon$, which implies that T is continuous on A. Similarly, we can see that T is continuous on B and this completes the proof. □

The next result follows immediately from Theorem 4.4.

Theorem 4.5. *Let (A, B) be a nonempty, closed, convex and proximinal pair in a complete* CAT(0) *space X with the property that the closed convex hull of every finite subset of X is compact. Then every noncyclic relatively u-continuous mapping $T : A \cup B \to A \cup B$ whose image $T(A)$ is relatively compact has a best proximity pair.*

Proof. It follows from Theorem 4.4 that T is continuous on $A \cup B$ and so $T|_A : A \to A$ is continuous. Now from Theorem 1.3 of [19], T has a fixed point in A. Let $p \in A$ be such that $p = Tp$. Since (A, B) is proximinal, there exists a point $q \in B$ for which $d(p, q) = \operatorname{dist}(A, B)$. In view of the fact that T is relatively u-continuous,

$$d(p, Tq) = d(Tp, Tq) = \operatorname{dist}(A, B),$$

which ensures that $q = Tq$. Thereby, (p, q) is a best proximity pair of T. □

Remark 4.6. *It is interesting to note that the considered pair (A, B) in Theorem 4.5 need not be compact. Thus, our result does not follow from Theorem 4.4 of [11].*

Let us illustrate Theorem 4.5 with the following example.

Example 4.2. Consider $X = \mathbb{R}^2$ with the *radial metric* defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} \rho((x_1, y_1), (x_2, y_2)) & \text{if} \quad (0,0), (x_1, y_1), (x_2, y_2) \text{ are collinear,} \\ \rho((x_1, y_1), (0,0)) + \rho((x_2, y_2), (0,0)) & \text{otherwise,} \end{cases}$$

where ρ denotes the usual Euclidean metric on $\mathbb{R}^2$. Then (X, d) is a complete CAT(0) space. (see [15]). Let $A = \{(x, 0) : x \geq 0\}$ and $B = \{(y, 1) : y \geq 0\}$. Then (A, B) is a convex and proximinal pair which is noncompact. Besides, for $((x, 0), (y, 1)) \in A \times B$ we have $d((x, 0), (y, 1)) = x + \sqrt{y^2 + 1}$ which implies that $\operatorname{dist}(A, B) = 1$. Consider the noncyclic mapping $T : A \cup B \to A \cup B$ defined by

$$T(x, 0) = (\frac{x}{x+1}, 0), \quad T(y, 1) = \begin{cases} (0, 1) \text{ if } y \in \mathbb{Q} \cap [0, \infty), \\ (y, 1) \text{ if } y \in \mathbb{Q}^c \cap [0, \infty). \end{cases}$$

In this case, we obtain

$$d(T(x, 0), T(y, 1)) = \begin{cases} \frac{x}{x+1} + 1 & \text{if} \quad y \in \mathbb{Q} \cap [0, \infty) \\ \frac{x}{x+1} + \sqrt{y^2 + 1} & \text{if} \quad y \in \mathbb{Q}^c \cap [0, \infty) \end{cases} \leq x + \sqrt{y^2 + 1} = d((x, 0), (y, 1)).$$

Also, $T(A) = [0, \frac{1}{2}] \times \{0\}$, that is, $T|_A$ is compact. Now by Theorem 4.5 T has a best proximity pair which is the point $((0, 0), (0, 1))$.

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