

Research Article

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Existence and stability of impulsive coupled system of fractional integrodifferential equations

<https://doi.org/10.1515/dema-2019-0035>

Received April 8, 2019; accepted August 19, 2019

Abstract: In this manuscript, we deal with a class and coupled system of implicit fractional differential equations, having some initial and impulsive conditions. Existence and uniqueness results are obtained by means of Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. Hyers–Ulam stability is investigated by using classical technique of nonlinear functional analysis. Finally, we provide illustrative examples to support our obtained results.

Keywords: Caputo fractional derivative; impulsive condition; existence and uniqueness theory; Hyers–Ulam stability

MSC: 26A33; 34A08; 35B40

1 Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order, but with this definition, many interesting questions will arise; for example, if the first derivative of a function gives you the slope of the function, what is the geometrical meaning of half derivative? In half order, which operator must be used twice to obtain the first derivative? The early history of these questions goes back to the birth of fractional calculus in 1695 when Gottfried Wilhelm Leibniz suggested the possibility of fractional derivatives for the first time [1].

Fractional differential equations (FDEs) have recently gained much importance and attention. It is the extension of classical calculus. FDEs as well as fractional integrodifferential equations appear naturally as generalizations to existing models with integer derivatives and they also present new models for many applications in physics, control theory, chemistry, biology, electrical circuits, mechanics, signal and image processing, heat conduction, computer analysis and economics etc., reader is referred to [2–7]. For example, in the last three fields, some important considerations such as modeling, curve fitting, filtering, pattern recognition, edge detection, identification, stability, controllability, observability and robustness are now linked to long-range dependence phenomena. Similar progress has been made in other fields listed here.

Recently, the study of existence and uniqueness of solutions (EUSs) to initial and boundary value problems for FDEs has attracted considerable attention, we refer to [8–17] and the references therein. A large number of methods are used in investigating the EUSs for FDEs, such as comparison methods, fixed point

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method and coincidence degree method etc [18, 19]. EUSs for FDEs in finite dimensional as well as infinite dimensional spaces were studied by several authors [20–25].

The study of coupled system (CS) of FDEs has also attracted some attention. Because mathematical models of various phenomena in the field of biology, physics and psychology etc. are in the form of CS of differential equations (DEs). For the study of CS of FDEs, we refer the reader to [26–31].

Another important class of DEs is known as impulsive differential equations (IDEs). This class plays the role of an effective mathematical tools for those evolution processes that are subject to abrupt changes in their states. There are many physical phenomena that exhibit impulsive behavior such as the maintenance of a species through periodic stocking or harvesting, mechanical systems subject to impacts, the thrust impulse maneuver of a spacecraft and the function of heart, we recommended [32–41] for more details on the theory of IDEs. It is well known that in the evolution processes the impulsive phenomena can be found in many situations. For example, change of the valve shutter speed in its transition from open to closed state [42], operation of a damper subjected to the percussive effects [43], disturbances in cellular neural networks [44, 45], relaxational oscillations of the electro mechanical systems [46], percussive systems with vibrations [47], using the radial acceleration, control of the satellite orbit [48], dynamic of system with automatic regulation [48], fluctuations of pendulum systems in the case of external impulsive effects [49], price fluctuations in commodity markets [50] and so on.

Furthermore, stability analysis plays an important role in the systems of DEs, numerical analysis, economics, optimization theory etc. In the literature, we can find different types of stability such as Mittag–Leffler stability, Lyapunov stability, Exponential stability and Hyers–Ulam (HU) stability [51–53]. Recently, HU stability is interesting for the stability analysis of FDEs. Many manuscripts are devoted to HU stability see [54–62].

In this manuscript, we study four different types of Ulam stability for implicit FDEs with impulses and initial conditions, which are HU stability, generalized HU stability, HU–Rassias stability and generalized HU–Rassias stability.

