



Research Article

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On trigonometric approximation of functions in the L^q norm

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Abstract: In this paper we obtain a degree of approximation of functions in L^q by operators associated with their Fourier series using integral modulus of continuity. These results generalize many known results and are proved under less stringent conditions on the infinite matrix.

Keywords: class $\text{Lip}(\beta, q)$, trigonometric approximation, L^q norm

MSC: 42A10, 41A25

1 Introduction

Let f be 2π periodic and $f \in L^q[0, 2\pi]$ for $1 \leq q < \infty$. Denote by

$$S_m(f) = S_m(f; x) = \frac{a_0}{2} + \sum_{k=1}^m (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^m U_k(f; x)$$

the partial sum of the first $(m+1)$ terms of the Fourier series of f at a point x , and by

$$\omega_q(f; \delta) = \sup_{0 \leq |h| \leq \delta} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x)|^q dx \right\}^{\frac{1}{q}};$$

the integral modulus of continuity of $f \in L^q$. If $\omega_q(f; \delta) = O(\delta^\beta)$ for $\beta \in (0, 1]$, then we will write $f \in \text{Lip}(\beta, q)$.

Throughout $\|\cdot\|_{L^q}$ will denote L^q norm, defined by

$$\|f\|_{L^q} = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^q dx \right\}^{\frac{1}{q}} \quad (f \in L^q (q \geq 1)).$$

In the present paper, we shall consider an approximation of $f \in L^q$ by trigonometric polynomials $T_m(f; x)$, where

$$T_m(f; x) = T_m(f, A; x) := \sum_{k=0}^m a_{m,k} S_k(f; x) \quad (m = 0, 1, 2, \dots)$$

and $A := (a_{m,k})_{k=0}^\infty$ is a lower triangular infinite matrix of real numbers such that:

$$a_{m,k} \geq 0 \text{ for } k \leq m \text{ and } a_{m,k} = 0 \text{ for } k > m \quad (k, m = 0, 1, 2, \dots) \quad (1.1)$$

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and

$$\sum_{k=0}^m a_{m,k} = 1 \quad (m = 0, 1, 2, \dots). \quad (1.2)$$

If $a_{m,k} = \frac{p_k}{P_m}$, where $P_m = p_0 + p_1 + \dots + p_m \neq 0$ ($m \geq 0$), then we shall define the trigonometric polynomials by

$$R_m(f; x) = \frac{1}{P_m} \sum_{k=0}^m p_k S_k(f; x) \quad (m = 0, 1, 2, \dots).$$

The case $a_{m,k} = \frac{1}{m+1}$ for $k \leq m$ and $a_{m,k} = 0$ for $k > m$ of $T_m(f; x)$ yields

$$\sigma_m(f; x) = \frac{1}{m+1} \sum_{k=0}^m S_k(f; x) \quad (m = 0, 1, 2, \dots).$$

We shall also use the notations

$$\Delta a_k = a_k - a_{k+1}, \quad \Delta_k a_{m,k} = a_{m,k} - a_{m,k+1}$$

and we shall write $I_1(m) \ll I_2(m)$ if there exists a positive constant K , independent of m , such that $I_1(m) \leq KI_2(m)$.

Let $C = (C_m)_{m=0}^\infty = \left(\frac{1}{m+1} \sum_{k=0}^m c_k \right)_{m=0}^\infty$, where $c := (c_m)_{m=0}^\infty$ is a sequence of nonnegative numbers. The sequence c is called a nondecreasing (resp. nonincreasing) mean sequence, briefly *NDMS* (resp. *NIMS*), if $C \in NDS$ (resp. $C \in NIS$), where *NDS* (resp. *NIS*) is the class of nonnegative and nondecreasing (resp. nonincreasing) sequences.

A nonnegative sequence $c = (c_m)_{m=0}^\infty$ is called almost monotone decreasing (*AMDS*) (resp. increasing (*AMIS*)) if there exists a constant $K := K(c)$, depending on the sequence c only, such that for all $m \geq \mu$

$$c_m \leq Kc_\mu \quad (\text{resp. } Kc_m \geq c_\mu).$$

Such sequences will be denoted by $c \in \text{AMDS}$ and $c \in \text{AMIS}$, respectively.

