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Generalizations of Darbo's fixed point theorem for new condensing operators with application to a functional integral equation

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Abstract: In this paper, we provide some generalizations of the Darbo's fixed point theorem associated with the measure of noncompactness and present some results on the existence of the coupled fixed point theorems for a special class of operators in a Banach space. To acquire this result, we define α-ψ and β-ψ condensing operators and using them we propose new fixed point results. Our results generalize and extend some comparable results from the literature. Additionally, as an application, we apply the obtained fixed point theorems to study the nonlinear functional integral equations.

Keywords: Darbo fixed point theorem, measure of noncompactness, fixed point, coupled fixed point, functional integral equation

MSC: 47H08, 47H09, 47H10

1 Introduction and preliminaries

The study of nonlinear integral equations, nowadays, is a subject of interest for many researchers in nonlinear functional analysis. Integral equations arise in many practical problems including potential theory and other physics-related problems. On the other hand, fixed point theory is one of the most effective and fruitful tool used in nonlinear analysis to solve functional integral equations. It's concerned with the conditions for the existence of one or more fixed points of a mapping T from a topological space X into itself. Brouwer [1] established a fixed point result what has become the well-known Brouwer's fixed point theorem for finite dimensional spaces. While in 1922, Banach [2] introduced his celebrated Banach contraction principle for complete metric spaces which guarantee the existence and uniqueness of fixed point. Afterwards, in 1930, Schauder [3] extended the Brouwer's fixed point theorem to infinite dimensional spaces using the condition of compactness. There are many developments in fixed point theory in various directions, one among them is single-valued mappings (see [4–10] and references therein). Furthermore, Kuratowski [11] in 1930, opened up a new direction of research with the introduction of the concept of a measure of noncompactness, which gives the degree of noncompactness for bounded sets. The measure of noncompactness can also be used in the study of single-valued and multivalued mappings, especially in metric and topological fixed point theory. The measure of noncompactness combining with some algebraic arguments is beneficial for studying mathematical formulations, especially solving the existence of solutions of some nonlinear problems under certain situations.

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The Kuratowski and Hausdorff measure of noncompactness [12, 13] in a metric space are well-known in the literature and the proof of the Darbo's fixed point result depends upon the technique of the measure of noncompactness. The Darbo's fixed point theorem is a very useful generalization of Schauder's fixed point theorem for noncompact operators which is very helpful to solve differential and integral equations. Due to this fact, researchers are always interested to find the extensions and generalizations of the Darbo's fixed point theorem. Up to now, several papers have been published on the generalization of Darbo's fixed point theorem (see [14–18] and references therein) and on the existence and behaviour of solutions of nonlinear differential and integral equations [19–27] using the technique of measure of noncompactnes.

On the other hand, in metric fixed point theory Samet [9] introduced a nice generalization of Banach fixed point theorem using the Definition 1.1, of α-admissible mappings. Inspired from the above mentioned, applied the concept of α-admissible mappings to the Darbo theorem. We generalized the Theorem 1.4, in a more general setting. To attain this result, we define α-ψ and β-ψ condensing operators and moreover, we used the introduced concepts to propose new fixed point results. We also supply some new coupled fixed point results through a measure of noncompactness for more general class of functions. The obtained results generalize and extend well-known results available in the literature. Moreover, some examples and an application to a functional integral equation are given to illustrate the usability of this idea.

Throughout this paper, we will work in a Banach space E with the norm $\|.\|$ and the zero element θ. Denote by $B(x, r)$ the closed ball centered at x with radius r. We use the standard notation λX and $X + Y$ to denote the algebraic operations on sets. Moreover, the symbol $\overline{X}$ stands for the closure of a set X while coX and $\overline{co}X$ denotes the convex hull and closed convex hull of X respectively. Finally, we denote $\mathfrak{M}_E$ by the family of all nonempty bounded subsets of the space E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets of E.

The remaining part of the paper is organized as follows. First, we recall some known definitions and basic tools which are useful to prove our main results with the corresponding references. In Section 2, we give our proposed fixed point results and their implementation to obtain a coupled fixed point results and in solving a functional integral equation. We begin by taking into account the axiomatic definition of the degree of noncompactness.

Definition 1.1. [4] A mapping $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+ = [0, +\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

MNC1. The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is a nonempty set and $ker\mu \subset \mathfrak{N}_E$;
MNC2. $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$;
MNC3. $\mu(X) = \mu(\overline{X})$;
MNC4. $\mu(coX) = \mu(X)$;
MNC5. $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, for all $\lambda \in [0, 1]$;
MNC6. If X_n is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \cdots$, and if $\lim_{n\to\infty} \mu(X_n) = 0$ then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \phi$.

It follows from Definition 1.1, that X_∞ is a member of the family $Ker\mu$. In view that $\mu(X_\infty) \leq \mu(X_n)$ for any n, we can deduce that $\mu(X_\infty) = 0$. This yields that $X_\infty \in ker\mu$.

Definition 1.2. (Compact operator)[28] An operator $T : X \to Y$ is referred to as compact if $T(S)$ is relatively compact in a Banach space Y for any bounded subset S in a Banach space X.

Theorem 1.3. *(Schauder's fixed point theorem)[3] Let C be a nonempty, bounded, closed and convex subset of a Banach space E. Then each continuous and compact map $T : C \to C$ has at least one fixed point in C.*

Theorem 1.4. *(Darbo's fixed point theorem)[29] Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \longrightarrow C$ is a continuous mapping such that there exists a constant $k \in [0, 1)$ such that*

$$\mu(TS) \leq k\mu(S),$$

for any nonempty subset S of C. Then T has a fixed point in the set C.

Denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\lim_{n\to\infty} \psi^n(t) = 0$, for each $t > 0$, where ψ^n is the n-th iteration of ψ.

Lemma 1.5. *For every function $\psi \in \Psi$ the following holds:*
if ψ is nondecreasing, then for each $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ implies $\psi(t) < t$.

Definition 1.6. [9] Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that T is α-admissible if for every $x, y \in X$

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Example 1.7. [9] Let $X = [0, +\infty)$. Define $T : X \to X$ by $Tx = \ln x, \forall x \in X$, and $\alpha : X \times X \to [0, +\infty)$ defined by

$$\alpha(x, y) = \begin{cases} 2 & \text{if} \quad x \geq y, \\ 0 & \text{if} \quad x < y. \end{cases}$$

Then T is α-admissible.

2 Main results

Definition 2.1. (α-ψ condensing operator) Let E be a Banach space and let $T : E \to E$ be a given operator. We say that T is an α-ψ condensing operator if there exist two functions $\alpha : E \times E \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, Tx)\mu(TX) \leq \psi(\mu(X)),$$

for any bounded subset X of E and $x \in X$ with μ an arbitrary measure of noncompactness.

Theorem 2.2. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and there exists $\alpha : E \times E \to [0, +\infty)$ such that $T : C \to C$ is a continuous, α-admissible and α-ψ condensing operator satisfying the following:*

(i) there exist closed and convex $X_0 \subseteq C$ and $x_0 \in X_0$ such that

$$TX_0 \subseteq X_0, \;\; \alpha(x_0, Tx_0) \geq 1, \tag{2.1}$$

where μ be an arbitrary measure of noncompactness and $\psi \in \Psi$. Then T has at least one fixed point in the set C.