In [63], Wang *et al.* developed sufficient conditions for the following problem:

$$\begin{cases} {}^c D^q u(t) = f(t, u(t)), & t \in [0, T], \quad \mathcal{D} := \{t_1, t_2, \dots, t_m\}, \\ u(0) = u_0, \\ \Delta u(t_i) = I_i(u(t_i)), \end{cases}$$

where $0 < q < 1$, $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $I_i : \mathbb{R} \rightarrow \mathbb{R}$ are nonlinear functions describing the jump size $I_i(u(t_i)) = u(t_i^+) - u(t_i^-)$ at t_i , $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$.

Tian *et al.* [64] investigated the existence of positive solutions to the following impulsive FDE:

$$\begin{cases} {}^c D^\tau u(t) = \Phi(t, u(t)), & 0 \leq t \leq 1, \quad t \neq t_k, \quad k = 1, 2, \dots, p, \\ \Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = \bar{I}_k(u(t_k)), & k = 1, 2, \dots, p, \\ u(0) = g(u), \quad u(1) = h(u), \end{cases}$$

where $1 < \tau \leq 2$, $\Phi : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and $I_k, \bar{I}_k : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions and $\Delta(u(t_k)) = u(t_k^+) - u(t_k^-)$, $\Delta(u'(t_k)) = u'(t_k^+) - u'(t_k^-)$ with $u(t_k^+)$, $u'(t_k^+)$, $u(t_k^-)$, $u'(t_k^-)$, are the respective right and left limits of $u(t_k)$ at $t = t_k$.

Zhang *et al.* [65] extended the work to CS of 2m–point boundary value problem for impulsive FDEs at resonance as:

$$\begin{cases} D_{0+}^\alpha u(t) = F(t, v(t), D^\beta v(t)), \quad D_{0+}^\beta v(t) = G(t, u(t), D^\alpha u(t)), & 0 < t < 1, \\ \Delta u(t_i) = A_i(v(t_i), D^\beta v(t_i)), \quad \Delta D_{0+}^\alpha u(t_i) = B_i(v(t_i), D^\beta v(t_i)), & i = 1, 2, \dots, k, \\ \Delta v(t_i) = C_i(u(t_i), D^\alpha u(t_i)), \quad \Delta D_{0+}^\beta v(t_i) = D_i(u(t_i), D^\alpha u(t_i)), & i = 1, 2, \dots, k, \\ D_{0+}^{\tau-1} u = \sum_{i=1}^m a_i D^{\tau-1} u(\xi_i), \quad u(1) = \sum_{i=1}^m b_i \eta_i^{2-\tau} u(\eta_i), \\ D_{0+}^{s-1} v = \sum_{i=1}^m c_i D^{s-1} v(\xi_i), \quad v(1) = \sum_{i=1}^m d_i \theta_i^{2-s} v(\theta_i), \end{cases}$$

where $1 < \tau, \varsigma < 2, \tau - q \geq 1, \varsigma - p \geq 1$ and $F, G : (0, 1) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions.

Shah *et al.* [66] investigated the EUSs of:

$$\begin{cases} {}^c D^\alpha x(t) = f(t, x(t), y(t)), & 0 \leq t \leq 1, t \neq t_j, j = 1, 2, \dots, m, \\ {}^c D^\beta y(t) = g(t, x(t), y(t)), & 0 \leq t \leq 1, t \neq t_i, i = 1, 2, \dots, n, \\ x(0) = h(x), x(1) = g(x) \text{ and } y(0) = k(y), y(1) = f(y), \\ \Delta x(t_j) = I_j(x(t_j)), \Delta x'(t_j) = \bar{I}_j(x(t_j)), & j = 1, 2, \dots, m, \\ \Delta y(t_i) = I_i(y(t_i)), \Delta y'(t_i) = \bar{I}_i(y(t_i)), & i = 1, 2, \dots, n, \end{cases}$$

where $\alpha, \beta \in (1, 2]$, $f, g : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions.