If $C \in \text{AMDS}$ (resp. $C \in \text{AMIS}$), then we shall say that c is an almost monotone decreasing (resp. increasing) mean sequence, briefly $c \in \text{AMDMS}$ (resp. $c \in \text{AMIMS}$).

When we write that a sequence $(a_{m,k})_{k=0}^\infty$ belongs to one of the above classes, it means that it satisfies the required conditions from the above definitions with respect to $k = 0, 1, 2, \dots, m$ for all m .

Now, we define two classes of sequences (see [1]).

First, a sequence $c = (c_m)_{m=0}^\infty$ of nonnegative numbers tending to zero is called a rest bounded variation sequence (resp. rest bounded variation mean sequence), or briefly $c \in \text{RBVS}$ (resp. $c \in \text{RBVMS}$), if it has the property

$$\sum_{k=\mu}^\infty |\Delta c_k| \leq K(c) c_\mu \quad \left(\text{resp. } \sum_{k=\mu}^\infty |\Delta C_k| \leq K(c) C_\mu \right) \quad (1.3)$$

for all natural numbers μ , where $K(c)$ is a constant depending only on c .

Secondly, a sequence $c = (c_m)_{m=0}^\infty$ of nonnegative numbers will be called a head bounded variation sequence (resp. head bounded variation mean sequence), or briefly $c \in \text{HBVS}$ (resp. $c \in \text{HBVMS}$), if it has the property

$$\sum_{k=0}^{\mu-1} |\Delta c_k| \leq K(c) c_\mu \quad \left(\text{resp. } \sum_{k=0}^{\mu-1} |\Delta C_k| \leq K(c) C_\mu \right) \quad (1.4)$$

for all natural numbers μ , or only for all $\mu \leq N$ if the sequence c has only a finite number of nonzero terms and the last nonzero terms is c_N .

Therefore we assume that the sequence $(K(\alpha_m))_{m=0}^\infty$ is bounded, that is, there exists a constant K such that

$$0 \leq K(\alpha_m) \leq K$$

holds for all m , where $K(\alpha_m)$ denote the sequence of constants appearing in the inequalities (1.3) or (1.4) for the sequence $\alpha_m = (a_{m,k})_{k=0}^\infty$. Now we can mention the conditions to be used later on. Let $A_{m,\mu} = \frac{1}{\mu+1} \sum_{k=0}^\mu a_{m,k}$. We assume that for all m and $0 \leq \mu \leq m$

$$\sum_{k=\mu}^\infty |\Delta_k a_{m,k}| \leq K a_{m,\mu} \quad \left(\text{resp. } \sum_{k=\mu}^\infty |\Delta_k A_{m,k}| \leq K A_{m,\mu} \right)$$

and

$$\sum_{k=0}^{\mu-1} |\Delta_k a_{m,k}| \leq K a_{m,\mu} \quad \left(\text{resp. } \sum_{k=0}^{\mu-1} |\Delta_k A_{m,k}| \leq K A_{m,\mu} \right)$$

hold if $\alpha_m = (a_{m,k})_{k=0}^\infty$ belongs to *RBVS* (resp. *RBVMS*) or *HBVS* (resp. *HBVMS*), respectively.

It is clear that

$$\begin{aligned} NIS &\subset RBVS \subset AMDS, \\ NIMS &\subset RBVMS \subset AMDMS \end{aligned}$$

and

$$\begin{aligned} NDS &\subset HBVS \subset AMIS, \\ NDMS &\subset HBVMS \subset AMIMS. \end{aligned}$$

In the present paper we shall show that $NIS \subset NIMS$, $AMDS \subset AMDMS$, $NDS \subset NDMS$ and $AMIS \subset AMIMS$, too.

In 1937 E. Quade [2] proved that, if $f \in \text{Lip}(\beta, q)$ for $0 < \beta \leq 1$, then $\|\sigma_m(f) - f\|_{L^q} = O(m^{-\beta})$ for either $q > 1$ and $0 < \beta \leq 1$ or $q = 1$ and $0 < \beta < 1$. He also showed that, if $q = \beta = 1$, then $\|\sigma_m(f) - f\|_{L^1} = O(m^{-1} \log(m+1))$.