Proof. Firstly, define the sequence of the sets $\{X_n\}$ and elements $\{x_n\}$ as follows:

$$X_n = \overline{co}(TX_{n-1}), \quad x_n = Tx_{n-1} \quad \forall\, n \geq 1.$$

Since $TX_0 \subseteq X_0$, thus

$$X_1 = \overline{co}(TX_0) \subseteq X_0,$$
$$X_2 = \overline{co}(TX_1) \subseteq \overline{co}(TX_0) = X_1.$$

Therefore, by continuing this process we obtain $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots X_n \supseteq X_{n+1} \supseteq \cdots$, and also

$$TX_n \subseteq TX_{n-1} \subseteq \overline{co}(TX_{n-1}) = X_n.$$

If there exists an integer $N \geq 0$ such that $\mu(X_N) = 0$, then X_N is a relatively compact set and also $TX_N \subseteq X_N$. Thus, Theorem 1.3, implies that T has fixed point. Next, we assume that $\mu(X_n) \neq 0$ for any $n \geq 0$. From equation (2.1) we have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$, and also T is a α-admissible operator implies that $\alpha(x_1, x_2) \geq 1$. Recursively, we get the following inequality

$$\alpha(x_n, x_{n+1}) \geq 1, \;\forall\, n \geq 0. \tag{2.2}$$

Furthermore, from our assumptions and equation (2.2), we have

$$\begin{aligned}\mu(X_{n+1}) &\le \alpha(x_n, x_{n+1})\mu(X_{n+1})\\ &= \alpha(x_n, Tx_n)\mu\big(\overline{co}(TX_n)\big)\\ &= \alpha(x_n, Tx_n)\mu(TX_n)\\ &\le \psi\big(\mu(X_n)\big),\end{aligned}$$

continuing in this manner, we reach the following inequality

$$\mu(X_{n+1}) \le \psi^n\big(\mu(X_0)\big). \tag{2.3}$$

Thus, equation (2.3) implies that $\mu(X_n) \to 0$ as $n \to \infty$. Since the sequence $\{X_n\}$ is nested so from Definition 1.1 (axiom MNC6), we deduce that the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty, closed and convex subset of the set X_0. On the other hand, $\mu(X_\infty) \le \mu(X_n)$, $\forall\, n \in N$ implies that $\mu(X_\infty) = 0$. Hence we get that X_∞ is a member of the $ker\mu$, which implies X_∞ is compact. Moreover, we have $X_\infty \subset X_n$ and $T(X_n) \subset X_n$ for all $n \in N$. Therefore, $T : X_\infty \to X_\infty$ is well defined. For any bounded $A \subset X_\infty$, we have $T(A) \subset X_\infty$ and $\overline{T(A)}$ is a compact subset of X_∞, implies that T is compact operator. Therefore, Theorem 1.3, completes the proof. □

Remark 2.3. In Theorem 2.2, we get Darbo's theorem if we take $\alpha(x, y) = 1$ and $\psi(t) = kt$ for all $t \ge 0$ and for $k \in [0, 1)$.

Now using the above theorem, we prove the following corollary which belongs to the classical metric fixed point theory.

Corollary 2.4. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and there exists $\alpha : E \times E \to [0, +\infty)$ such that $T : C \to C$ is a continuous and α-admissible operator satisfying the following:*

(i) for any $x, y, u \in X$, we have

$$\alpha(u, Tu)\|Tx - Ty\| \le \psi\big(\|x - y\|\big); \tag{2.4}$$

(ii) there exist closed and convex $X_0 \subseteq C$ and $x_0 \in X_0$ such that

$$TX_0 \subseteq X_0, \;\; \alpha(x_0, Tx_0) \ge 1, \tag{2.5}$$

where $\psi \in \Psi$. Then T has at least one fixed point in C.

Proof. Let $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ be defined as

$$\mu(X) = diamX,$$

where $diamX = \sup\{\|x - y\| : x, y \in X\}$ stands for the diameter of X. It is easy to see that μ is a measure of noncompactness in a space E in the sense of Definition 1.1. Furthermore, from equation (2.4) and ψ being nondecreasing we have

$$\alpha(u, Tu) \sup_{x,y\in X} \|Tx - Ty\| \le \psi\big(\sup_{x,y\in X} \|x - y\|\big),$$

which implies that

$$\alpha(u, Tu)\mu(TX) \le \psi\big(\mu(X)\big),$$

so from Theorem 2.2, we get the desired result. □

Proposition 2.5. *If $\alpha(x, x) \ge 1$ for all $x \in E$, then the set of all fixed points of T in Theorem 2.2, is a compact set.*

Proof. Let $F = \{x \in C : Tx = x\}$ be the set of all fixed points of T and $\mu(F) \ne 0$, then by α-ψ condensing operator of T we have

$$\alpha(x, Tx)\mu(TF) \le \psi\big(\mu(F)\big) < \mu(F),$$

which is a contradiction from above inequality since $T(F) = F$. This implies that F is a relatively compact set. Now taking into account any convergent sequence $\{x_n\} \subset F$ and $x_n \to x^\star$, we have $x^\star \in C$, because C is closed. The continuity of T implies that $x_n = Tx_n \to Tx^\star$ and $Tx^\star = x^\star$, which means that $x^\star \in F$, i.e. F is a compact set. □

Example 2.6. The operator $T : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ defined by

$$Tx(t) = \begin{cases} \frac{x}{2} & \text{if } \quad \|x\| \le 1, \\ 2x - 2 & \text{if } \quad \|x\| > 1, \end{cases}$$

and let $BC(\mathbb{R}_+)$ denote the space of all real-valued bounded and continuous functions on $\mathbb{R}_+$. First, we observe that Theorem 1.4, cannot be applied in the case when $\|x\|, \|y\| > 1$ and we obtain

$$\|Tx(t) - Ty(t)\| = \|2x(t) - 2 - 2y(t) + 2\| = 2\|x(t) - y(t)\|,$$

and by taking supremum value on both sides we have

$$\mu(TX) = 2\mu(X).$$

Now, we define the mapping $\alpha : BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \longrightarrow BC(\mathbb{R}_+)$ by

$$\alpha(x, y) = \begin{cases} 1 & \|x\| \le 1, \\ 0 & \text{otherwise}. \end{cases}$$

Clearly, T is a α-ψ condensing operator with $\psi(t) = \frac{t}{2}$ for $t \ge 0$, and $\mu(X) = diamX$.

In order to prove the next results, we need the following definitions.

Definition 2.7. (β-admissible) Let $T : E \to E$ and $\beta : 2^E \to [0, +\infty)$. We say that T is β-admissible operator if for every $X \in 2^E$, we have

$$\beta(X) \ge 1 \Rightarrow \beta(\overline{co}\ TX) \ge 1.$$

Example 2.8. Let $T : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ be defined by $Tx(t) = \lambda x(t)$ for $\lambda \ge 1$, and also there exist $\beta : 2^{BC(\mathbb{R}_+)} \to \mathbb{R}_+$ such that

$$\beta(X) = diam(X)$$

for every $X \subset BC(\mathbb{R}_+)$. Then T is β-admissible operator.

Example 2.9. Let $T : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ and there exist $\beta : 2^{BC(\mathbb{R}_+)} \to \mathbb{R}_+$ such that

$$Tx(t) = e^{x(t)} \ \ and \ \ \beta(X) = \sup\{\|x\| : x \in X\}$$

for every $x \in BC(\mathbb{R}_+)$, $X \subset BC(\mathbb{R}_+)$ respectively. Then T is β-admissible operator.

Definition 2.10. (β-ψ condensing operator) Let E be a Banach space and let $T : E \to E$ be a given operator. We say that T is β-ψ condensing operator if there exist two functions $\beta : 2^E \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\beta(X)\mu(TX) \le \psi\big(\mu(X)\big), \tag{2.6}$$

for any bounded subset X of E with μ an arbitrary measure of noncompactness.