Benchohra and Lazreg [67] investigated the implicit FDEs:

$$\begin{cases} {}^c D^\alpha u(t) = \Phi(t, u(t), {}^c D^\alpha u(t)), & 0 \leq t \leq 1, 0 < \alpha \leq 1, \\ u(0) = u_0, \end{cases}$$

where ${}^c D^\alpha$ is the Caputo derivative of fractional order α and $\Phi : (0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous function, $u_0 \in \mathbb{R}$.

Zada *et al.* [68] studied the EUSs to the implicit FDE of the form:

$$\begin{cases} {}^c D_{0,t}^\beta y(t) = \mathcal{G}(t, y(t), {}^c D_{0,t}^\beta y(t)), & t \in (t_k, s_k], k = 0, 1, \dots, m, \beta \in (0, 1], \\ y(t) = I_{s_{k-1}, t_k}^\beta (\xi_k(t, y(t))), & t \in (s_{k-1}, t_k], k = 1, 2, \dots, m, \\ y(0) = I_{0,T}^\beta \eta(t, y(t)), \end{cases}$$

where ${}^c D_{0,t}^\beta$ is the Caputo fractional derivative of order β with lower limit 0 and $\mathcal{G} : (0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function.

In this manuscript, we study the existence, uniqueness and HU stability results of the implicit FDE with impulsive condition as:

$$\begin{cases} {}^c D^\tau x(t) = \mathcal{F}(t, x(t), {}^c D^\tau x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^\tau x(\xi)) d\xi, & t \in \mathcal{J}, t \neq t_i \text{ for } i = 1, 2, \dots, m, \\ x(0) = h(x), \\ \Delta x(t_i) = I_i(x(t_i)), & i = 1, 2, \dots, m, \end{cases} \quad (1.1)$$

where $0 < \tau < 1, \mathcal{J} = [0, T]$ with $T > 0, \sigma, \delta > 0$. The functions $\mathcal{F}, \mathcal{G} : \mathcal{J} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $h : \mathcal{X} \rightarrow \mathbb{R}$ are continuous functions. Also we investigate the aforementioned analysis for the proposed implicit CS:

$$\begin{cases} {}^c D^\tau x(t) = \mathcal{F}(t, y(t), {}^c D^\tau x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), {}^c D^\tau x(\xi)) d\xi, & t \in \mathcal{J}, t \neq t_i \text{ for } i = 1, 2, \dots, m, \\ {}^c D^\varsigma y(t) = \mathcal{F}'(t, x(t), {}^c D^\varsigma y(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), {}^c D^\varsigma y(\xi)) d\xi, & t \in \mathcal{J}, t \neq t_j \text{ for } j = 1, 2, \dots, n, \\ x(0) = h(x), y(0) = g(y), \\ \Delta x(t_i) = I_i(x(t_i)), & i = 1, 2, \dots, m, \\ \Delta y(t_j) = I_j(y(t_j)), & j = 1, 2, \dots, n, \end{cases} \quad (1.2)$$

where $0 < \tau, \varsigma < 1, \sigma, \delta > 0$ and $\mathcal{J} = [0, T]$ with $T > 0$. The functions $\mathcal{F}, \mathcal{G}, \mathcal{F}', \mathcal{G}' : \mathcal{J} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $h : \mathcal{X} \rightarrow \mathbb{R}, g : \mathcal{Y} \rightarrow \mathbb{R}$ are continuous.

Notation: Let $\mathcal{J} = [0, T]$. We denote $PC(\mathcal{J}, \mathbb{R})$ by \mathcal{M} i.e the space of all piecewise continuous functions. The interval $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \mathcal{J}_2 \cup \dots \cup \mathcal{J}_i$, where $\mathcal{J}_0 = [t_0, t_1], \mathcal{J}_1 = (t_1, t_2], \mathcal{J}_2 = (t_2, t_3], \dots, \mathcal{J}_i = (t_i, t_{i+1}], i =$

$1, 2, \dots, m$ and $\mathcal{J}' = \mathcal{J} - \{t_1, t_2, t_3, \dots, t_i\}$.

