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Degrees of the approximations by some special matrix means of conjugate Fourier series

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Abstract: In this paper we will present the pointwise and normwise estimations of the deviations considered by W. Łenski, B. Szal, [Acta Comment. Univ. Tartu. Math., 2009, 13, 11-24] and S. Saini, U. Singh, [Boll. Unione Mat. Ital., 2016, 9, 495-504] under general assumptions on the class considered sequences defining the method of the summability. We show that the obtained estimations are the best possible for some subclasses of L^p by constructing the suitable type of functions.

Keywords: degree of approximation, Fourier series, matrix means

MSC: 42A24

1 Introduction

Let L^p $(1 \leq p < \infty)$ be the class of all 2π–periodic real–valued functions, integrable in the Lebesgue sense, with p–th power over $Q = [-\pi, \pi]$ with the norm

$$\|f\| = \|f(\cdot)\|_{L^p} = \left(\int_Q | f(t) |^p \, dt \right)^{1/p} \quad \text{when } 1 \leq p < \infty. \tag{1}$$

Consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

and its conjugate

$$\widetilde{S}f(x) := \sum_{\nu=1}^{\infty} (b_\nu(f) \cos \nu x - a_\nu(f) \sin \nu x)$$

with the partial sums $\widetilde{S}_k f$. We know that if $f \in L^p$ then

$$\widetilde{f}(x) := -\frac{1}{\pi} \int_0^{\pi} \psi_x(t) \frac{1}{2} \cot \frac{t}{2} dt = \lim_{\epsilon \to 0} \widetilde{f}(x, \epsilon),$$

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where

$$\widetilde{f}(x,\epsilon) := -\frac{1}{\pi}\int_{\epsilon}^{\pi} \psi_x(t)\,\frac{1}{2}\cot\frac{t}{2}\,dt$$

with

$$\psi_x(t) := f(x+t) - f(x-t),$$

which exists for almost all x [1, Theorem (3.1)IV].

Let $A := (a_{n,k})$ be an infinite lower triangular matrix of real numbers such that

$$a_{n,k} \geq 0 \text{ when } k = 0,1,2,\ldots,n, \;\; a_{n,k} = 0 \text{ when } k > n,$$

$$\sum_{k=0}^{n} a_{n,k} = 1, \text{ where } n = 0,1,2,\ldots,$$

and let, for $m = 0,1,2,\ldots,n$,

$$A_{n,m} = \sum_{k=0}^{m} a_{n,k} \quad\text{and}\quad \overline{A}_{n,m} = \sum_{k=m}^{n} a_{n,k}.$$

Let the A-transformation of $\left(\widetilde{S}_k f\right)$ be given by

$$\widetilde{T}_{n,A} f(x) := \sum_{k=0}^{n} a_{n,k}\widetilde{S}_k f(x) \quad (\,n = 0,1,2,\ldots).$$

Following Leindler [2] (see also [3]), we assume that for every n and $0 \leq m < n$

$$\sum_{k=m+1}^{n-1} |a_{n,r} - a_{n,r+1}| \leq K\frac{1}{m+1}\sum_{r \geq m/2}^{m} a_{n,r}$$

or

$$\sum_{r=0}^{n-m-1} |a_{n,r} - a_{n,r+1}| \leq K\frac{1}{m+1}\sum_{r=n-m}^{n} a_{n,r}$$

hold if $(a_{n,r})_{r=0}^{n}$ belongs to $MRBVS$ (Mean Rest Bounded Variation Sequence) or $MHBVS$ (Mean Head Bounded Variation Sequence), for $n = 1,2,\ldots,$ respectively, and let

$$|A|_{n,m} = \begin{cases} A_{n,m}, & \text{when } (a_{n,r})_{r=0}^{n} \in MRBVS, \\ \overline{A}_{n,n-m}, & \text{when } (a_{n,r})_{r=0}^{n} \in MHBVS. \end{cases}$$

As a measure of approximation, we will use the generalized modulus of continuity of function f in the space L^p defined for $\beta \geq 0$ by the formula

$$\widetilde{\omega}_\beta f(\delta)_{L^p} := \sup_{0 \leq |t| \leq \delta}\left\{\left|\sin\frac{t}{2}\right|^{\beta p}\int_0^{\pi} |\psi_x(t)|^p\,dx\right\}^{\frac{1}{p}}.$$

It is clear that for $\beta > \alpha \geq 0$

$$\widetilde{\omega}_\beta f(\delta)_{L^p} \leq \widetilde{\omega}_\alpha f(\delta)_{L^p},$$

and it is easily seen that $\widetilde{\omega}_0 f(\cdot)_{L^p} = \widetilde{\omega} f(\cdot)_{L^p}$ is the classical modulus of continuity.

Let us consider a function $\widetilde{\omega}$ of modulus of continuity type on the interval $[0,2\pi]$, i.e. a nondecreasing continuous function having the following properties: $\widetilde{\omega}(0) = 0$, $\widetilde{\omega}(\delta_1+\delta_2) \leq \widetilde{\omega}(\delta_1) + \widetilde{\omega}(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1+\delta_2 \leq 2\pi$. It is easy to conclude that the function $\delta^{-1}\widetilde{\omega}(\delta)$ is a quasi nonincreasing function of δ. Namely the subadditivity implies $\widetilde{\omega}(n\delta) \leq n\widetilde{\omega}(\delta)$, whence $\widetilde{\omega}(\lambda\delta) \leq (\lambda+1)\widetilde{\omega}(\delta)$ and therefore $\frac{\widetilde{\omega}(\delta_2)}{\delta_2} \leq 2\frac{\widetilde{\omega}(\delta_1)}{\delta_1}$ since

$$\widetilde{\omega}(\delta_2) \leq \left(\frac{\delta_2}{\delta_1}+1\right)\widetilde{\omega}(\delta_1) \leq 2\frac{\delta_2}{\delta_1}\widetilde{\omega}(\delta_1),$$

where $n \in \mathbb{N}_0$, $\lambda \geq 0$ and $0 \leq \delta_1 \leq \delta_2$.