Theorem 2.11. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and there exists $\beta : 2^E \to [0, +\infty)$ such that $T : C \to C$ is a continuous, β-admissible and β-ψ condensing operator satisfying the following:*

(i) there exist closed and convex $X_0 \subseteq C$ such that

$$TX_0 \subseteq X_0, \quad \beta(X_0) \ge 1, \tag{2.7}$$

where μ be an arbitrary measure of noncompactness and $\psi \in \Psi$. Then T has at least one fixed point in C.

Proof. Similarly, as in the proof of Theorem 2.2, we define the following sequence:

$$X_n = \overline{co}(TX_{n-1}).$$

Since $TX_0 \subseteq X_0$, thus

$$X_1 = \overline{co}(TX_0) \subseteq X_0,$$
$$X_2 = \overline{co}(TX_1) \subseteq \overline{co}(TX_0) = X_1.$$

Therefore, by continuing this process we obtain

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots X_n \supseteq X_{n+1} \supseteq \cdots$$

and

$$TX_n \subseteq TX_{n-1} \subseteq \overline{co}(TX_{n-1}) = X_n.$$

If there exists an integer $N_0 \geq 0$ such that $\mu(X_{N_0}) = 0$, implies X_{N_0} is relatively compact and also $TX_{N_0} \subseteq X_{N_0}$. Thus, Theorem 1.3, implies that T has fixed point. Moreover, we assume that $\mu(X_n) \neq 0$ for all $n \geq 0$. For T to be a β-admissible operator and from equation (2.7), we obtain $\beta(X_1) = \beta(\overline{co}\ TX_0) \geq 1$. Recursively, we obtain the following inequality

$$\beta(X_n) \geq 1, \ \forall\, n \geq 0. \tag{2.8}$$

Furthermore, from equation (2.8) we have

$$\begin{aligned}\mu(X_{n+1}) &\leq \beta(X_n)\mu(X_{n+1})\\ &= \beta(X_n)\mu\big(\overline{co}(TX_n)\big)\\ &= \beta(X_n)\mu(TX_n)\\ &\leq \psi\big(\mu(X_n)\big),\end{aligned}$$

continuing in this manner, we reach at the following inequality

$$\mu(X_{n+1}) \leq \psi^n\big(\mu(X_0)\big). \tag{2.9}$$

Equation (2.9) implies that $\mu(X_n) \to 0$ as $n \to \infty$. Since the sequence $\{X_n\}$ is nested and in view of Definition 1.1(axiom MNC6), we deduce that the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty, closed and convex subset of the set X_0. Hence, we get that X_∞ is a member of the $ker\mu$ and T maps X_∞ into itself and taking into account Theorem 1.3, gives the desired result. □

Remark 2.12. From Theorem 2.11, we get Darbo's theorem if we take $\beta(X) = 1$ and $\psi(t) = kt$ for all $t \geq 0$ and for some $k \in [0, 1)$.

Now let us pay attention to the following corollary from the above theorem which belongs to the classical metric fixed point theory.

Corollary 2.13. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and there exists $\beta : 2^E \to [0, +\infty)$ such that $T : C \to C$ is a continuous and β-admissible operator satisfying the following:*

(i) for any $X \in \mathfrak{M}_C$ and $x, y \in X$, we have

$$\beta(X)\|Tx - Ty\| \leq \psi\big(\|x - y\|\big); \tag{2.10}$$

(ii) there exist closed and convex $X_0 \subseteq C$ such that

$$TX_0 \subseteq X_0, \quad \beta(X_0) \geq 1 \tag{2.11}$$

where $\psi \in \Psi$. Then T has at least one fixed point in the set C.

Proof. Let $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ be defined as:

$$\mu(X) = diamX$$

where $diamX = \sup\{\|x - y\| : x, y \in X\}$ stands for the diameter of X. Clearly, from Definition 1.1, μ is a measure of noncompactness in the space E. Further, from (2.10) and ψ to be nondecreasing we have

$$\beta(X) \sup_{x,y\in X} \|Tx - Ty\| \leq \psi\big(\sup_{x,y\in X} \|x - y\|\big),$$

implies that

$$\beta(X)\mu(TX) \leq \psi\big(\mu(X)\big).$$

So from Theorem 2.11, we get desired result. □

Proposition 2.14. *If $\beta(X) \geq 1$ for all $X \in 2^E$ having $T(X) = X$, then the set of all fixed points of T in Theorem 2.11, is a compact set.*

Proof. Let $F = \{x \in C : Tx = x\}$ be the set of all fixed points of T and $\mu(F) \neq 0$, from the assumption of Theorem 2.11, we have

$$\beta(F)\mu(TF) \leq \psi\big(\mu(F)\big) < \mu(F),$$

which is a contradiction from above inequality since $T(F) = F$. This implies that F is a relatively compact set. Now taking into account any convergent sequence $\{x_n\} \subset F$ and $x_n \to x^*$, we have $x^* \in C$, because C is closed. The continuity of T implies that $x_n = Tx_n \to Tx^*$ and $Tx^* = x^*$, which means that $x^* \in F$, i.e. F is a compact set. □

Example 2.15. The operator $T : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ defined by

$$Tx(t) = \begin{cases} \frac{x}{2} & \text{if} \quad \|x\| \leq 1, \\ 2x - \frac{3}{2} & \text{if} \quad \|x\| > 1. \end{cases}$$

At first we take a look at that Theorem 1.4, cannot be carried out in the case while $\|x\|, \|y\| > 1$ and we obtain

$$\|Tx(t) - Ty(t)\| = \|2x(t) - \frac{3}{2} - 2y(t) + \frac{3}{2}\| = 2\|x(t) - y(t)\|.$$

Through taking supremum value on both sides we have

$$\mu(TX) = 2\mu(X).$$

However, if we define $\beta : 2^{BC(\mathbb{R}_+)} \to [0, +\infty)$ by

$$\beta(X) = \begin{cases} 1 & \text{if} \quad \|x\| \leq 1, \forall x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Certainly T is an β-ψ condensing operator with $\psi(t) = \frac{t}{2}$ for $t \geq 0$, and $\mu(X) = diamX$.

Coupled fixed point theorems

In this section, we prove some coupled fixed point theorems using α-admissible and β-admissible operators. Before that let us take into account some basic definitions about coupled fixed points and the measure of noncompactness in product spaces.

Definition 2.16. (Coupled fixed point)[10] An element (x, y) in E^2 is called a coupled fixed point of a mapping $T : E^2 \to E$ if $T(x, y) = x$ and $T(y, x) = y$.

Lemma 2.17. *[30] Suppose that $\mu_1, \mu_2, \ldots, \mu_n$ are the measures of noncompactness in Banach spaces $E_1, E_2, \cdots, E_n$ respectively. Moreover, assume that the function $F : [0, \infty)^n \longrightarrow [0, \infty)$ is convex and $F(x_1, x_2, \cdots, x_n) = 0$ if and only if each $x_i = 0$ for all $i = 1, 2, \cdots, n$. Then we define a measure of noncompactness on $E_1 \times E_2 \times, \cdots, \times E_n$ as follows:*

$$\mu(S) = F(\mu_1(S_1), \mu_2(S_2), \ldots, \mu_n(S_n)),$$

where S_i denotes the natural projections of S into E_i for $i = 1, 2, \cdots, n$.

Additionally, as a result of Lemma 2.17, we present the following examples.

Example 2.18. Let μ be a measure of noncompactness on a Banach space E, and let the function $F : [0, +\infty)^2 \to [0, +\infty)$ be convex with $F(x_1, x_2) = 0$ if and only if $x_i = 0$ for $i = 1, 2$. Then

$$\mu^*(X) = F(\mu(X_1), \mu(X_2)),$$

defines a measure of noncompactness in $E \times E$, where X_i denote the natural projections of X into E.