We define $\mathcal{M} = \{x : \mathcal{J} \rightarrow \mathbb{R} : x \in C(\mathcal{J}_i, \mathbb{R}) \text{ and } x(t_i^+), x(t_i^-) \text{ exist so that } \Delta x(t_i) = x(t_i^+) - x(t_i^-) \text{ for } i = 1, 2, \dots, m\}$.

The rest of this paper is arranged as follows: In section 2, we present some basic notions needed to prove our main results. In section 3, we setup some adequate conditions for the EUSs, by applying some standard fixed point principles to the proposed system (1.1) and (1.2), respectively. In section 4, we setup applicable results under which the solution of the considered problems (1.1) and (1.2), respectively, fulfills the conditions of different kinds of Ulam stability. The establish results are illustrated by examples in section 5.

2 Supplementary results

The following definitions and lemmas are adopted from [18].

Definition 2.1. The integral of a function $u \in L^1(\mathcal{J}, \mathbb{R})$ of order $\tau \in (0, \infty)$ is defined by

$$I^\tau u(t) = \frac{1}{\Gamma(\tau)} \int_0^t (t - \xi)^{\tau-1} u(\xi) d\xi,$$

where

$$\Gamma(\tau) = \int_0^\infty t^{\tau-1} e^{-t} dt; \tau > 0.$$

Definition 2.2. The Caputo arbitrary order τ derivative of function $u \in C^{(n)}((0, \infty), \mathbb{R})$ is defined by

$${}^c D^\tau u(t) = \frac{1}{\Gamma(\rho - \tau)} \int_0^t (t - \xi)^{\rho-\tau-1} u^{(n)}(\xi) d\xi,$$

where $\rho = [\tau] + 1$ in which $[\tau]$ represents the integer part of τ .

Lemma 2.3. For $\tau > 0$, the following result hold:

$$I^\tau [{}^c D^\tau u(t)] = u(t) - \sum_{\rho=0}^{n-1} \frac{u^{(\rho)}(0)}{\rho!} t^\rho,$$

where $n = [\tau] + 1$.

Lemma 2.4. For $\tau > 0$, the solution of FDE

$${}^c D^\tau u(t) = \beta(t)$$

is given by

$$u(t) = I^\tau \beta(t) + \sum_{\rho=0}^{n-1} \frac{u^{(\rho)}(0)}{\rho!} t^\rho,$$

where $n = [\tau] + 1$.

Theorem 2.5. [69] Let \mathcal{M} be a Banach space, $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{M}$ be a completely continuous operator, and the set $\Omega = \{x \in \mathcal{M} : x = \mathfrak{N}\mathcal{T}x, 0 < \mathfrak{N} < 1\}$ is bounded, then \mathcal{T} has at least one fixed point in \mathcal{M} .

Theorem 2.6. [69] Let \mathcal{B} be a non-empty closed subset of a Banach space \mathcal{M} . Then any contraction mapping \mathcal{T} of \mathcal{B} into itself has a unique fixed point.

Theorem 2.7. [70] Let \mathcal{H} be a convex closed and non-empty subset of Banach space $\mathcal{X} \times \mathcal{Y}$ and \mathbb{F}, \mathbb{G} be the operators so that

(i) $\mathbb{F}x + \mathbb{G}y \in \mathcal{H}$ whenever $x, y \in \mathcal{H}$.

(ii) \mathbb{F} is compact and continuous and \mathbb{G} is contraction mapping.

Then $\exists z \in \mathcal{H}$ so that $z = \mathbb{F}z + \mathbb{G}z$, where $z = (x, y) \in \mathcal{X} \times \mathcal{Y}$.

3 Existence and uniqueness

Here we present our result about the existence of at least one solution to considered problem (1.1).