There are several generalizations of the above result for $q > 1$ (see, for example [3–5], [6] and [7]). In [8], P. Chandra extended the work of E. Quade and proved the following theorems:

Theorem 1. Let $f \in \text{Lip}(\beta, q)$ and let $(p_m)_{m=0}^\infty$ be positive. Suppose that

- either
 - (i) $q > 1$, $0 < \beta \leq 1$, and
 - (ii) $\sum_{k=0}^{m-1} \left| \Delta_k \left(\frac{p_k}{k+1} \right) \right| = O\left(\frac{p_m}{m+1} \right)$
- or
 - (i) $q = 1$, $0 < \beta < 1$, and
 - (ii) $(p_m)_{m=0}^\infty$ is nondecreasing

and

$$(m+1)p_m = O(P_m). \quad (1.5)$$

Then

$$\|R_m(f) - f\|_{L^q} = O(m^{-\beta}).$$

Theorem 2. Let $f \in \text{Lip}(1, 1)$ and let $(p_m)_{m=0}^\infty$ with (1.5) be positive, and that

$$((m+1)^{-\eta} p_m) \in NDS \text{ for some } \eta > 0.$$

Then

$$\|R_m(f) - f\|_{L^1} = O(m^{-1}).$$

In [9] M. Mittal, B. Rhoades, V. Mishra and U. Singh obtained the same degree of approximation as in above theorems, for a more general class of lower triangular matrices satisfying (1.1), and deduced some of the results of P. Chandra. Namely, they proved the following theorem:

Theorem 3. Let $f \in \text{Lip}(\beta, q)$ and $(a_{m,k})_{k=0}^{\infty}$ satisfy (1.1). Suppose $(a_{m,k})_{k=0}^{\infty} \in \text{NDS}$ or $(a_{m,k})_{k=0}^{\infty} \in \text{NIS}$ and

$$\left| \sum_{k=0}^m a_{m,k} - 1 \right| = O(m^{-\beta}).$$

(i) If $q > 1$, $0 < \beta < 1$, $(m+1) \max\{a_{m,0}, a_{m,r}\} = O(1)$, where $r := [\frac{m}{2}]$, then

$$\|T_m(f) - f\|_{L^q} = O(m^{-\beta}). \quad (1.6)$$

(ii) If $q > 1$, $\beta = 1$, then (1.6) is satisfied.

(iii) If $q = 1$, $0 < \beta < 1$, and $(m+1) \max\{a_{m,0}, a_{m,m}\} = O(1)$, then (1.6) is satisfied.

In this paper we shall prove that the above mentioned theorems are valid with less stringent assumptions. Similar results in the case of the Nörlund type summability in [10] were obtained.

2 Statement of the results

Our first theorem deals with some embedding results.

Theorem 4. The following embedding relations are valid:

- (i) $\text{NIS} \subset \text{NIMS}$,
- (ii) $\text{NDS} \subset \text{NDMS}$,
- (iii) $\text{AMDS} \subset \text{AMDMS}$,
- (iv) $\text{AMIS} \subset \text{AMIMS}$.

Our next theorem deals with the degree of convergence of operators involving an infinite matrix.

Theorem 5. Assume that $f \in \text{Lip}(\beta, q)$, and both (1.1) and (1.2) hold. If one of the following conditions

- (i) $q > 1$, $0 < \beta < 1$ and $(a_{m,k})_{k=0}^{\infty} \in \text{AMIMS}$,
- (ii) $q > 1$, $0 < \beta < 1$, $(a_{m,k})_{k=0}^{\infty} \in \text{AMDMS}$ and $(m+1)a_{m,0} = O(1)$,
- (iii) $q > 1$, $\beta = 1$ and $\sum_{k=0}^{m-1} |\Delta_k a_{m,k}| = O(m^{-1})$,
- (iv) $q = 1$, $0 < \beta < 1$, $\sum_{k=0}^{m-1} |\Delta_k a_{m,k}| = O(m^{-1})$ and $(m+1)a_{m,m} = O(1)$,
- (v) $q = 1$, $0 < \beta < 1$, $(a_{m,k})_{k=0}^{\infty} \in \text{RBVS}$ and $(m+1)a_{m,0} = O(1)$,
- (vi) $q = \beta = 1$, $((k+1)^{-\gamma} a_{m,k})_{k=0}^{\infty} \in \text{HBVS}$ for some $\gamma > 0$ and $(m+1)a_{m,m} = O(1)$

holds, then

$$\|T_m(f) - f\|_{L^q} = O(m^{-\beta}). \quad (2.1)$$

Remark 1. Let $f \in \text{Lip}(\beta, q)$, (1.1) and

$$\left| \sum_{k=0}^m a_{m,k} - 1 \right| = O(m^{-\beta})$$

hold. Under the assumptions of Theorem 5 (i) – (vi) we can observe that the estimation (2.1) is true, too.