Example 2.19. Let μ be a measure of noncompactness on a Banach space E, considering $F(x, y) = x+y$ for any $(x, y) \in [0, +\infty)^2$. Then we see that F is convex and $F(x, y) = 0$ if and only if $x = y = 0$, hence all the conditions of Lemma 2.17, are satisfied. Therefore, $\mu^*(X) = \mu(X_1) + \mu(X_2)$ defines a measure of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X into E.

Example 2.20. Let μ be a measure of noncompactness on a Banach space E. If we define $F(x, y) = \max\{x, y\}$ for any $(x, y) \in [0, +\infty)^2$, then all the conditions of Lemma 2.17, are satisfied and $\mu^*(X) = \max\{\mu(X_1), \mu(X_2)\}$ is a measure of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X into E.

Theorem 2.21. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $\gamma : E^2 \times E^2 \to [0, +\infty)$, let $T : C \times C \to C$ be continuous and also fulfilling the following conditions:*

(i) let $X_1 \times X_2 \subseteq C \times C$ and $T(x_1, y_1) = x_2$, $T(y_1, x_1) = y_2$ such that

$$\gamma\big((x_1, y_1), (x_2, y_2)\big)\mu\big(T(X_1 \times X_2)\big) \le \frac{1}{2}\psi\big(\mu(X_1) + \mu(X_2)\big), \tag{2.12}$$

for any $(x_1, y_1) \in X_1 \times X_2$;

(ii) for all $(x, y), (u, v) \in C \times C$ and $\gamma\big((x, y), (u, v)\big) \ge 1$ we have

$$\gamma\Big(\big(T(x, y), T(y, x)\big), \big(T(u, v), T(v, u)\big)\Big) \ge 1; \tag{2.13}$$

(iii) further, there exist closed and convex $X_0, Y_0 \subseteq C$ such that $T(X_0 \times Y_0) \subseteq X_0$, $T(Y_0 \times X_0) \subseteq Y_0$ and also there exists $(x_0, y_0) \in X_0 \times Y_0$ such that

$$\gamma\Big((x_0, y_0), \big(T(x_0, y_0), T(y_0, x_0)\big)\Big) \ge 1, \tag{2.14}$$

and

$$\gamma\Big((y_0, x_0), \big(T(y_0, x_0), T(x_0, y_0)\big)\Big) \ge 1, \tag{2.15}$$

where μ be an arbitrary measure of noncompactness on E and $\psi \in \Psi$. Then T has at least one coupled fixed point in $C \times C$.

Proof. To prove this theorem, we need to define $G : C \times C \to C \times C$ by

$$G(x, y) = \big(T(x, y), T(y, x)\big). \tag{2.16}$$

From Example 2.19, we take μ^* is a measure of noncompactness on E^2 as follows:

$$\mu^*(X) = \mu(X_1) + \mu(X_2), \tag{2.17}$$

where X_1 and X_2 are the natural projections of X on E. Further, we define $\alpha : X^2 \times X^2 \to [0, +\infty)$ as follows:

$$\alpha\big((x_1, y_1), (x_2, y_2)\big) = \min\Big\{\gamma\big((x_1, y_1), (x_2, y_2)\big), \gamma\big((y_1, x_1), (y_2, x_2)\big)\Big\}.$$

We need to show that G satisfies all the conditions of Theorem 2.2, then G has a fixed point in $C \times C$, which is the coupled fixed point of the operator T. We list the following conditions which we want to meet for the desired result:

- G is continuous and α-admissible operator;
- $\alpha\big((x_1, y_1), G(x_1, y_1)\big)\mu^*(G(X)) \le \psi\big(\mu^*(X)\big)$, for $(x_1, y_1) \in X$;
- There exists closed and convex subset $A_0 \subseteq C \times C$, and $(x_0, y_0) \in A_0$ such that

$$GA_0 \subseteq A_0,\ \alpha\big((x_0, y_0), G(x_0, y_0)\big) \ge 1.$$

We can easily see from equation (2.16) that G is continuous. Now from equations (2.13) and (2.16), it is clear that whenever $\alpha\big((x_1, y_1), (x_2, y_2)\big) \ge 1$, we have $\alpha\big(G(x_1, y_1), G(x_2, y_2)\big) \ge 1$, which shows that G is α-admissible. Also, from equation (2.12) we have

$$\gamma\big((x_1, y_1), (x_2, y_2)\big)\mu\big(T(X_1 \times X_2)\big) \le \frac{1}{2}\psi\big(\mu(X_1) + \mu(X_2)\big), \tag{2.18}$$

$$\gamma\big((y_1, x_1), (y_2, x_2)\big)\mu\big(T(X_2 \times X_1)\big) \le \frac{1}{2}\psi\big(\mu(X_2) + \mu(X_1)\big). \tag{2.19}$$

From equations (2.17), (2.18) and (2.19), we obtain

$$\alpha\big((x_1, y_1), (x_2, y_2)\big)\Big(\mu\big(T(X_1 \times X_2)\big) + \mu\big(T(X_2 \times X_1)\big)\Big) \le \psi\big(\mu^*(X_1 \times X_2)\big).$$

Eventually, from equations (2.16) and (2.17) we have

$$\alpha\big((x_1, y_1), (x_2, y_2)\big)\mu^*\big(G(X)\big) \le \psi\big(\mu^*(X)\big).$$

Now allow taking $A_0 = X_0 \times Y_0$ and from equations (2.14) and (2.15) we have $G(A_0) \subseteq A_0$ and $\alpha\big((x_0, y_0), G(x_0, y_0)\big) \ge 1$. So all the conditions of the Theorem 2.2 are satisfied and G has a fixed point in $C \times C$. □

Corollary 2.22. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $\gamma : E^2 \times E^2 \to [0, +\infty)$, let $T : C \times C \to C$ be continuous and also satisfying the following conditions:*

(i) let $X_1 \times X_2 \subseteq C \times C$ and $T(x_1, y_1) = x_2$, $T(y_1, x_1) = y_2$ such that

$$\gamma\big((x_1, y_1), (x_2, y_2)\big)\mu\big(T(X_1 \times X_2)\big) \le \frac{1}{2}\psi\big(\max\{\mu(X_1), \mu(X_2)\}\big), \tag{2.20}$$

for any $(x_1, y_1) \in X_1 \times X_2$;

(ii) for all $(x, y), (u, v) \in C \times C$ and $\gamma\big((x, y), (u, v)\big) \ge 1$ we have

$$\gamma\Big(\big(T(x, y), T(y, x)\big), \big(T(u, v), T(v, u)\big)\Big) \ge 1; \tag{2.21}$$

(iii) also, there exist closed and convex $X_0, Y_0 \subseteq C$ such that $T(X_0 \times Y_0) \subseteq X_0$, $T(Y_0 \times X_0) \subseteq Y_0$ and also there exists $(x_0, y_0) \in X_0 \times Y_0$ such that

$$\gamma\Big((x_0, y_0), \big(T(x_0, y_0), T(y_0, x_0)\big)\Big) \ge 1, \tag{2.22}$$

and

$$\gamma\Big((y_0, x_0), \big(T(y_0, x_0), T(x_0, y_0)\big)\Big) \ge 1, \tag{2.23}$$

where μ be an arbitrary measure of noncompactness on E and $\psi \in \Psi$. Then T has at least one coupled fixed point in $C \times C$.