Theorem 3.1. Let $0 < \tau \leq 1$ and $\alpha \in \mathcal{M}$ be a continuous function, then a function $x \in \mathcal{M}$ is solution to problem

$$\begin{cases} {}^c D^\tau x(t) = \alpha(t), & t \in \mathcal{J}, t \neq t_i, \text{ for } i = 1, 2, \dots, m, \\ x(0) = h(x), \\ \Delta x(t_i) = I_i(x(t_i)), & i = 1, 2, \dots, m, \end{cases} \quad (3.1)$$

where

$$\alpha(t) = \mathcal{F}(t, x(t), {}^c D^\tau x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^\tau x(\xi)) d\xi,$$

if and only if x satisfies

$$x(t) = \begin{cases} \frac{1}{\Gamma(\tau)} \int_0^t (t-\xi)^{\tau-1} \alpha(\xi) d\xi + h(x), & t \in \mathcal{J}_0, \\ \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} \alpha(\xi) d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} \alpha(\xi) d\xi + I_i(x(t_i)) \right] + h(x), & t \in \mathcal{J}_i, i = 1, 2, \dots, m. \end{cases} \quad (3.2)$$

Proof. Let x be the solution of problem (3.1), then using Lemma 2.4, for each $t \in \mathcal{J}_0$, we have

$$x(t) = I^\tau \alpha(t) + x(0) = \frac{1}{\Gamma(\tau)} \int_0^t (t-\xi)^{\tau-1} \alpha(\xi) d\xi + x(0). \quad (3.3)$$

Using the initial condition $x(0) = h(x)$, we get from (3.3)

$$x(t) = \frac{1}{\Gamma(\tau)} \int_0^t (t-\xi)^{\tau-1} \alpha(\xi) d\xi + h(x), \quad t \in \mathcal{J}_0. \quad (3.4)$$

Similarly, for $t \in \mathcal{J}_1$,

$$x(t) = \frac{1}{\Gamma(\tau)} \int_{t_1}^t (t-\xi)^{\tau-1} \alpha(\xi) d\xi + x(t_1), \quad (3.5)$$

we have

$$x(t_1^-) = \frac{1}{\Gamma(\tau)} \int_0^{t_1} (t-\xi)^{\tau-1} \alpha(\xi) d\xi + h(x), \quad x(t_1^+) = x(t_1).$$

From

$$\Delta x(t_1) = x(t_1^+) - x(t_1^-) = I_1(x(t_1)),$$

we get

$$x(t_1) = \frac{1}{\Gamma(\tau)} \int_0^t (t - \xi)^{\tau-1} \alpha(\xi) d\xi + h(x) + I_1(x(t_1)), \quad t \in \mathcal{J}_1.$$

Putting for $x(t_1)$, (3.5) implies

$$x(t) = \frac{1}{\Gamma(\tau)} \int_{t_1}^t (t - \xi)^{\tau-1} \alpha(\xi) d\xi + \frac{1}{\Gamma(\tau)} \int_0^t (t - \xi)^{\tau-1} \alpha(\xi) d\xi + h(x) + I_1(x(t_1)), \quad t \in \mathcal{J}_1.$$

Generalizing in this way, for $t \in \mathcal{J}_i$, we have

$$x(t) = \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t - \xi)^{\tau-1} \alpha(\xi) d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{\tau-1} \alpha(\xi) d\xi + I_i(x(t_i)) \right] + h(x), \quad i = 1, 2, \dots, m.$$

Using (3.4) and (3.6), we obtain (3.2).

Conversely, let x be the solution of integral equation (3.2), then the τ th order derivative of (3.2) will lead us to the first equation in (3.1). Further, it is easy to obtain the initial and impulsive conditions of (3.1). \square

Corollary 3.2. *In light of Theorem 3.1, problem (1.1) has the following solution:*

$$x(t) = \begin{cases} \frac{1}{\Gamma(\tau)} \int_0^t (t - \xi)^{\tau-1} \alpha(\xi) d\xi + h(x), & t \in \mathcal{J}_0, \\ \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t - \xi)^{\tau-1} \alpha(\xi) d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{\tau-1} \alpha(\xi) d\xi + I_i(x(t_i)) \right] + h(x), & t \in \mathcal{J}_i, \quad i = 1, 2, \dots, m, \end{cases} \quad (3.6)$$

where

$$\alpha(t) = \mathcal{F}(t, x(t), {}^c D^\tau x(t)) + \int_0^t \frac{(t - \xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^\tau x(\xi)) d\xi.$$