Let

$$L^p\left(\widetilde{\omega}\right)_\beta = \left\{f \in L^p : \widetilde{\omega}_\beta f\left(\delta\right)_{L^p} \leq \widetilde{\omega}\left(\delta\right)\right\},$$

where $\widetilde{\omega}$ is a function of modulus of continuity type. It is clear that for $\beta > \alpha \geq 0$

$$L^p\left(\widetilde{\omega}\right)_\alpha \subset L^p\left(\widetilde{\omega}\right)_\beta.$$

The deviation $\widetilde{T}_{n,A}f - \widetilde{f}$ was estimated by Qureshi [4] (with a special matrix A), the norm estimates we can find in the works of Lal and Nigam [5], Dhakal [6], Lal and Singh [7], Mishra, Khari et al. [8], Mishra and Mishra [9], Nigam and Sharma [10], Rhoades [11], Sonker and Singh [12] and Qureshi [13]. The next generalization was obtained by Łenski and Szal [14] in the following form:

Theorem A. Let $f \in L^p\left(\widetilde{\omega}\right)_\beta$ with $\beta < 1 - \frac{1}{p}$, $\left(a_{n,k}\right)_{k=0}^n \in HBVS$ (Head Bounded Variation Sequence) or $\left(a_{n,k}\right)_{k=0}^n \in RBVS$ (Rest Bounded Variation Sequence), respectively, and let $\widetilde{\omega}$ be such that

$$\left\{\int_0^{\frac{\pi}{n+1}} \left(\frac{t\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p} = O_x\left(\left(n+1\right)^{-1}\right) \tag{2}$$

and

$$\left\{\int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{-\gamma}\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p} = O_x\left(\left(n+1\right)^{\gamma}\right) \tag{3}$$

hold with $0 < \gamma < \beta + \frac{1}{p}$. Then

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right)\right| = O_x\left(\left(n+1\right)^{\beta+\frac{1}{p}} a_n(n+1)\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right)$$

for all x, where

$$a_n = \begin{cases} a_{n,0}, & \text{when } \left(a_{n,k}\right)_{k=0}^n \in RBVS, \\ a_{n,n}, & \text{when } \left(a_{n,k}\right)_{k=0}^n \in HBVS. \end{cases}$$

Theorem B. Let $f \in L^p\left(\widetilde{\omega}\right)_\beta$ with $\beta < 1 - \frac{1}{p}$, $\left(a_{n,k}\right)_{k=0}^n \in HBVS$ (or $\left(a_{n,k}\right)_{k=0}^n \in RBVS$) and let $\widetilde{\omega}$ satisfy (3) with $0 < \gamma < \beta + \frac{1}{p}$,

$$\left\{\int_0^{\frac{\pi}{n+1}} \left(\frac{\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p} = O_x\left(\left(n+1\right)^{-\frac{1}{p}}\right) \tag{4}$$

and

$$\left\{\int_0^{\frac{\pi}{n+1}} \left(\frac{\widetilde{\omega}(t)}{t\sin^{\beta}\frac{t}{2}}\right)^q dt\right\}^{1/q} = O_x\left(\left(n+1\right)^{\beta+\frac{1}{p}}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right) \tag{5}$$

where $q = p(p-1)^{-1}$. Then

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right| = O_x\left(\left(n+1\right)^{\beta+\frac{1}{p}} a_n(n+1)\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right)$$

for all x such that $\widetilde{f}(x)$ exists, where

$$a_n = \begin{cases} a_{n,0}, & \text{when } \left(a_{n,k}\right)_{k=0}^n \in RBVS, \\ a_{n,n}, & \text{when } \left(a_{n,k}\right)_{k=0}^n \in HBVS. \end{cases}$$

Recently, Saini and Singh [15] have proved the following theorem:

Theorem C. Let f be 2π - periodic function belonging to $Lip\left(\widetilde{\omega}(t), p\right)$- class with $p \geq 1$ and let $A = (a_{n,k})$ be a lower triangular regular matrix with nonnegative and nondecreasing (with respect to $0 \leq k \leq n$) entries and $A_{n,0} = 1$. Then the degree of approximation of $\widetilde{f}$, conjugate of f, by matrix means of its conjugate Fourier series is given by

$$\left\|\widetilde{T_n}(f;x) - \widetilde{f}(x)\right\|_p = O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{\widetilde{\omega}(t)}{t^{2+1/p}}dt\right),$$

provided $\widetilde{\omega}(t)$ is a positive increasing function satisfying the condition

$$\int_0^v \frac{\widetilde{\omega}(t)}{t^{1+1/p}}dt = O\left(\frac{\widetilde{\omega}(v)}{v^{1/p}}\right),$$

where $0 < v < \pi$.

We shall write $J_1 \ll J_2$, if there exists a positive constant C, depending on some parameters, such that $J_1 \leq CJ_2$.

2 Statement of the results

In this paper, we will present the estimations of the deviations $\widetilde{T}_{n,A}f(\cdot) - \widetilde{f}(\cdot)$ and $\widetilde{T}_{n,A}f(\cdot) - \widetilde{f}\left(\cdot, \frac{\pi}{n+1}\right)$ under general assumptions and we will show that the obtained degrees of approximations are the best for some subclasses of L^p.

Theorem 1. *Let $\left(a_{n,k}\right)_{k=0}^n \in MHBVS \cup MRBVS$ with the condition $|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right)$, where $\tau = [\pi/t]$, $\left(\frac{\pi}{n+1} \leq t \leq \pi\right)$. Let $f \in L^p$ and let $\widetilde{\omega}$ be such that*

$$\left\{\int_0^{\frac{\pi}{n+1}}\left(\frac{t\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p} = O_x\left((n+1)^{-1-\frac{1}{p}}\right) \tag{6}$$

and

$$\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{t^{-\gamma}\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p} = O_x\left((n+1)^{\gamma-\frac{1}{p}}\right) \tag{7}$$

hold with $0 < \gamma < \beta + \frac{1}{p}$. Then

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right)\right| = O_x\left((n+1)^{\beta}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right) \tag{8}$$

holds for all x.