Theorem 2.23. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $\gamma : 2^{E_1 \times E_2} \to [0, +\infty)$, let $T : C \times C \to C$ be continuous and satisfying the following conditions:*

(i) for any $X_1 \times X_2 \subseteq C \times C$, we have

$$\gamma(X_1 \times X_2)\mu(T(X_1 \times X_2)) \leq \frac{1}{2}\psi\big(\mu(X_1) + \mu(X_2)\big); \tag{2.24}$$

(ii) for any $U \times V \subseteq C \times C$ and $\gamma(U \times V) \geq 1$, we have

$$\gamma\Big(\overline{co}\big(T(U \times V) \times T(V \times U)\big)\Big) \geq 1; \tag{2.25}$$

(iii) there exist closed and convex $X_0, Y_0 \subseteq C$ such that $T(X_0 \times Y_0) \subseteq X_0$ and $T(Y_0 \times X_0) \subseteq Y_0$ such that

$$\gamma(X_0 \times Y_0) \geq 1 \quad \textit{and} \quad \gamma(Y_0 \times X_0) \geq 1, \tag{2.26}$$

where μ be an arbitrary measure of noncompactness and $\psi \in \Psi$. Then T has at least one coupled fixed point in $C \times C$.

Proof. Firstly, we define $G : C \times C \to C \times C$ such that

$$G(x, y) = \big(T(x, y), T(y, x)\big), \tag{2.27}$$

also take μ^* is measure of noncompactness on E^2 as follows:

$$\mu^*(X) = \mu(X_1) + \mu(X_2), \tag{2.28}$$

where X_1 and X_2 are the natural projections of X on E. Further, we define $\beta : 2^{E_1 \times E_2} \to [0, +\infty)$ in the following way:

$$\beta(X) = \min\Big\{\gamma(X_1 \times X_2), \gamma(X_2 \times X_1)\Big\},$$

where X_1 and X_2 are the natural projections of X on E. To get required result we need to show that G satisfies all the conditions of Theorem 2.11, which are following:

- G is continuous and β-admissible operator;
- $\beta(X)\mu^*(G(X)) \leq \psi\big(\mu^*(X)\big)$;
- There exists closed and convex subset $A_0 \subseteq C \times C$ such that $GA_0 \subseteq A_0$ and $\beta(A_0) \geq 1$.

Clearly G is continuous and also whenever $\beta(X_1 \times X_2) \geq 1$, we have $\beta\big(\overline{co}\ G(X_1 \times X_2)\big) \geq 1$, which shows that G is β-admissible. From our hypothesis we have

$$\gamma(X_1 \times X_2)\mu(T(X_1 \times X_2)) \leq \frac{1}{2}\psi\big(\mu(X_1) + \mu(X_2)\big), \tag{2.29}$$

$$\gamma(X_2 \times X_1)\mu(T(X_2 \times X_1)) \leq \frac{1}{2}\psi\big(\mu(X_2) + \mu(X_1)\big). \tag{2.30}$$

From equations (2.28), (2.29) and (2.30) we obtain

$$\beta(X)\Big(\mu\big(T(X_1 \times X_2)\big) + \mu\big(T(X_2 \times X_1)\big)\Big) \leq \psi\big(\mu^*(X_1 \times X_2)\big).$$

Finally, from equations (2.27) and (2.28), we get the following inequality

$$\beta(X)\mu^*\big(G(X)\big) \leq \psi\big(\mu^*(X)\big).$$

In the end, from equation (2.26) we take $A_0 = X_0 \times Y_0$ and we have $G(A_0) \subseteq A_0$ and also $\beta(A_0) \geq 1$. all the conditions of the Theorem 2.11 are satisfied and G has a fixed point in $C \times C$. □

Corollary 2.24. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $\gamma : 2^{E_1 \times E_2} \to [0, +\infty)$, let $T : C \times C \to C$ be continuous and satisfying the following conditions:*

(i) for any $X_1 \times X_2 \subseteq C \times C$, we have

$$\gamma(X_1 \times X_2)\mu(T(X_1 \times X_2)) \leq \frac{1}{2}\psi\Big(\max\left\{\mu(X_1), \mu(X_2)\right\}\Big); \tag{2.31}$$

(ii) for any $U \times V \subseteq C \times C$ and $\gamma(U \times V) \geq 1$ we have

$$\gamma\Big(\overline{co}\big(T(U \times V) \times T(V \times U)\big)\Big) \geq 1; \tag{2.32}$$

(iii) there exist closed and convex $X_0, Y_0 \subseteq C$ such that $T(X_0 \times Y_0) \subseteq X_0$ and $T(Y_0 \times X_0) \subseteq Y_0$ such that

$$\gamma(X_0 \times Y_0) \geq 1 \quad \textit{and} \quad \gamma(Y_0 \times X_0) \geq 1, \tag{2.33}$$

where μ be an arbitrary measure of noncompactness and $\psi \in \Psi$. Then T has at least one coupled fixed point in $C \times C$.

A functional integral equation

In this section, we are going to present an application of Theorem 2.11, a study of existence of solution for an integral equation defined on the Banach spaces $BC(\mathbb{R}_+)$, which includes all continuous real-valued and bounded functions on $\mathbb{R}_+$ and equipped with the norm, i.e.

$$\|x\| = \sup\left\{|x(t)| : t > 0\right\}.$$

The measure of noncompactness on $BC(\mathbb{R}_+)$ [21, 30–32] for a positive fixed t on $\mathfrak{M}_{BC(\mathbb{R}_+)}$ is defined as follows:

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty}\, diam\, X(t), \tag{2.34}$$

where $diam\, X(t) = \sup\left\{|x(t) - y(t)| : x, y \in X\right\}$ and $X(t) = \left\{x(t) : x \in X\right\}$. Before defining the $\omega_0(X)$, we first need to define the modulus of continuity for any $x \in X$ and $\epsilon > 0$. The modulus of the continuity of x on the interval $[0, T]$ denoted by $\omega^T(x, \epsilon)$, i.e.

$$\omega^T(x, \epsilon) = \sup\left\{\big|x(t) - x(s)\big| : t, s \in [0, T],\ |t - s| \leq \epsilon\right\},$$

where

$$\omega^T(X, \epsilon) = \sup\left\{\omega^T(x, \epsilon) : x \in X\right\},$$
$$\omega_0^T(X) = \lim_{\epsilon\to 0} \omega^T(X, \epsilon),$$
$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X).$$

As an application of the Theorem 2.11, we are going to have a look at the existence of the solution for the following integral equation:

$$x(t) = A(t) + h\big(t, x(\xi(t))\big) + f\left(t, x(\xi(t)), \varphi\Big(\int_0^{\beta(t)} g\big(t, s, x(\eta(s))\big)\, ds\Big)\right). \tag{2.35}$$

For this cause, we assume the following conditions:

i) the function $A : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and bounded with $M_1 = \sup\{|A(t)| : t \in \mathbb{R}_+\}$;
ii) $\xi, \eta, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions and $\xi(t) \to \infty$ as $t \to \infty$;
iii) the function φ is continuous and there exist $\alpha, \delta > 0$, such that

$$|\varphi(t_1) - \varphi(t_2)| \leq \delta |t_1 - t_2|^{\alpha}, \tag{2.36}$$

for any $t_1, t_2 \in \mathbb{R}_+$ and moreover, $\varphi(0) = 0$;

iv) the functions $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and $\psi \in \Psi$, and there exists nondecreasing continuous function $\theta : \mathbb{R}_+ \to \mathbb{R}$ with $\theta(0) = 0$. Also, there exists $\zeta : R^2 \to R$ with $\zeta(x_1, y_1) \geq 0$ such that

$$|h(t, x_1) - h(t, y_1)| \leq \frac{1}{2}\psi(|x_1 - y_1|), \tag{2.37}$$

and

$$|f(t, x_1, d_1) - f(t, y_1, d_2)| \leq \frac{1}{2}\big(\psi\,(|x_1 - y_1|)\big) + \theta(|d_1 - d_2|), \tag{2.38}$$

for all $x_1, y_1 \in \mathbb{R}$ for any $t \geq 0$;

v) the functions defined by $t \to |h(t, 0)|$ and $t \to |f(t, 0, 0)|$ are bounded on $\mathbb{R}_+$; i.e.