For simplicity, we use the following notation:

$$\begin{aligned} \tilde{x}(t) &= \mathcal{F}(t, x(t), {}^c D^\tau x(t)) + \int_0^t \frac{(t - \xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^\tau x(\xi)) d\xi \\ &= \mathcal{F}(t, x(t), \tilde{x}(t)) + \int_0^t \frac{(t - \xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), \tilde{x}(\xi)) d\xi. \end{aligned}$$

Let $\mathcal{M} = PC(\mathcal{J}, \mathbb{R})$ be a Banach space endowed with norm

$$\|x\|_{\mathcal{M}} = \max\{|x(t)| : t \in \mathcal{J}\}.$$

If x is a solution of the problem (1.1), then

$$x(t) = \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t - \xi)^{\tau-1} \alpha(\xi) d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t - \xi)^{\tau-1} \alpha(\xi) d\xi + I_i(x(t_i)) \right] + h(x), \quad \forall t \in \mathcal{J}_i, \quad i = 1, 2, \dots, m.$$

To transform problem (1.1) into a fixed point problem, we define an operator $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{M}$ by

$$(\mathcal{T}x)(t) = \begin{cases} \frac{1}{\Gamma(\tau)} \int_0^t (t-\xi)^{\tau-1} \tilde{x}(\xi) d\xi + h(x), & t \in \mathcal{J}_0, \\ \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} \tilde{x}(\xi) d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} \tilde{x}(\xi) d\xi + I_i(x(t_i)) \right] + h(x), & t \in \mathcal{J}_i, i = 1, 2, \dots, m, \end{cases} \quad (3.7)$$

where

$$\tilde{x}(t) = \mathcal{F}(t, x(t), \tilde{x}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), \tilde{x}(\xi)) d\xi.$$

For our next results, we put the following hypotheses.

Assume that

- $[A_1]$ there exist constants $M_1 > 0$ and $0 < N_1 < 1$ so that for each $t \in \mathcal{J}$ and for all $u, \bar{u} \in \mathcal{M}$ and $w, \bar{w} \in \mathbb{R}$, the following relation holds

$$|\mathcal{F}(t, u, w) - \mathcal{F}(t, \bar{u}, \bar{w})| \leq M_1 |u - \bar{u}| + N_1 |w - \bar{w}|;$$

Similarly, there exist constants $M_2 > 0$ and $0 < N_2 < 1$ so that for each $t \in \mathcal{J}$ and for all $u, \bar{u} \in \mathcal{M}$ and $w, \bar{w} \in \mathbb{R}$ the following relation holds

$$|\mathcal{G}(t, u, w) - \mathcal{G}(t, \bar{u}, \bar{w})| \leq M_2 |u - \bar{u}| + N_2 |w - \bar{w}|;$$

- $[A_2]$ for any $u, \bar{u} \in \mathcal{M}$, \exists a constant $A_{I_i} > 0$ so that

$$|I_i(u(t_i)) - I_i(\bar{u}(t_i))| \leq A_{I_i} |u(t_i) - \bar{u}(t_i)|;$$

- $[A_3]$ for any $u, w \in \mathbb{R}$, \exists a constant $A_h > 0$ so that

$$|h(u) - h(w)| \leq A_h |u - w|;$$

- $[A_4]$ there exist bounded functions $a_1, b_1, c_1 \in \mathcal{M}$ so that

$$|\mathcal{F}(t, u(t), w(t))| \leq a_1(t) + b_1(t) |u(t)| + c_1(t) |w(t)|$$

with $a_1^* = \sup_{t \in \mathcal{J}} a_1(t)$, $b_1^* = \sup_{t \in \mathcal{J}} b_1(t)$, $c_1^* = \sup_{t \in \mathcal{J}} c_1(t) < 1$;