Theorem 2. *Let $\left(a_{n,k}\right)_{k=0}^n \in MHBVS \cup MRBVS$ with the condition $|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right)$, where $\tau = [\pi/t]$ $\left(\frac{\pi}{n+1} \leq t \leq \pi\right)$. Let $f \in L^p$ and let $\widetilde{\omega}$ be such that (7) and*

$$\left\{\int_0^{\frac{\pi}{n+1}}\left(\frac{\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p} = O_x\left((n+1)^{-\frac{1}{p}}\right), \tag{9}$$

$$\left\{\int_0^{\frac{\pi}{n+1}}\left(\frac{\widetilde{\omega}(t)}{t^{\beta+1}}\right)^q dt\right\}^{1/q} = O\left((n+1)^{\beta+\frac{1}{p}}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right), \tag{10}$$

hold with $0 < \gamma < \beta + \frac{1}{p}$. *Then*

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right| = O_x\left((n+1)^{\beta}\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right) \tag{11}$$

holds for all x such that $\widetilde{f}(x)$ *exists.*

Theorem 3. *Let* $(a_{n,k})_{k=0}^{n} \in MHBVS \cup MRBVS$ *with the condition* $|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right)$, *where* $\tau = [\pi/t]$, $\left(\frac{\pi}{n+1} \le t \le \pi\right)$ *and let* $\widetilde{\omega}$ *satisfy the conditions* (6) *and* (7) *with* $\frac{1}{p} < \gamma < \beta + \frac{1}{p}$. *If the function* $t^{-\beta}\widetilde{\omega}(t)$ *is nondecreasing, then*

$$(n+1)^{\beta}\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right) \ll \sup_{f\in L^p(\widetilde{\omega})_\beta}\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right)\right| = O_x(1)\,(n+1)^{\beta}\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right) \tag{12}$$

with $0 < \beta < 1 - \frac{1}{p}$ *for all x.*

Theorem 4. *Let* $(a_{n,k})_{k=0}^{n} \in MHBVS \cup MRBVS$ *with the condition* $|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right)$, *where* $\tau = [\pi/t]$, $\left(\frac{\pi}{n+1} \le t \le \pi\right)$ *and let* $\widetilde{\omega}$ *satisfy the conditions* (7), (9) *and* (10) *with* $\frac{1}{p} < \gamma < \beta + \frac{1}{p}$.

If the function $t^{-\beta}\widetilde{\omega}(t)$ *is nondecreasing and concave then*

$$(n+1)^{\beta}\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right) \ll \sup_{f\in L^p(\widetilde{\omega})_\beta}\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right| = O_x(1)\,(n+1)^{\beta}\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right) \tag{13}$$

with $\beta > 0$, *for all x such that* $\widetilde{f}(x)$ *exists .*

Remark 1. *If we consider* $\widetilde{\omega}(t) = t^{\alpha}$ *with* $\beta < \alpha < 1 + \beta$, *then* $t^{-\beta}\widetilde{\omega}(t)$ *is a nondecreasing and concave function of t.*

Theorem 5. *Let* $(a_{n,k})_{k=0}^{n} \in MHBVS \cup MRBVS$ *with the condition* $|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right)$, *where* $\tau = [\pi/t]$, $\left(\frac{\pi}{n+1} \le t \le \pi\right)$. *Let* $f \in L^p(\widetilde{\omega})_\beta$ *where* $\beta < 1 - \frac{1}{p}$ *and* $0 < \gamma < \beta + \frac{1}{p}$. *Then*

$$\left\|\widetilde{T}_{n,A}f(\cdot) - \widetilde{f}\left(\cdot, \frac{\pi}{n+1}\right)\right\|_{L^p} = O\left((n+1)^{\beta}\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right). \tag{14}$$

Theorem 6. *Let* $(a_{n,k})_{k=0}^{n} \in MHBVS \cup MRBVS$ *with the condition* $|A|_{n,\tau} = O\left(\frac{\tau}{n+1}\right)$, *where* $\tau = [\pi/t]$, $\left(\frac{\pi}{n+1} \le t \le \pi\right)$. *Let* $f \in L^p(\widetilde{\omega})_\beta$ *and* $\widetilde{\omega}$ *be such that* (10) *holds with* $0 < \gamma < \beta + \frac{1}{p}$. *Then*

$$\left\|\widetilde{T}_{n,A}f(\cdot) - \widetilde{f}(\cdot)\right\|_{L^p} = O\left((n+1)^{\beta}\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right). \tag{15}$$

Remark 2. *If* $f \in L^p(\widetilde{\omega})_\beta$, *where* $\widetilde{\omega}(t) = t^{\alpha}$ *with* $\beta < \alpha \le 1 + \beta$, $\beta > 0$, *then the conditions of our theorems are satisfied. Putting* $A_0 = (a_{n,k})$, *where* $a_{n,k} = \frac{1}{n+1}$, *when* $k = 0, 1, 2, \dots, n$ *and* $a_{n,k} = 0$, *when* $k > n$, *in our theorems, we obtain the following degree of approximation* $\pi^{\alpha}(n+1)^{\beta-\alpha}$.

3 Corollaries

Finally, we give some corollaries as an application of our results.

Corollary 1. *Under the assumptions of Theorems 1 and 2 we can obtain better orders of approximations than these in Theorems A and B.*

Corollary 2. *From Theorems 5 and 6 the result of Saini and Singh follows with more general assumptions on the matrix A.*

4 Auxiliary results

We begin this section with some notation following Zygmund [1, Section 5 of Chapter II].