$$M_2 = \sup\{|h(t, 0)| : t \in \mathbb{R}_+\} < \infty, \tag{2.39}$$

$$M_3 = \sup\{|f(t, 0, 0)| : t \in \mathbb{R}_+\} < \infty; \tag{2.40}$$

vi) the function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists positive solution r_0 of the inequality

$$M_1 + \psi(r_0) + M_2 + M_3 + \theta(\delta M_4) < r_0, \tag{2.41}$$

where M_4 is a positive constant defined by the following equalities

$$M_4 = \sup\left\{\Big|\int_0^{\beta(t)} g\big(t, s, x(\eta(s))\big)\,ds\Big|^{\alpha} : t \in \mathbb{R}_+ \; and \; x \in BC(\mathbb{R}_+)\right\}, \tag{2.42}$$

and

$$\lim_{t\to\infty} \int_0^{\beta(t)} \Big|g\big(t, s, x(\eta(s))\big) - g\big(t, s, u(\eta(s))\big)\Big|\,ds = 0, \tag{2.43}$$

uniformly with respect to $x, u \in BC(\mathbb{R}_+)$;

vii) for $\zeta(x(t), y(t)) \geq 0$, for all $x, y \in X \subseteq BC(\mathbb{R}_+)$ and for any $t \in \mathbb{R}_+$, implies that $\zeta(u(t), v(t)) \geq 0$, for all $u, v \in \overline{co}T(X)$ and for any $t \in \mathbb{R}_+$. Moreover, $\zeta(x_0(t), y_0(t)) \geq 0$ for all $x_0, y_0 \in B_{r_0}$(ball of radius of r_0 in $BC(\mathbb{R}_+)$), for any $t \in \mathbb{R}_+$.

Theorem 2.25. *Suppose that* (*i*)-(*vii*) *holds; then the system of integral equation*

$$x(t) = A(t) + h\big(t, x(\xi(t))\big) + f\left(t, x(\xi(t)), \varphi\Big(\int_0^{\beta(t)} g\big(t, s, x(\eta(s))\big)\,ds\Big)\right) \tag{2.44}$$

has at least one solution in the space $BC(\mathbb{R}_+)$.

Proof. Let $T : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ be an operator defined by

$$(Tx)(t) = A(t) + h\big(t, x(\xi(t))\big) + f\left(t, x(\xi(t)), \varphi\Big(\int_0^{\beta(t)} g\big(t, s, x(\eta(s))\big)\,ds\Big)\right). \tag{2.45}$$

Moreover, the space $BC(\mathbb{R}_+)$ is equipped the following norm:

$$\|(x)\|_{BC(\mathbb{R}_+)} = \|x\|_\infty. \tag{2.46}$$

We can easily show that the solution of equation (2.44) in $BC(\mathbb{R}_+)$ is equivalent to the fixed point of T. Obviously Tx is continuous function for any $x \in BC(\mathbb{R}_+)$. Furthermore, using the triangular inequality and

$\zeta(x(t), 0) \geq 0$ for $t \in \mathbb{R}_+$, and additionally by means of our assumptions we obtain

$$\begin{aligned}
\left|T(x)(t)\right| &\leq \left|A(t)\right| + \left|h\big(t, x(\xi(t))\big) - h(t,0)\right| + \left|h(t,0)\right| + \left|f(t,0,0)\right| \\
&\quad + \left|f\left(t, x(\xi(t)), \varphi\Big(\int_0^{\beta(t)} g\big(t,s,x(\eta(s))\big)ds\Big)\right) - f(t,0,0)\right| \\
&\leq M_1 + \frac{1}{2}\psi\big(|x(\xi(t))|\big) + M_3 + \frac{1}{2}\psi\big(|x(\xi(t))|\big) \\
&\quad + \theta\left(\left|\varphi\Big(\int_0^{\beta(t)} g\big(t,s,x(\eta(s))\big)ds\Big) - \varphi(0)\right|\right) + M_2 \\
&\leq M_1 + \psi\big(|x(\xi(t))|\big) + M_3 + M_2 + \theta\left(\delta\left|\Big(\int_0^{\beta(t)} g\big(t,s,x(\eta(s))\big)ds\Big)\right|^{\alpha}\right).
\end{aligned} \tag{2.47}$$

So, from above equation and making use of equations (2.41) and (2.42), we have

$$\|Tx\|_\infty \leq M_1 + M_2 + M_3 + \psi(\|x\|_\infty) + \theta(\delta M_4) \leq r_0. \tag{2.48}$$

Thus, T is well-defined and we obtain $T(\bar{B}_{r_0}) \subset \bar{B}_{r_0}$. Further, we prove that the mapping $T : \bar{B}_{r_0} \to \bar{B}_{r_0}$ is continuous. Let $x, u \in \bar{B}_{r_0}$ such that $\zeta(x(t), u(t)) \geq 0$ for $t \in \mathbb{R}_+$ and for $\epsilon > 0$, $\|x - u\|_{\bar{B}_{r_0}} < \frac{\epsilon}{2}$, then we have

$$\begin{aligned}
\left|T(x)(t) - T(u)(t)\right| &= \left|h\big(t, x(\xi(t))\big) + f\left(t, x(\xi(t)), \varphi\Big(\int_0^{\beta(t)} g\big(t,s,x(\eta(s))\big)ds\Big)\right)\right. \\
&\quad \left. - h\big(t, u(\xi(t))\big) - f\left(t, u(\xi(t)), \varphi\Big(\int_0^{\beta(t)} g\big(t,s,u(\eta(s))\big)ds\Big)\right)\right| \\
&\leq \frac{1}{2}\psi\big(|x(\xi(t)) - u(\xi(t))|\big) + \frac{1}{2}\psi\big(|x(\xi(t)) - u(\xi(t))|\big) \\
&\quad + \theta\left(\left|\varphi\Big(\int_0^{\beta(t)} g\big(t,s,x(\eta(s))\big)ds\Big) - \varphi\Big(\int_0^{\beta(t)} g\big(t,s,u(\eta(s))\big)ds\Big)\right|\right) \\
&\leq \psi(\|x - u\|) + \theta\left(\delta\left|\int_0^{\beta(t)} \Big(g\big(t,s,x(\eta(s))\big) - g\big(t,s,u(\eta(s))\big)\Big)ds\right|^{\alpha}\right).
\end{aligned} \tag{2.49}$$

Now using the equation (2.43) there exists $T > 0$ such that if $t > T$, Then we have

$$\theta\left(\delta\left|\int_0^{\beta(t)} \Big(g\big(t,s,x(\eta(s))\big) - g\big(t,s,u(\eta(s))\big)\Big)ds\right|^{\alpha}\right) \leq \frac{\epsilon}{2}, \tag{2.50}$$

for any $x, u \in BC(\mathbb{R}_+)$. Now we consider the following two cases:
Case 1. If $t > T$, then from equations (2.49) and (2.50) we get

$$\left|T(x)(t) - T(u)(t)\right| \leq \psi\Big(\frac{\epsilon}{2}\Big) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{2.51}$$