Similarly, there exist bounded functions $a_2, b_2, c_2 \in \mathcal{M}$ so that

$$|\mathcal{G}(t, u(t), w(t))| \leq a_2(t) + b_2(t) |u(t)| + c_2(t) |w(t)|$$

with $a_2^* = \sup_{t \in \mathcal{J}} a_2(t)$, $b_2^* = \sup_{t \in \mathcal{J}} b_2(t)$, $c_2^* = \sup_{t \in \mathcal{J}} c_2(t) < 1$;

- $[A_5]$ \exists a constant $\lambda > 0$ so that $|h(u)| \leq \lambda$, for all $u \in \mathcal{M}$;
- $[A_6]$ for each $u \in \mathbb{R}$, the function $I_i : \mathbb{R} \rightarrow \mathbb{R}$; $i = 1, 2, \dots, m$ are assumed to be continuous and for constants $\mathcal{K}, \mathcal{L} > 0$, the inequality $|I_i(u(t_i))| \leq \mathcal{K} |u(t)| + \mathcal{L}$, holds.

The existence of solution for the problem (1.1) is based on Theorem 2.5.

Theorem 3.3. *Problem (1.1) has at least one solution if the hypothesis $[A_1]$ – $[A_6]$ are satisfied.*

Proof. Let the operator \mathcal{T} is defined in (3.7). We need to prove that (1.1) has at least one solution.

Let the operator \mathcal{T} be continuous. Consider a sequence $\{x_n\}$ so that $x_n \rightarrow x \in \mathcal{M}$, $t \in \mathcal{J}$, then

$$|(\mathcal{T}x_n)(t) - (\mathcal{T}x)(t)| \leq \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} |\tilde{x}_n(\xi) - \tilde{x}(\xi)| d\xi + \sum_{i=1}^m \frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} |\tilde{x}_n(\xi) - \tilde{x}(\xi)| d\xi$$

$$+ \sum_{i=1}^m |I_i(x_n(t_i)) - I_i(x(t_i))| + |h(x_n) - h(x)|, \quad i = 1, 2, \dots, m, \quad (3.8)$$

where $\tilde{x}_n, \tilde{x} \in \mathcal{M}$ and are given by

$$\tilde{x}_n(t) = \mathcal{F}(t, x_n(t), \tilde{x}_n(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x_n(\xi), \tilde{x}_n(\xi)) d\xi$$

and

$$\tilde{x}(t) = \mathcal{F}(t, x(t), \tilde{x}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), \tilde{x}(\xi)) d\xi.$$

By utilizing $[A_1]$, we have

$$\begin{aligned} |\tilde{x}_n(t) - \tilde{x}(t)| &= \left| \mathcal{F}(t, x_n(t), \tilde{x}_n(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x_n(\xi), \tilde{x}_n(\xi)) d\xi \right. \\ &\quad \left. - \mathcal{F}(t, x(t), \tilde{x}(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), \tilde{x}(\xi)) d\xi \right| \\ &\leq |\mathcal{F}(t, x_n(t), \tilde{x}_n(t)) - \mathcal{F}(t, x(t), \tilde{x}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}(\xi, x_n(\xi), \tilde{x}_n(\xi)) - \mathcal{G}(\xi, x(\xi), \tilde{x}(\xi))| d\xi \\ &\leq M_1 |x_n(t) - x(t)| + N_1 |\tilde{x}_n(t) - \tilde{x}(t)| + \frac{t^\sigma}{\sigma \Gamma(\delta)} \left(M_2 |x_n(t) - x(t)| + N_2 |\tilde{x}_n(t) - \tilde{x}(t)| \right). \end{aligned}$$

Then

$$|\tilde{x}_n(t) - \tilde{x}(t)| \leq \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} \right) |x_n(t) - x(t)|. \quad (3.9)$$