It is clear that

$$\widetilde{S}_k f(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\widetilde{D}_k(t)\,dt$$

and

$$\widetilde{T}_{n,A} f(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\sum_{k=0}^{n} a_{n,k}\widetilde{D_k}(t)\,dt,$$

where

$$\widetilde{D_k}(t) = \sum_{\nu=0}^{k}\sin\nu t = \frac{\cos\frac{t}{2} - \cos\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}.$$

Hence

$$\widetilde{T}_{n,A} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) = -\frac{1}{\pi}\int_{0}^{\frac{\pi}{n+1}} \psi_x(t)\sum_{k=0}^{n} a_{n,k}\widetilde{D_k}(t)\,dt + \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi} \psi_x(t)\sum_{k=0}^{n} a_{n,k}\widetilde{D_k^{\circ}}(t)\,dt$$

and

$$\widetilde{T}_{n,A} f(x) - \widetilde{f}(x) = \frac{1}{\pi}\int_{0}^{\pi} \psi_x(t)\sum_{k=0}^{n} a_{n,k}\widetilde{D_k^{\circ}}(t)\,dt,$$

where

$$\widetilde{D_k^{\circ}}(t) = \frac{\cos\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}.$$

Now, we formulate some estimates of the considered kernel.

Lemma 1. *(see [1]) If* $0 < |t| \le \pi/2$, *then*

$$\left|\widetilde{D_k^{\circ}}(t)\right| \le \frac{\pi}{2|t|} \quad \text{and} \quad \left|\widetilde{D_k}(t)\right| \le \frac{\pi}{|t|}$$

and for any real t, *we have*

$$\left|\widetilde{D_k}(t)\right| \le \frac{1}{2}k(k+1)|t| \quad \text{and} \quad \left|\widetilde{D_k}(t)\right| \le k+1.$$

Lemma 2. *(see [16]) If* $(a_{n,k})_{k=0}^{n} \in MHBVS$, *then*

$$\left|\sum_{k=0}^{n} a_{n,k}\widetilde{D_k^{\circ}}(t)\right| = O\left(t^{-1}\overline{A}_{n,n-2\tau}\right)$$

and if $(a_{n,k})_{k=0}^{n} \in MRBVS$, *then*

$$\left|\sum_{k=0}^{\infty} a_{n,k}\widetilde{D_k^{\circ}}(t)\right| = O\left(t^{-1}A_{n,\tau}\right),$$

for $\frac{2\pi}{n} \le t \le \pi$ $(n = 2, 3, \ldots)$, *where* $\tau = [\pi/t]$.

Lemma 3. *If* $t^{-\beta}\widetilde{\omega}(t)$ *is a concave and nondecreasing function of* t, *then the function*

$$f_0(x) = \sum_{k=2}^{\infty}\left[k^{\beta}\widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^{\beta}\frac{k-1}{k}\widetilde{\omega}\left(\frac{1}{k-1}\right)\right]\sin kx, \quad (x \in [0.\pi])$$

belongs to $L^p(\widetilde{\omega})_{\beta}$.

Proof. Let $k = 2, 3, \ldots$ and

$$a_k := k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^\beta \frac{k-1}{k} \widetilde{\omega}\left(\frac{1}{k-1}\right).$$

First, we show that $a_k \geq a_{k+1}$, i.e.

$$k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^\beta \frac{k-1}{k} \widetilde{\omega}\left(\frac{1}{k-1}\right) \geq (k+1)^\beta \widetilde{\omega}\left(\frac{1}{k+1}\right) - k^\beta \frac{k}{k+1} \widetilde{\omega}\left(\frac{1}{k}\right) \tag{16}$$

and $0 \leq k a_k \leq k^\beta \widetilde{\omega}\left(\frac{1}{k}\right)$, i.e.

$$0 \leq k\left[k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^\beta \frac{k-1}{k} \widetilde{\omega}\left(\frac{1}{k-1}\right)\right] \leq k^\beta \widetilde{\omega}\left(\frac{1}{k}\right). \tag{17}$$

From the relations (see equation [17])

$$\frac{1}{k} = \frac{k+1}{2k+1} \cdot \frac{1}{k+1} + \frac{(k+1)(k-1)}{k(2k+1)} \cdot \frac{1}{k-1},$$

$$\frac{k+1}{2k+1} + \frac{(k+1)(k-1)}{k(2k+1)} < 1,$$

and using the concavity of the function $t^{-\beta}\widetilde{\omega}(t)$, we obtain

$$k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) \geq \frac{k+1}{2k+1}(k+1)^\beta \widetilde{\omega}\left(\frac{1}{k+1}\right) + \frac{(k+1)(k-1)}{k(2k+1)}(k-1)^\beta \widetilde{\omega}\left(\frac{1}{k-1}\right), \tag{18}$$

i.e.

$$\frac{2k+1}{k+1} k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) \geq (k+1)^\beta \widetilde{\omega}\left(\frac{1}{k+1}\right) + \frac{k-1}{k}(k-1)^\beta \widetilde{\omega}\left(\frac{1}{k-1}\right).$$

Thus, we get

$$k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) + \frac{k}{k+1} k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) \geq (k+1)^\beta \widetilde{\omega}\left(\frac{1}{k+1}\right) + \frac{k-1}{k}(k-1)^\beta \widetilde{\omega}\left(\frac{1}{k-1}\right).$$

Hence, we finally obtain estimation (16).

We know (see inequality [17]) that from the concavity of the function $t^{-\beta}\widetilde{\omega}(t)$, we have

$$k k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) \geq (k-1)(k-1)^\beta \widetilde{\omega}\left(\frac{1}{k-1}\right),$$

which implies immediately the left side of inequality (17). Using the monotonicity of the function $t^{-\beta}\widetilde{\omega}(t)$ we get

$$k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^\beta \frac{k-1}{k} \widetilde{\omega}\left(\frac{1}{k-1}\right) \leq k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) - k^\beta \frac{k-1}{k} \widetilde{\omega}\left(\frac{1}{k}\right) = \frac{1}{k} k^\beta \widetilde{\omega}\left(\frac{1}{k}\right),$$

which gives the right side of inequality (17).