Case 2. Similarly for $t \in [0, T]$, we have

$$\begin{aligned}
\left|T(x)(t) - T(u)(t)\right| &\leq \psi\Big(\frac{\epsilon}{2}\Big) + \theta\left(\delta\left|\int_0^{\beta(t)} \Big(g\big(t,s,x(\eta(s))\big) - g\big(t,s,u(\eta(s))\big)\Big)ds\right|^{\alpha}\right) \\
&< \frac{\epsilon}{2} + \theta\Big(\delta\big(\beta_T\beta(\epsilon)\big)^{\alpha}\Big),
\end{aligned} \tag{2.52}$$

where $\beta_T = \sup\{\beta(t) : t \in [0,T]\}$, and

$$\beta(\epsilon) = \sup\left\{\left|g(t,s,x) - g(t,s,u)\right| : t \in [0,T], s \in [0,\beta_T], x,u \in [-r_0, r_0], \|x-u\|_{BC(\mathbb{R}_+)} < \frac{\epsilon}{2}\right\}. \tag{2.53}$$

Since g is continuous on $[0,T]\times[0,\beta_T]\times[-r_0,r_0]$, we have $\beta(\epsilon) \to 0$ as $\epsilon \to 0$ and by continuity of θ we obtain

$$\theta\Big(\delta\big(\beta_T\beta(\epsilon)\big)^{\alpha}\Big) \to 0.$$

Finally, from equations (2.51) and (2.52), we conclude that T is a continuous function from $\bar{B}_{r_0}$ into $\bar{B}_{r_0}$. Now we show that the map T satisfies all the conditions of Theorem 2.11. To do this, for an arbitrary $T > 0$ and $\epsilon > 0$, assume that X_1 are arbitrary nonempty subsets of $\bar{B}_{r_0}$ and $t_1, t_2 \in [0,T]$, with $|t_1 - t_2| \le \epsilon$. Without loss of generality let $\beta(t_1) \le \beta(t_2)$, and $\zeta(x(t_1), x(t_2)) \ge 0$, for any arbitrary $x \in X_1$, we have

$$\begin{aligned}
&\left|Gx(t_1) - Gx(t_2)\right| \\
&= \left|A(t_1) - A(t_2)\right| + \left|h\big(t_2, x(\xi(t_2))\big) - h\big(t_2, x(\xi(t_1))\big)\right| + \left|h\big(t_2, x(\xi(t_1))\big) - h\big(t_1, x(\xi(t_1))b\big)\right| \\
&\quad + \left|f\Big(t_2, x(\xi(t_2)), \varphi\Big(\int_0^{\beta(t_2)} g\big(t_2, s, x(\eta(s))\big)ds\Big)\Big) - f\Big(t_2, x(\xi(t_1)), \varphi\Big(\int_0^{\beta(t_2)} g\big(t_2, s, x(\eta(s))\big)ds\Big)\Big)\right| \\
&\quad + \left|f\Big(t_2, x(\xi(t_1)), \varphi\Big(\int_0^{\beta(t_2)} g\big(t_2, s, x(\eta(s))\big)ds\Big)\Big) - f\Big(t_1, x(\xi(t_1)), \varphi\Big(\int_0^{\beta(t_2)} g\big(t_2, s, x(\eta(s))\big)ds\Big)\Big)\right| \\
&\quad + \left|f\Big(t_1, x(\xi(t_1)), \varphi\Big(\int_0^{\beta(t_2)} g\big(t_2, s, x(\eta(s))\big)ds\Big)\Big) - f\Big(t_1, x(\xi(t_1)), \varphi\Big(\int_0^{\beta(t_2)} g\big(t_1, s, x(\eta(s))\big)ds\Big)\Big)\right| \\
&\quad + \left|f\Big(t_1, x(\xi(t_1)), \varphi\Big(\int_0^{\beta(t_2)} g\big(t_1, s, x(\eta(s))\big)ds\Big)\Big) - f\Big(t_1, x(\xi(t_1)), \varphi\Big(\int_0^{\beta(t_1)} g\big(t_1, s, x(\eta(s))\big)ds\Big)\Big)\right|.
\end{aligned} \tag{2.54}$$

Now we make the following substitutions

$$\begin{cases}
\omega^T(A,\epsilon) = \sup\left\{\left|A(t_1) - A(t_2)\right| : t_1, t_1 \in [0,T], |t_1 - t_2| \le \epsilon\right\}, \\
\omega^T_{r_0}(h,\epsilon) = \sup\left\{\left|h(t_2, x) - h(t_1, x)\right| : t_1, t_2 \in [0,T], |t_1 - t_2| \le \epsilon, x \in [-r_0, r_0]\right\}, \\
\omega^T(\xi,\epsilon) = \sup\left\{\left|\xi(t_1) - \xi(t_2)\right| : t_1, t_2 \in [0,T], |t_1 - t_2| \le \epsilon\right\}, \\
\omega^T\big(x, \omega^T(\xi,\epsilon)\big) = \sup\left\{\left|x(t_1) - x(t_2)\right| : t_1, t_2 \in [0,T],\ |t_1 - t_2| \le \omega^T(\xi,\epsilon)\right\}, \\
D_1 = \beta_T \sup\left\{\left|g(t,s,x)\right| : t \in [0,T], s \in [0,\beta_T], x \in [-r_0, r_0]\right\}, \\
\omega^T_{r_0,K}(f,\epsilon) = \sup\Big\{\left|f(t_2, x, d) - f(t_1, x, d)\right| : t_1, t_2 \in [0,T], |t_1 - t_2| \le \epsilon, \\
\qquad\qquad\qquad\qquad\qquad x \in [-r_0, r_0], d \in [-D_1, D_1]\Big\}, \\
\omega^T_{r_0}(g,\epsilon) = \sup\Big\{\Big|g(t_1, s, x) - g(t_2, s, x)\Big| : t_1, t_2 \in [0,T],\ |t_1 - t_2| \le \epsilon, \\
\qquad\qquad\qquad\qquad\qquad x \in [-r_0, r_0],\ s \in [0,\beta_T]\}, \\
\omega^T(q,\epsilon) = \sup\left\{\left|q(t_1) - q(t_2)\right| : t_1, t_2 \in [0,T],\ |t_1 - t_2| \le \epsilon\right\}, \\
U^T_{r_0} = \sup\left\{\left|g(t,s,x)\right| : t \in [0,T], s \in [0,\beta_T], x \in [-r_0, r_0]\right\}.
\end{cases} \tag{2.55}$$

Now using equations (2.54) and (2.55) we have

$$\begin{aligned}
&\big|Tx(t_1) - Tx(t_2)\big| \\
&\le \omega^T(A,\epsilon) + \frac{1}{2}\psi\big(|x(\xi(t_2)) - x(\xi(t_1))|\big) + \omega^T_{r_0}(h,\epsilon) + \frac{1}{2}\psi\big(|x(\xi(t_2)) - x(\xi(t_1))|\big) + \omega^T_{r_0,K}(f,\epsilon) \\
&\quad + \theta\Bigg(\bigg|\varphi\Big(\int_0^{\beta(t_2)} g\big(t_2,s,x(\eta(s))\big)ds\Big) - \varphi\Big(\int_0^{\beta(t_2)} g\big(t_1,s,x(\eta(s))\big)ds\Big)\bigg|\Bigg) \\
&\quad + \theta\Bigg(\bigg|\varphi\Big(\int_0^{\beta(t_2)} g\big(t_1,s,x(\eta(s))\big)ds\Big) - \varphi\Big(\int_0^{\beta(t_1)} g\big(t_1,s,x(\eta(s))\big)ds\Big)\bigg|\Bigg) \\
&\le \omega^T(A,\epsilon) + \omega^T_{r_0}(h,\epsilon) + \psi\big(\omega^T(x,\omega^T(\xi,\epsilon))\big) + \omega^T_{r_0,K}(f,\epsilon) \\
&\quad + \theta\Bigg(\delta\bigg|\int_0^{\beta(t_2)}\Big(g\big(t_2,s,x(\eta(s))\big) - g\big(t_1,s,x(\eta(s))\big)\Big)ds\bigg|^{\alpha}\Bigg) + \theta\Bigg(\delta\bigg|\int_{\beta_{(t_1)}}^{\beta(t_2)}\Big(g\big(t_1,s,x(\eta(s))\big)\bigg|^{\alpha}\Bigg) \\
&\le \omega^T(A,\epsilon) + \omega^T_{r_0}(h,\epsilon) + \psi\big(\omega^T(x,\omega^T(\xi,\epsilon))\big) + \omega^T_{r_0,K}(f,\epsilon) + \theta\big(\delta(\beta_T\omega^T_{r_0}(g,\epsilon))^{\alpha}\big) + \theta\big(\delta K\omega^T(\beta,\epsilon))^{\alpha}\big).
\end{aligned} \tag{2.56}$$