In the special cases in which $a_{m,k} = \frac{p_k}{P_m}$, where $P_m = p_0 + p_1 + \dots + p_m \neq 0$, we can derive from Theorem 5 the following corollary:

Corollary 1. Assume that $f \in \text{Lip}(\beta, q)$ and $(p_k)_{k=0}^\infty$ is a positive sequence. If one of the following conditions

- (i) $q > 1, 0 < \beta < 1$ and $(p_k)_{k=0}^\infty \in \text{AMIMS}$,
- (ii) $q > 1, 0 < \beta < 1, (p_k)_{k=0}^\infty \in \text{AMDMS}$ and $(m+1) = O(P_m)$,
- (iii) $q > 1, \beta = 1$ and $\sum_{k=0}^{m-1} \left| \Delta_k \frac{p_k}{k+1} \right| = O\left(\frac{p_m}{m}\right)$,
- (iv) $q = 1, 0 < \beta < 1, \sum_{k=0}^{m-1} |\Delta_k p_k| = O\left(\frac{p_m}{m}\right)$ and $(m+1)p_m = O(P_m)$,
- (v) $q = 1, 0 < \beta < 1, (p_k)_{k=0}^\infty \in \text{RBVS}$ and $(m+1) = O(P_m)$,
- (vi) $q = \beta = 1, ((k+1)^{-\gamma} p_k)_{k=0}^\infty \in \text{HBVS}$ for some $\gamma > 0$ and $(m+1)p_m = O(P_m)$,

holds, then

$$\|R_m(f) - f\|_{L^q} = O\left(m^{-\beta}\right).$$

Remark 2. We can observe that from Remark 1 and Theorem 4 we can obtain Theorem 3. Analogously, the first case of Theorem 1 ($q > 1$) follows from Theorem 4 and Corollary 1 ((i), (iii)) but the second case of Theorem 1 ($q = 1$) follows from Corollary 1 (iv). Moreover, since $\text{NDS} \subset \text{HBVS}$, we can derive from Corollary 1 (vi) an analogous estimate, as in Theorem 2, for the deviation $R_m(f) - f$ in the L^q norm.

3 Auxiliary results

We shall use the following lemmas for the proof of our theorems:

Lemma 1. [2, Theorem 4] Let $f \in \text{Lip}(\beta, q)$, $q \geq 1, 0 < \beta \leq 1$. Then, for any positive integer m , f may be approximated in L^q space by a trigonometric polynomial t_m of degree m such that

$$\|f - t_m\|_{L^q} = O\left(m^{-\beta}\right).$$

Lemma 2. [2, Theorem 5 (i)] If $f \in \text{Lip}(\beta, q)$ for $0 < \beta \leq 1$ and $q > 1$, then

$$\|S_m(f) - f\|_{L^q} = O\left(m^{-\beta}\right).$$

Lemma 3. [2, p. 541, last line] Let $f \in \text{Lip}(1, q)$ ($q > 1$). Then

$$\|\sigma_m(f) - S_m(f)\|_{L^q} = O\left(m^{-1}\right).$$

Lemma 4. [2, Theorem 6 (i), p. 541] Let $f \in \text{Lip}(\beta, 1)$, $0 < \beta < 1$. Then

$$\|\sigma_m(f) - f\|_{L^1} = O\left(m^{-\beta}\right).$$

Lemma 5. Let (1.1) and (1.2) hold. If $(a_{m,k})_{k=0}^\infty \in \text{AMIMS}$ or $(a_{m,k})_{k=0}^\infty \in \text{AMDMS}$ and $(m+1)a_{m,0} = O(1)$, then, for $0 < \beta < 1$,

$$\sum_{k=0}^m (k+1)^{-\beta} a_{m,k} = O\left((m+1)^{-\beta}\right)$$

holds.