Let us denote

$$\psi_x^0(t) = f_0(x+t) - f_0(x) - f_0(x-t) + f_0(x) = S_x(t) - S_x(-t).$$

Hence, we get

$$\begin{aligned}
\widetilde{\omega}_\beta f_0(\delta)_{L^p} &:= \sup_{0 \leq |t| \leq \delta} \left\{ \left|\sin \frac{t}{2}\right|^{\beta p} \cdot \int_0^\pi \left|\psi_x^0(t)\right|^p dx \right\}^{\frac{1}{p}} \\
&\leq \sup_{0 \leq |t| \leq \delta} \left[\left\{ \left|\frac{t}{2}\right|^\beta \cdot \int_0^\pi |f_0(x+t) - f_0(x)|^p dx \right\}^{\frac{1}{p}} + \left\{ \left|\frac{t}{2}\right|^\beta \cdot \int_0^\pi |f_0(x-t) - f_0(x)|^p dx \right\}^{\frac{1}{p}} \right] \\
&\leq 2 \sup_{0 \leq |t| \leq \delta} \left|\frac{t}{2}\right|^{\beta p} \cdot \left\{ \int_0^\pi |S_x(t)|^p dx \right\}^{\frac{1}{p}}.
\end{aligned}$$

Let $\frac{\pi}{m} < t < \frac{\pi}{m-1}$, $t \le x \le \pi - t$. We have

$$\begin{aligned}
|S_x(t)| &= \left|\sum_{k=2}^{\infty}\left[k^{\beta}\widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^{\beta}\frac{k-1}{k}\widetilde{\omega}\left(\frac{1}{k-1}\right)\right](\sin k(x+t) - \sin kx)\right| \\
&\le \left|\sum_{k=2}^{m}\left[k^{\beta}\widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^{\beta}\frac{k-1}{k}\widetilde{\omega}\left(\frac{1}{k-1}\right)\right](\sin k(x+t) - \sin kx)\right| \\
&+ \left|\sum_{k=m+1}^{\infty}\left[k^{\beta}\widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^{\beta}\frac{k-1}{k}\widetilde{\omega}\left(\frac{1}{k-1}\right)\right]\sin k(x+t)\right| \\
&+ \left|\sum_{k=m+1}^{\infty}\left[k^{\beta}\widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^{\beta}\frac{k-1}{k}\widetilde{\omega}\left(\frac{1}{k-1}\right)\right]\sin kx\right| =: |s_1| + |s_2| + |s_3|.
\end{aligned}$$

Using the mean value theorem and the left side of inequality (17) , we get for $x < z < x + t$

$$\begin{aligned}
|s_1| &\le |t|\sum_{k=2}^{m} k\left|k^{\beta}\widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^{\beta}\frac{k-1}{k}\widetilde{\omega}\left(\frac{1}{k-1}\right)\right||\cos kz| \\
&\le |t|\sum_{k=2}^{m}\left[kk^{\beta}\widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)(k-1)^{\beta}\widetilde{\omega}\left(\frac{1}{k-1}\right)\right].
\end{aligned}$$

Thus by summation, we obtain the estimate

$$|s_1| \le |t|mm^{\beta}\widetilde{\omega}\left(\frac{1}{m}\right) \le |t|^{-\beta}\widetilde{\omega}(|t|).$$

For the terms $|s_2|$ and $|s_3|$, with $x \ne 0$, we get using the left side of inequality (17) and following Totik (see estimation [17])

$$\begin{aligned}
|s_2| &\le \sum_{k=m+1}^{\infty}\left|k^{\beta}\widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^{\beta}\frac{k-1}{k}\widetilde{\omega}\left(\frac{1}{k-1}\right)\right||\sin k(x+t)| \\
&\le \frac{4}{x+t}\left[(m+1)^{\beta}\widetilde{\omega}\left(\frac{1}{m+1}\right) - m^{\beta}\frac{m}{m+1}\widetilde{\omega}\left(\frac{1}{m}\right)\right],
\end{aligned}$$

$$\begin{aligned}
|s_3| &\le \sum_{k=m+1}^{\infty}\left|k^{\beta}\widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^{\beta}\frac{k-1}{k}\widetilde{\omega}\left(\frac{1}{k-1}\right)\right||\sin kx| \\
&\le \frac{4}{x}\left[(m+1)^{\beta}\widetilde{\omega}\left(\frac{1}{m+1}\right) - m^{\beta}\frac{m}{m+1}\widetilde{\omega}\left(\frac{1}{m}\right)\right].
\end{aligned}$$

Thus by inequalities (16) and (17) , we obtain

$$\begin{aligned}
|s_2| &\le 4m\left[(m+1)^{\beta}\widetilde{\omega}\left(\frac{1}{m+1}\right) - m^{\beta}\frac{m}{m+1}\widetilde{\omega}\left(\frac{1}{m}\right)\right] \\
&\le 4m\left[m^{\beta}\widetilde{\omega}\left(\frac{1}{m}\right) - (m-1)^{\beta}\frac{m-1}{m}\widetilde{\omega}\left(\frac{1}{m-1}\right)\right] \\
&\le 4m^{\beta}\widetilde{\omega}\left(\frac{1}{m}\right) \le 4|t|^{-\beta}\widetilde{\omega}(|t|)
\end{aligned}$$

and analogously

$$|s_3| \le 4|t|^{-\beta}\widetilde{\omega}(|t|).$$

If $x = 0$, then we can prove that $|s_1| \ll t^{-\beta}\widetilde{\omega}(|t|)$, $|s_2| \ll t^{-\beta}\widetilde{\omega}(|t|)$ and $|s_3| = 0$. Collecting these estimates, we get $|S_x(t)| \ll |t|^{-\beta}\widetilde{\omega}(|t|)$. Hence

$$\widetilde{\omega}_{\beta}f_0(\delta)_{L^p} \ll \widetilde{\omega}(\delta).$$

Thus we have proved that f_0 belongs to $L^p(\widetilde{\omega})_{\beta}$. □

Lemma 4. *If $t^{-\beta}\widetilde{\omega}(t)$ is a nondecreasing function of t, then the function*

$$f_1(x) = (n+1)^{\beta}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\sin x, \quad (x \in [0.\pi])$$

belongs to $L^p(\widetilde{\omega})_\beta$.