Since x is an arbitrary element of X_1 the above expression implies that

$$\begin{aligned}
\omega^L\big(T(X_1),\epsilon\big) \le \omega^T(A,\epsilon) + \omega^T_{r_0}(h,\epsilon) + \psi\big(\omega^T(X_1,\omega^T(\xi,\epsilon))\big) + \omega^T_{r_0,K}(f,\epsilon) \\
+ \theta\big(\delta(\beta_T\omega^T_{r_0}(g,\epsilon))^{\alpha}\big) + \theta\big(\delta(K\omega^T(\beta,\epsilon))^{\alpha}\big).
\end{aligned} \tag{2.57}$$

Moreover, by the uniform continuity of f, g and h on the compact sets $[0,T]\times[-r_0,r_0]\times[-D_1,D_1]$, $[0,T]\times[0,\beta_T]\times[-r_0,r_0]$ and $[0,T]\times[-r_0,r_0]$ respectively. We get $\omega^T_{r_0,K}(f,\epsilon)\to 0$, $\omega^T_{r_0}(g,\epsilon)\to 0$ and $\omega^T_{r_0,K}(h,\epsilon)\to 0$ as $\epsilon\to 0$. Also due to the uniform continuity of ξ, q and A on $[0,T]$, we get $\omega^T(\xi,\epsilon)\to 0$, $\omega^T(\beta,\epsilon)\to 0$ and $\omega^T(A,\epsilon)\to 0$ as $\epsilon\to 0$. Moreover, θ is a nondecreasing continuous function with $\theta(0)=0$ and K is finite, hence we have $\theta\big(\delta(\beta_T\omega^T_{r_0}(g,\epsilon))^{\alpha}\big) + \theta\big(\delta(K\omega^T(\beta,\epsilon))^{\alpha}\big) \longrightarrow 0$, as $\epsilon\to 0$. Now taking the limit $\epsilon\to 0$ in equation (2.57), we get

$$\omega^L_0\big(T(X_1)\big) \le \psi\big(\omega^T_0(X_1)\big), \tag{2.58}$$

and taking the limit $T\to\infty$ in aforementioned equation (2.58), we obtain

$$\omega_0\big(T(X_1)\big) \le \psi\big(\omega_0(X_1)\big). \tag{2.59}$$

Further, in addition $\zeta(x(t),u(t))\ge 0$ for arbitrary $x,u,\in X_1$

$$\begin{aligned}
\big|Tx(t) - Tu(t)\big| &\le \big|h\big(t,x(\xi(t))\big) - h\big(t,u(\xi(t))\big)\big| \\
&\quad + \bigg|f\Big(t,x(\xi(t)),\varphi\Big(\int_0^{\beta(t)} g\big(t,s,x(\eta(s))\big)ds\Big)\Big) - f\Big(t,u(\xi(t)),\varphi\Big(\int_0^{\beta(t)} g\big(t,s,u(\eta(s))\big)ds\Big)\Big)\bigg| \\
&\le \frac{1}{2}\psi\big(|x(\xi(t)) - u(\xi(t))|\big) + \frac{1}{2}\psi\big(|x(\xi(t)) - u(\xi(t))|\big) \\
&\quad + \theta\Bigg(\bigg|\varphi\Big(\int_0^{\beta(t)} g\big(t,s,x(\eta(s))\big)ds\Big) - \varphi\Big(\int_0^{\beta(t)} g\big(t,s,u(\eta(s))\big)ds\Big)\bigg|\Bigg) \\
&\le \psi\big(diam X_1(\xi(t))\big) + \theta\Bigg(\delta\bigg|\int_0^{\beta(t)}\Big(g\big(t,s,x(\eta(s))\big) - g\big(t,s,u(\eta(s))\big)\Big)ds\bigg|^{\alpha}\Bigg).
\end{aligned} \tag{2.60}$$

Since x, u and t are arbitrary in equation (2.60) we conclude that

$$diam T(X_1) \le \psi\big(diam X_1(\xi(t))\big) + \theta\Bigg(\delta\bigg|\int_0^{\beta(t}\Big(g\big(t,s,x(\eta(s))\big) - g\big(t,s,u(\eta(s))\big)\Big)ds\bigg|^{\alpha}\Bigg). \tag{2.61}$$

Taking $t \to 0$ in equation (2.61) and using equation (2.43), we have

$$\limsup_{t\to\infty} diam\ TX_1(t) \leq \psi\big(\limsup_{t\to\infty} diam X_1(\xi(t))\big). \tag{2.62}$$

Now from equations (2.59), (2.62) and taking into account the superadditivity of the function ψ, we conclude that

$$\begin{aligned}\omega_0\big(T(X_1\big) + \limsup_{t\to\infty} diam T(X_1)(t) &\leq \psi\big(\omega_0(X_1)\big) + \psi\big(\limsup_{t\to\infty} diam X_1(\xi(t))\big) \\ &\leq \psi\big(\omega_0(X_1) + \limsup_{t\to\infty} diam X_1(\xi(t))\big)\end{aligned} \tag{2.63}$$

Finally, from equation (2.34), we get

$$\mu\big(T(X_1)\big) \leq \psi\big(\mu(X_1)\big). \tag{2.64}$$

Furthermore, we define the function $\beta : BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \longrightarrow [0, +\infty)$, by

$$\beta(X_1) = \begin{cases} 1 & \zeta(x(t), y(t)) \geq 0,\ \forall\, x, y \in X_1, \\ 0 & \text{otherwise,} \end{cases}$$

which implies that for any $X_1 \subseteq B_{r_0}$, we have

$$\beta(X_1)\mu\big(T(X_1)\big) \leq \psi\big(\mu(X_1)\big).$$

Let $\beta(X) \geq 1$, which implies that $\zeta(x(t), y(t)) \geq 0$, and from our assumption $(viii)$ implies that $\beta(\overline{co}TX) \geq 1$, so T is β-admissible. Since from hypothesis $\zeta(x_0(t), y_0(t)) \geq 0$, for all $x_0, y_0 \in B_{r_0}$, implies that $\beta(B_{r_0}) \geq 1$. Thus by Theorem 2.11, T has atleast one fixed point in $BC(\mathbb{R}_+)$. □

3 Conclusion

Taking into account its interesting applications, looking for newly fixed point theorems concerning the new setup of contractive type conditions has acquired considerable attention over the last few decades. In this regard, the main purpose of this paper is to provide new ideas of α-ψ and β-ψ condensing operators and make use of them to establish a new fixed point and coupled fixed point theorems. An application to a solution of the functional integral equation is illustrated to the usability of the obtained fixed point results.

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