Proof. Let $r = \left[\frac{m}{2}\right]$. If (1.1) and (1.2) hold, then

$$\begin{aligned} \sum_{k=0}^m (k+1)^{-\beta} a_{m,k} &\leq \sum_{k=0}^r (k+1)^{-\beta} a_{m,k} + (r+1)^{-\beta} \sum_{k=r+1}^m a_{m,k} \\ &\leq \sum_{k=0}^r (k+1)^{-\beta} a_{m,k} + (r+1)^{-\beta}. \end{aligned}$$

By the Abel transformation, we obtain

$$\begin{aligned} \sum_{k=0}^m (k+1)^{-\beta} a_{m,k} &\leq \sum_{k=0}^{r-1} \left\{ (k+1)^{-\beta} - (k+2)^{-\beta} \right\} \sum_{i=0}^k a_{m,i} \\ &+ (r+1)^{-\beta} \sum_{k=0}^r a_{m,k} + (r+1)^{-\beta} \leq \sum_{k=0}^{r-1} \frac{(k+2)^{\beta} - (k+1)^{\beta}}{(k+1)^{\beta-1} (k+2)^{\beta}} A_{m,k} + (r+1)^{-\beta}. \end{aligned}$$

Using the Lagrange mean value theorem to the function $f(x) = x^{\beta}$ ($0 < \beta < 1$) on the interval $(k+1, k+2)$ we obtain

$$\sum_{k=0}^m (k+1)^{-\beta} a_{m,k} \leq \sum_{k=0}^{r-1} \frac{\beta}{(k+2)^{\beta}} A_{m,k} + (r+1)^{-\beta}.$$

If $(a_{m,k})_{k=0}^{\infty} \in AMIMS$, then

$$\begin{aligned} \sum_{k=0}^m (k+1)^{-\beta} a_{m,k} &\ll A_{m,r} \sum_{k=0}^{r-1} \frac{1}{(k+2)^{\beta}} + (r+1)^{-\beta} \\ &\ll (r+1)^{-\beta} \sum_{k=0}^r a_{m,k} + (r+1)^{-\beta} \ll (m+1)^{-\beta}. \end{aligned}$$

When $(a_{m,k})_{k=0}^{\infty} \in AMDMS$ and $(m+1)a_{m,0} = O(1)$ we get

$$\begin{aligned} \sum_{k=0}^m (k+1)^{-\beta} a_{m,k} &\ll A_{m,0} \sum_{k=0}^{r-1} \frac{1}{(k+2)^{\beta}} + (r+1)^{-\beta} \\ &\ll (r+1)^{1-\beta} a_{m,0} + (r+1)^{-\beta} \ll (m+1)^{-\beta}. \end{aligned}$$

This completes our proof. □

4 Proofs of the results

4.1 Proof of Theorem 4

(i) If $(a_m)_{m=0}^{\infty} \in NIS$, then

$$\begin{aligned} (m+2) \sum_{k=0}^m a_k &= (m+1) \sum_{k=0}^{m+1} a_k + \sum_{k=0}^m a_k - (m+1) a_{m+1} \\ &\geq (m+1) \sum_{k=0}^{m+1} a_k + (m+1)(a_m - a_{m+1}) \geq (m+1) \sum_{k=0}^{m+1} a_k. \end{aligned}$$

Thus

$$\frac{1}{m+2} \sum_{k=0}^{m+1} a_k \leq \frac{1}{m+1} \sum_{k=0}^m a_k$$

and $(a_m)_{m=0}^{\infty} \in NIMS$.

(ii) Let $(a_m)_{m=0}^{\infty} \in NDS$. Hence

$$(m+2) \sum_{k=0}^m a_k = (m+1) \sum_{k=0}^{m+1} a_k + \sum_{k=0}^m a_k - (m+1) a_{m+1}$$

$$\leq (m+1) \sum_{k=0}^{m+1} a_k + (m+1)(a_m - a_{m+1}) \leq (m+1) \sum_{k=0}^{m+1} a_k.$$

Therefore

$$\frac{1}{m+1} \sum_{k=0}^m a_k \leq \frac{1}{m+2} \sum_{k=0}^{m+1} a_k$$

and $(a_m)_{m=0}^\infty \in NDMS$.