Proof. We have

$$\widetilde{\omega}_\beta f_1(\delta)_{L^p} := \sup_{0\le|t|\le\delta}\left\{\left|\sin\frac{t}{2}\right|^{\beta p}\cdot\int_0^\pi\left|\psi_x^1(t)\right|^p dx\right\}^{\frac{1}{p}}.$$

Let $\frac{\pi}{n+1} < t < \frac{\pi}{n}$, $t \le x \le \pi - t$. We have

$$\int_0^\pi\left|\psi_x^1(t)\right|^p dx = \int_0^\pi\left|f_1(x+t) - f_1(x-t)\right|^p dx = \int_0^\pi\left[(n+1)^\beta\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right]^p|2\sin t\cos x|^p dx$$
$$\ll\left[(n+1)^\beta\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right]^p \ll \left[|t|^{-\beta}\,\widetilde{\omega}(|t|)\right]^p.$$

Hence, we get

$$\widetilde{\omega}_\beta f_1(\delta)_{L^p} \ll \sup_{0\le|t|\le\delta}\left\{|t|^{\beta p}\cdot|t|^{-\beta p}\,\widetilde{\omega}(|t|)^p\right\}^{\frac{1}{p}} \ll \widetilde{\omega}(\delta).$$

Thus, we have proved that f_1 belongs to $L^p(\widetilde{\omega})_\beta$. □

5 Proofs of the results

5.1 Proof of Theorem 1

Let us start with the obvious relations

$$\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) = -\frac{1}{\pi}\int_0^{\frac{\pi}{n+1}}\psi_x(t)\sum_{k=0}^n a_{nk}\widetilde{D_k}(t)\,dt + \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\psi_x(t)\sum_{k=0}^n a_{nk}\widetilde{D_k^\circ}(t) =: \widetilde{I_1} + \widetilde{I_2^\circ}$$

and

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right)\right| \le \left|\widetilde{I_1}\right| + \left|\widetilde{I_2^\circ}\right|.$$

By the Hölder inequality $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, Lemma 1 and equation (6)

$$\left|\widetilde{I_1}\right| \le (n+1)^2\int_0^{\frac{\pi}{n+1}} t\,|\psi_x(t)|\,dt$$
$$\le (n+1)^2\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{t\,|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{\widetilde{\omega}(t)}{\sin^\beta\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}}$$
$$\ll (n+1)^{1-\frac{1}{p}}\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{\widetilde{\omega}(t)}{t^\beta}\right]^q dt\right\}^{\frac{1}{q}} \ll (n+1)^\beta\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right),$$

for $\beta < 1 - \frac{1}{p}$.

By the Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 2, monotonicity of the function $t^{-1}\widetilde{\omega}(t)$ and equation (7)

$$\begin{aligned}\left|\widetilde{I_2^{\circ}}\right| &\le \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\psi_x(t)|}{t}|A_{n,\tau}|dt \le (n+1)^{-1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\psi_x(t)|}{t^2}dt\\ &= (n+1)^{-1}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{t^{-\gamma}|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^{\beta}\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{\widetilde{\omega}(t)}{t^{2-\gamma}\sin^{\beta}\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}}\\ &\ll (n+1)^{-1}(n+1)^{\gamma-\frac{1}{p}}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{\widetilde{\omega}(t)}{t^{2+\beta-\gamma}}\right]^q dt\right\}^{\frac{1}{q}}\\ &\le (n+1)^{\gamma-\frac{1}{p}}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left[\frac{1}{t^{1+\beta-\gamma}}\right]^q dt\right\}^{\frac{1}{q}}\\ &\ll (n+1)^{\beta}\widetilde{\omega}\left(\frac{\pi}{n+1}\right),\end{aligned}$$

for $0<\gamma<\beta+\frac{1}{p}$.

Collecting these estimates, we obtain the desired result. □

5.2 Proof of Theorem 2

As usual, let us start with the obvious relations

$$\widetilde{T}_{n,A}f(x)-\widetilde{f}(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{n+1}}\psi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D_k^{\circ}}(t)\,dt+\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\psi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D_k^{\circ}}(t)\,dt =: \widetilde{I_1^{\circ}}+\widetilde{I_2^{\circ}}$$

and

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)\right| \le \left|\widetilde{I_1^{\circ}}\right|+\left|\widetilde{I_2^{\circ}}\right|.$$

By the Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 1, equations (9) and (10),

$$\begin{aligned}\left|\widetilde{I_1^{\circ}}\right| &\le \frac{1}{\pi}\int_0^{\frac{\pi}{n+1}}\frac{|\psi_x(t)|}{t}dt = \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^{\beta}\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{\widetilde{\omega}(t)}{t\sin^{\beta}\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}}\\ &\ll (n+1)^{-\frac{1}{p}}\left\{\int_0^{\frac{\pi}{n+1}}\left[\frac{\widetilde{\omega}(t)}{t^{1+\beta}}\right]^q dt\right\}^{\frac{1}{q}} \ll (n+1)^{\beta}\widetilde{\omega}\left(\frac{\pi}{n+1}\right).\end{aligned}$$

By the previous proof

$$\left|\widetilde{I_2^{\circ}}\right| \ll (n+1)^{\beta}\widetilde{\omega}\left(\frac{\pi}{n+1}\right),$$

for $0<\gamma<\beta+\frac{1}{p}$.

Collecting these estimates, we obtain the desired result. □

5.3 Proof of Theorem 3

Let us fix a point x and let us consider the class $L^p(\widetilde{\omega})_{\beta}$, with $\beta>0$, of all functions $f\in L^p$ such that $\widetilde{\omega}_{\beta}f(\delta)_{L^p}\le\widetilde{\omega}(\delta)$, $(0\le\delta\le 2\pi)$.

Then Theorem 1 implies the following estimate

$$\sup_{f\in L^p(\widetilde{\omega})_\beta}\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{\pi}{n+1}\right)\right|\ll(n+1)^\beta\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right)$$

for $\beta<1-\frac{1}{p}$.

On the other hand, the function

$$f_1(x)=(n+1)^\beta\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\sin x$$

by Lemma 4 belongs to class $L^p(\widetilde{\omega})_\beta$, if $t^{-\beta}\widetilde{\omega}(t)$ is a nondecreasing function of t, and f_1 satisfies the conditions (6) and (7) of Theorem 1. Indeed, we have

$$\left\{\int_0^{\frac{\pi}{n+1}}\left(\frac{t\left|\psi_x^1(t)\right|}{\widetilde{\omega}(|t|)}\right)^p\sin^{\beta p}\frac{t}{2}dt\right\}^{1/p}\ll\left\{\int_0^{\frac{\pi}{n+1}}\left(\frac{t\,|t|^{-\beta}\,\widetilde{\omega}(|t|)}{\widetilde{\omega}(|t|)}\right)^p\sin^{\beta p}\frac{t}{2}dt\right\}^{1/p}=O_x(n+1)^{-1-\frac{1}{p}}.$$

Moreover, there exists γ such that $\frac{1}{p}<\gamma<\beta+\frac{1}{p}$ and

$$\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{t^{-\gamma}\left|\psi_x^1(t)\right|}{\widetilde{\omega}(|t|)}\right)^p\sin^{\beta p}\frac{t}{2}dt\right\}^{1/p}\ll\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{t^{-\gamma}\,|t|^{-\beta}\,\widetilde{\omega}(|t|)}{\widetilde{\omega}(|t|)}\right)^p\sin^{\beta p}\frac{t}{2}dt\right\}^{1/p}$$

$$\ll\left\{\int_{\frac{\pi}{n+1}}^{\pi}t^{-\gamma p}dt\right\}^{1/p}=O_x(n+1)^{\gamma-\frac{1}{p}}.$$

Let $\frac{\pi}{n}<t<\frac{\pi}{n-1}$. We have

$$\begin{aligned}\widetilde{f}_1\left(x,\frac{\pi}{n+1}\right)&=-\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\left[f_1(x+t)-f_1(x-t)\right]\frac{\cos\frac{t}{2}}{2\sin\frac{t}{2}}dt\\&=-\frac{1}{\pi}(1+n)^\beta\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\int_{\frac{\pi}{n+1}}^{\pi}2\sin t\cos x\frac{\cos\frac{t}{2}}{2\sin\frac{t}{2}}dt\\&=-\frac{2}{\pi}(1+n)^\beta\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\cos x\int_{\frac{\pi}{n+1}}^{\pi}\cos^2\frac{t}{2}d\\&=-\frac{2}{\pi}(1+n)^\beta\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\cos x\left[\frac{\pi}{2}-\frac{\pi}{2(n+1)}-\frac{1}{2}\sin\frac{\pi}{n+1}\right].\end{aligned}$$

We get

$$\begin{aligned}\sup_{f\in L^p(\widetilde{\omega})_\beta}\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{\pi}{n+1}\right)\right|&\geq\left|\widetilde{T}_{n,A}f_1(x)-\widetilde{f}_1\left(x,\frac{\pi}{n+1}\right)\right|\\&=\left|\sum_{k=2}^{n}a_{n,k}(1+n)^\beta\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\cos x+n(n+1)^{\beta-1}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\cos x\right.\\&\quad\left.-\frac{1}{\pi}(1+n)^\beta\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\cos x\sin\frac{\pi}{n+1}\right|.\end{aligned}$$

Thus in a special case, for $x=0$, we get

$$\left|(1+n)^\beta\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\left[1+\frac{\pi}{2}-\frac{\pi}{2(n+1)}-\frac{1}{2}\sin\frac{\pi}{n+1}\right]\right|\gg(1+n)^\beta\widetilde{\omega}\left(\frac{\pi}{n+1}\right).$$

Hence, we finally obtain equation (12).

When $x=x_0$, we can consider the function $f_{x_0}(\cdot)=f_1(\cdot-x_0)$ instead of $f_1(\cdot)$. Thus our proof is complete. □

5.4 Proof of Theorem 4

Let us fix a point x and let us consider the class $L^p(\widetilde{\omega})_\beta$, with $\beta < 1 - \frac{1}{p}$, of all functions $f \in L^p$ such that $\widetilde{\omega}_\beta f(\delta)_{L^p} \le \widetilde{\omega}(\delta)$, $(0 \le \delta \le 2\pi)$.

The Theorem 2 implies the estimate

$$\sup_{f\in L^p(\widetilde{\omega})_\beta} \left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right| \ll (n+1)^\beta \widetilde{\omega}\left(\frac{\pi}{n+1}\right).$$

On the other hand, the function

$$f_0(x) = \sum_{k=2}^{\infty}\left[k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^\beta \frac{k-1}{k}\widetilde{\omega}\left(\frac{1}{k-1}\right)\right]\sin kx,$$

by Lemma 3 belongs to class $L^p(\widetilde{\omega})_\beta$, if $t^{-\beta}\widetilde{\omega}(t)$ is a concave and nondecreasing function of t.

We can see that the function f_0 satisfies the condition (7) with $\frac{1}{p} < \gamma < \beta + \frac{1}{p}$ and the condition (9). Indeed, using the estimation (18) obtained in the proof of Lemma 3, we have

$$|\Psi_x^0(t)| = |f_0(x+t) - f_0(x-t)| \ll |t|^{-\beta}\widetilde{\omega}(|t|).$$

Thus, we easily get

$$\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{t^{-\gamma}\left|\psi_x^0(t)\right|}{\widetilde{\omega}(|t|)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p} \ll \left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{t^{-\gamma}|t|^{-\beta}\widetilde{\omega}(|t|)}{\widetilde{\omega}(|t|)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p}$$

$$\ll \left\{\int_{\frac{\pi}{n+1}}^{\pi} t^{-\gamma p}dt\right\}^{1/p} = O_x(n+1)^{\gamma-\frac{1}{p}}$$

and

$$\left\{\int_0^{\frac{\pi}{n+1}}\left(\frac{\left|\psi_x^0(t)\right|}{\widetilde{\omega}(|t|)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p} \ll \left\{\int_0^{\frac{\pi}{n+1}}\left(\frac{|t|^{-\beta}\widetilde{\omega}(|t|)}{\widetilde{\omega}(|t|)}\right)^p t^{\beta p}dt\right\}^{1/p} = O_x(n+1)^{-\frac{1}{p}}.$$

Hence, by Theorem 2 the estimation (11) holds for the function f_0.