(iii) Suppose that $(a_m)_{m=0}^\infty \in AMDS$. We have for $\mu \leq l$

$$\begin{aligned} (l+1) \sum_{i=0}^{\mu} a_i &= (\mu+1) \sum_{i=0}^{\mu} a_i + (l-\mu) \sum_{i=0}^{\mu} a_i \\ &\geq (\mu+1) \left\{ \sum_{i=0}^{\mu} a_i + \frac{1}{K} (l-\mu) a_{\mu} \right\} \geq (\mu+1) \left\{ \sum_{i=0}^{\mu} a_i + \frac{1}{K^2} \sum_{i=\mu+1}^l a_i \right\} \\ &\geq \min \left\{ 1, \frac{1}{K^2} \right\} (\mu+1) \sum_{i=0}^l a_i. \end{aligned}$$

Hence

$$\frac{1}{\min \left\{ 1, \frac{1}{K^2} \right\}} \cdot \frac{1}{\mu+1} \sum_{i=0}^{\mu} a_i \geq \frac{1}{l+1} \sum_{i=0}^l a_i$$

and $(a_m)_{m=0}^\infty \in AMDMS$.

(iv) If $(a_m)_{m=0}^\infty \in AMIS$, then for $\mu \leq l$ we get

$$\begin{aligned} (l+1) \sum_{i=0}^{\mu} a_i &\leq (\mu+1) \left\{ \sum_{i=0}^{\mu} a_i + K(l-\mu) a_{\mu} \right\} \\ &\leq (\mu+1) \left\{ \sum_{i=0}^{\mu} a_i + K^2 \sum_{i=\mu+1}^l a_i \right\} \leq \max \left\{ 1, K^2 \right\} (\mu+1) \sum_{i=0}^l a_i. \end{aligned}$$

Thus

$$\frac{1}{\mu+1} \sum_{i=0}^{\mu} a_i \leq \max \left\{ 1, K^2 \right\} \frac{1}{l+1} \sum_{i=0}^l a_i$$

and $(a_m)_{m=0}^\infty \in AMIMS$.

The proof is now complete. \square

4.2 Proof of Theorem 5

We prove the cases (i) and (ii) together utilizing Lemmas 4 and 5. Since

$$T_m(f; x) - f(x) = \sum_{k=0}^m a_{m,k} (S_k(f; x) - f(x)),$$

there

$$\|T_m(f) - f\|_{L^q} \leq \sum_{k=0}^m a_{m,k} \|S_k(f) - f\|_{L^q} \ll \sum_{k=0}^m (k+1)^{-\beta} a_{m,k} = O(m^{-\beta})$$

and this is the statemet of (2.1).

Next we consider the case (iii).

Using twice the Abel transformation and (1.2) we obtain

$$\begin{aligned}
 T_m(f; x) - f(x) &= \sum_{k=0}^m a_{m,k} (S_k(f; x) - f(x)) \\
 &= \sum_{k=0}^{m-1} (S_k(f; x) - S_{k+1}(f; x)) \sum_{i=0}^k a_{m,i} + S_m(f; x) - f(x) \\
 &= S_m(f; x) - f(x) - \sum_{k=0}^{m-1} (k+1) U_{k+1}(f; x) A_{m,k} \\
 &= S_m(f; x) - f(x) - \sum_{k=0}^{m-2} (A_{m,k} - A_{m,k+1}) \sum_{i=0}^k (i+1) U_{i+1}(f; x) \\
 &\quad - A_{m,m-1} \sum_{k=0}^{m-1} (k+1) U_{k+1}(f; x) = S_m(f; x) - f(x) \\
 &\quad - \sum_{k=0}^{m-2} (A_{m,k} - A_{m,k+1}) \sum_{i=0}^k (i+1) U_{i+1}(f; x) - \frac{1}{m} \sum_{i=0}^{m-1} a_{m,i} \sum_{k=0}^{m-1} (k+1) U_{k+1}(f; x).
 \end{aligned}$$