On the other hand using the fact

$$\widetilde{f_0}(x) = \sum_{k=2}^{\infty}\left[k^\beta \widetilde{\omega}\left(\frac{1}{k}\right) - (k-1)^\beta \frac{k-1}{k}\widetilde{\omega}\left(\frac{1}{k-1}\right)\right]\cos kx$$

we get

$$\sup_{f\in L^p(\omega)_\beta}\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right| \ge \left|\widetilde{T}_{n,A}f_0(x) - \widetilde{f_0}(x)\right| = \left|\sum_{k=2}^{\infty}a_{n,k}\left(\widetilde{S}_k f_0(x) - \widetilde{f_0}(x)\right)\right|$$

$$= \left|\sum_{k=2}^{\infty}a_{n,k}\sum_{l=k+1}^{\infty}\left(\left[l^\beta\widetilde{\omega}\left(\frac{1}{l}\right) - (l-1)^\beta\frac{l-1}{l}\widetilde{\omega}\left(\frac{1}{l-1}\right)\right]\cos lx\right)\right|.$$

Thus in a special case, for $x = 0$, we get

$$\left|\sum_{k=2}^{n}a_{n,k}\left(\widetilde{S}_k f_0(0) - \widetilde{f_0}(0)\right)\right| = \sum_{k=2}^{n}a_{n,k}k^\beta\frac{k}{k+1}\widetilde{\omega}\left(\frac{1}{k}\right) \ge \frac{1}{2}\sum_{k=2}^{n}a_{n,k}k^\beta\widetilde{\omega}\left(\frac{1}{k}\right).$$

Using the inequality (18) , we have

$$k^\beta\widetilde{\omega}\left(\frac{1}{k}\right) \ge \frac{k+1}{2k+1}(k+1)^\beta\widetilde{\omega}\left(\frac{1}{k+1}\right) + \frac{(k+1)(k-1)}{k(2k+1)}(k-1)^\beta\widetilde{\omega}\left(\frac{1}{k-1}\right)$$

$$\ge \frac{1}{2}(k+1)^\beta\widetilde{\omega}\left(\frac{1}{k+1}\right),$$

which implies

$$\sup_{f \in L^p(\omega)_\beta} \left| \widetilde{T}_{n,A} f(x) - \widetilde{f}(x) \right| \geq \frac{1}{4} \sum_{k=2}^{n} a_{n,k} (k+1)^\beta \widetilde{\omega} \left(\frac{1}{k+1} \right) \gg (n+1)^\beta \widetilde{\omega} \left(\frac{1}{n+1} \right).$$

Hence, we finally obtain (13) .

When $x = x_0$, we can consider the function $f_{x_0}(\cdot) = f_0(\cdot - x_0)$ instead of $f_0(\cdot)$. Thus our proof is complete. □

5.5 Proof of Theorem 5

Note that if $f \in L^p(\widetilde{\omega})_\beta$, then Theorem 1 implies

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \ll (n+1)^\beta \, \widetilde{\omega} \left(\frac{\pi}{n+1} \right).$$

Thus, we easily get

$$\begin{aligned} \left\| \widetilde{T}_{n,A} f(\cdot) - \widetilde{f}\left(\cdot, \frac{\pi}{n+1}\right) \right\|_{L^p} &= \left\{ \int_{-\pi}^{\pi} \left| \widetilde{T}_{n,A} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) \right|^p dx \right\}^{\frac{1}{p}} \\ &\ll (n+1)^\beta \, \widetilde{\omega} \left(\frac{\pi}{n+1} \right) = O\left((n+1)^\beta \, \widetilde{\omega} \left(\frac{\pi}{n+1} \right) \right). \end{aligned}$$

The conditions (6) and (7) from Theorem 1 are satisfied in the following form

$$\left\| \left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{t \, |\psi_\cdot(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \right\|_{L^p} \leq \left\{ \int_0^{\frac{\pi}{n+1}} \frac{t^p \left\| \psi_\cdot(t) \sin^\beta \frac{t}{2} \right\|_{L^p}^p}{(\widetilde{\omega}(t))^p} dt \right\}^{1/p} = O\left((n+1)^{-1-\frac{1}{p}} \right)$$

$$\left\| \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{-\gamma} \, |\psi_\cdot(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \right\|_{L^p} \leq \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \frac{t^{-\gamma p} \left\| \psi_\cdot(t) \sin^\beta \frac{t}{2} \right\|_{L^p}^p}{(\widetilde{\omega}(t))^p} dt \right\}^{1/p} = O\left((n+1)^{\gamma-\frac{1}{p}} \right).$$

Hence our proof is complete. □

5.6 Proof of Theorem 6

Similarly to the previous proof, note that if $f \in L^p(\widetilde{\omega})_\beta$, then Theorem 2 implies

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}(x) \right| \ll (n+1)^\beta \, \widetilde{\omega} \left(\frac{\pi}{n+1} \right).$$

Thus, we easily get

$$\left\| \widetilde{T}_{n,A} f(\cdot) - \widetilde{f}(\cdot) \right\|_{L^p} = \left\{ \int_{-\pi}^{\pi} \left| \widetilde{T}_{n,A} f(x) - \widetilde{f}(x) \right|^p dx \right\}^{\frac{1}{p}} \ll (n+1)^\beta \, \widetilde{\omega} \left(\frac{\pi}{n+1} \right) = O\left((n+1)^\beta \, \widetilde{\omega} \left(\frac{\pi}{n+1} \right) \right).$$

We know by the previous proof that the condition (7) is satisfied and the condition (9) , from Theorem 2, is satisfied in the following form

$$\left\| \left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{|\psi_\cdot(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \right\|_{L^p} \leq \left\{ \int_0^{\frac{\pi}{n+1}} \frac{\left\| \psi_\cdot(t) \sin^\beta \frac{t}{2} \right\|_{L^p}^p}{(\widetilde{\omega}(t))^p} dt \right\}^{1/p} = O\left((n+1)^{-\frac{1}{p}} \right).$$

Hence, our proof is complete. □

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