Hence

$$\begin{aligned}
 \|T_m(f) - f\|_{L^q} &\leq \|S_m(f) - f\|_{L^q} \\
 &\quad + \sum_{k=0}^{m-2} |A_{m,k} - A_{m,k+1}| \left\| \sum_{i=1}^{k+1} i U_i(f) \right\|_{L^q} + \frac{1}{m} \left\| \sum_{k=1}^m k U_k(f; x) \right\|_{L^q}.
 \end{aligned} \tag{4.1}$$

Since

$$\sigma_m(f; x) - S_m(f; x) = \frac{1}{m+1} \sum_{k=1}^m k U_k(f; x),$$

thus by Lemma 3

$$\left\| \sum_{k=1}^m k U_k(f) \right\|_{L^q} = (m+1) \|\sigma_m(f) - S_m(f)\|_{L^q} = O(1). \tag{4.2}$$

By (4.1), (4.2) and Lemma 4 we get

$$\|T_m(f) - f\|_{L^q} \ll \frac{1}{m} + \sum_{k=0}^{m-1} |A_{m,k} - A_{m,k+1}|.$$

If $\sum_{k=0}^{m-1} |\Delta_k A_{m,k}| = O(m^{-1})$, then

$$\|T_m(f) - f\|_{L^q} = O(m^{-1})$$

and (2.1) holds.

The cases (iv) and (v) we also prove together. By the Abel transformation

$$\begin{aligned}
 T_m(f; x) - f(x) &= \sum_{k=0}^m a_{m,k} (S_k(f; x) - f(x)) \\
 &= \sum_{k=0}^{m-1} (a_{m,k} - a_{m,k+1}) \sum_{i=0}^k (S_i(f; x) - f(x)) + a_{m,m} \sum_{k=0}^m (S_k(f; x) - f(x)) \\
 &= \sum_{k=0}^{m-1} (a_{m,k} - a_{m,k+1}) (k+1) (\sigma_k(f; x) - f(x)) + a_{m,m} (m+1) (\sigma_m(f; x) - f(x)).
 \end{aligned}$$

Using Lemma 2 we get

$$\begin{aligned} \|T_m(f) - f\|_{L^1} &\leq \sum_{k=0}^{m-1} |a_{m,k} - a_{m,k+1}| (k+1) \|\sigma_k(f) - f\|_{L^1} \\ &+ a_{m,m} (m+1) \|\sigma_m(f) - f\|_{L^1} \ll \sum_{k=0}^{m-1} |a_{m,k} - a_{m,k+1}| (k+1)^{1-\beta} \\ &+ a_{m,m} (m+1)^{1-\beta} \leq (m+1)^{1-\beta} \left(\sum_{k=0}^{m-1} |a_{m,k} - a_{m,k+1}| + a_{m,m} \right). \end{aligned}$$

When the assumptions of (iv) hold we get

$$\|T_m(f) - f\|_{L^1} = O(m^{-\beta}).$$

If $(a_{m,k})_{k=0}^\infty \in RBVS$, then $(a_{m,k})_{k=0}^\infty \in AMDS$. Thus

$$\|T_m(f) - f\|_{L^1} \ll (m+1)^{1-\beta} (a_{m,0} + a_{m,m}) \ll (m+1)^{1-\beta} a_{m,0}.$$

Hence, if $(m+1)a_{m,0} = O(1)$, then (2.1) holds. This ends the proof of the cases (iv) and (v).

Finally, we prove the case (vi). Let t_m be a trigonometric polynomial of degree m . Then for $\mu \leq m$,

$$S_\mu(t_m; x) = t_\mu \quad \text{and} \quad S_\mu(f; x) - t_\mu = S_\mu(f - t_m; x).$$

Thus

$$T_m(f; x) - \sum_{k=0}^m a_{m,k} t_k(x) = \sum_{k=0}^m a_{m,k} S_k(f - t_m; x),$$

where

$$S_k(f - t_m; x) = \frac{1}{\pi} \int_0^{2\pi} \{f(x+u) - t_m(x+u)\} \frac{\sin(k + \frac{1}{2})u}{2 \sin \frac{u}{2}} du.$$

By a general form of the Minkowski inequality we get

$$\begin{aligned} \left\| T_m(f) - \sum_{k=0}^m a_{m,k} t_k \right\|_{L^1} &\leq \frac{1}{2\pi^2} \int_0^{2\pi} |K_m(u)| \int_0^{2\pi} |f(x+u) - t_m(x+u)| dx du \\ &= \frac{1}{2\pi^2} \int_0^{2\pi} |K_m(u)| du \int_0^{2\pi} |f(x) - t_m(x)| dx = \frac{1}{\pi} \|f - t_m\|_{L^1} \int_0^{2\pi} |K_m(u)| du \\ &= \frac{2}{\pi} \|f - t_m\|_{L^1} \int_0^\pi |K_m(u)| du = \frac{2}{\pi} \|f - t_m\|_{L^1} \left(\int_0^{\pi/m} |K_m(u)| du + \int_{\pi/m}^\pi |K_m(u)| du \right) \\ &= \frac{2}{\pi} \|f - t_m\|_{L^1} (I_1 + I_2), \end{aligned} \tag{4.3}$$

where

$$K_m(u) = \sum_{k=0}^m a_{m,k} \frac{\sin(k + \frac{1}{2})u}{2 \sin \frac{u}{2}}.$$

Now we estimate the quantities I_1 and I_2 . By (1.2)

$$I_1 \ll \int_0^{\pi/m} \sum_{k=0}^m (k+1) a_{m,k} du = O(1). \tag{4.4}$$

If $((k+1)^{-\gamma} a_{m,k})_{k=0}^{\infty} \in HBVS$, then $((k+1)^{-\gamma} a_{m,k})_{k=0}^{\infty} \in AMIS$. Hence, for $0 \leq l \leq \mu \leq m$,

$$K a_{m,\mu} \geq a_{m,l} \left(\frac{\mu+1}{l+1} \right)^{\gamma} \geq a_{m,l}.$$

Thus $(a_{m,k})_{k=0}^{\infty} \in AMIS$. Using this and the assumption $(m+1) a_{m,m} = O(1)$ we obtain that

$$I_2 \ll a_{m,m} \int_{\pi/m}^{\pi} u^{-2} du = O(1). \quad (4.5)$$

Combining (4.3)-(4.5) we have

$$\left\| T_m(f) - \sum_{k=0}^m a_{m,k} t_k \right\|_{L^1} \ll \|f - t_m\|_{L^1}. \quad (4.6)$$

Further, by using (4.6) and Lemma 1 for $q = \beta = 1$, we get

$$\begin{aligned} \|T_m(f) - f\|_{L^1} &\leq \left\| T_m(f) - \sum_{k=0}^m a_{m,k} t_k \right\|_{L^1} + \left\| \sum_{k=0}^m a_{m,k} t_k - f \right\|_{L^1} \\ &\ll \frac{1}{m} + \left\| \sum_{k=0}^m a_{m,k} t_k - f \right\|_{L^1} \leq \frac{1}{m} + \sum_{k=0}^m a_{m,k} \|t_k - f\|_{L^1} \ll \frac{1}{m} + \sum_{k=0}^m (k+1)^{-1} a_{m,k}. \end{aligned}$$

By the Abel transformation

$$\begin{aligned} \|T_m(f) - f\|_{L^1} &\ll \frac{1}{m} + \sum_{k=0}^{m-1} \left| \frac{a_{m,k}}{(k+1)^{\gamma}} - \frac{a_{m,k+1}}{(k+2)^{\gamma}} \right| \sum_{i=0}^k (i+1)^{\gamma-1} \\ &\quad + \frac{a_{m,m}}{(m+1)^{\gamma}} \sum_{k=0}^m (k+1)^{\gamma-1} \ll \frac{1}{m} + (m+1)^{\gamma} \sum_{k=0}^{m-1} \left| \frac{a_{m,k}}{(k+1)^{\gamma}} - \frac{a_{m,k+1}}{(k+2)^{\gamma}} \right| + a_{m,m}. \end{aligned}$$

Since $((k+1)^{-\gamma} a_{m,k})_{k=0}^{\infty} \in HBVS$ and $(m+1) a_{m,m} = O(1)$, then

$$\|T_m(f) - f\|_{L^1} = O(m^{-1})$$

and (2.1) holds.

This completes the proof of Theorem 5. □

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