Thus, using hypothesis $[A_1]$ – $[A_3]$ and (3.9), inequality (3.8) implies

$$\begin{aligned} |(\mathcal{J}x_n)(t) - (\mathcal{J}x)(t)| &\leq \left[\left(\frac{(1+m)t^\tau}{\Gamma(\tau+1)} \right) \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} \right) \right. \\ &\quad \left. + mA_{I_i} + A_h \right] |x_n(t) - x(t)|, \quad i = 1, 2, \dots, m. \end{aligned}$$

Since for each $t \in \mathcal{J}$, the sequence $x_n \rightarrow x$ as $n \rightarrow \infty$, hence by Lebesgue dominated convergence theorem, (3.8) implies that

$$|(\mathcal{J}x_n)(t) - (\mathcal{J}x)(t)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence

$$\|\mathcal{J}x_n - \mathcal{J}x\|_{\mathcal{M}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So \mathcal{J} is continuous on \mathcal{J} .

Now, we have to show that \mathcal{J} is bounded in \mathcal{M} . So, for any $\wp > 0$, there is $R_E > 0$, so that

$$\mathbf{E} = \{x \in \mathcal{M} : \|x\|_{\mathcal{M}} \leq \wp\}$$

then, we have

$$\|\mathcal{J}x\|_{\mathcal{M}} \leq R_E.$$

For $t \in \mathcal{J}_i$, consider

$$|(\mathcal{J}x)(t)| \leq \frac{1}{\Gamma(\mathfrak{r})} \int_{t_i}^t (t-\xi)^{\mathfrak{r}-1} |\tilde{x}(\xi)| d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\mathfrak{r})} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\mathfrak{r}-1} |\tilde{x}(\xi)| d\xi + |I_i(x(t_i))| \right] + |h(x)|, \quad i = 1, 2, \dots, m. \quad (3.10)$$

Using $[A_4]$, we have

$$\begin{aligned} |\tilde{x}(t)| &\leq |\mathcal{F}(t, x(t), \tilde{x}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}(\xi, x(\xi), \tilde{x}(\xi))| d\xi \\ &\leq a_1(t) + b_1(t)|x(t)| + c_1(t)|\tilde{x}(t)| + \frac{t^\sigma}{\sigma\Gamma(\delta)} \left(a_2(t) + b_2(t)|x(t)| + c_2(t)|\tilde{x}(t)| \right) \\ &\leq a_1^* + b_1^* \|x\|_{\mathcal{M}} + c_1^* \|\tilde{x}\|_{\mathcal{M}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \left(a_2^* + b_2^* \|x\|_{\mathcal{M}} + c_2^* \|\tilde{x}\|_{\mathcal{M}} \right). \end{aligned}$$

Therefore, we get

$$\|\tilde{x}(t)\| \leq \|\tilde{x}\|_{\mathcal{M}} \leq \frac{a_1^* + b_1^* \|x\|_{\mathcal{M}}}{1 - c_1^* - c_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \frac{a_2^* + b_2^* \|x\|_{\mathcal{M}}}{1 - c_1^* - c_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} = \hbar. \quad (3.11)$$

By using (3.11) and $[A_4]$ – $[A_6]$, (3.10) becomes

$$\begin{aligned} |\mathcal{J}x(t)| &\leq \frac{\hbar}{\Gamma(\mathfrak{r})} \int_{t_i}^t (t-\xi)^{\mathfrak{r}-1} d\xi + \frac{\hbar}{\Gamma(\mathfrak{r})} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\mathfrak{r}-1} d\xi + m(\mathcal{K}|x| + \mathcal{L}) + \lambda \\ &\leq \frac{\hbar(1+m)t^\mathfrak{r}}{\Gamma(\mathfrak{r}+1)} + m(\mathcal{K}_{\mathcal{J}} + \mathcal{L}) + \lambda = \eta. \end{aligned}$$

Thus

$$\|\mathcal{J}x\|_{\mathcal{M}} \leq \eta.$$

Similarly, for $t \in \mathcal{J}_0$ we can verify that

$$\|\mathcal{J}x\|_{\mathcal{M}} \leq \eta.$$

Now we have to show that the operator \mathcal{J} is equicontinuous in \mathbf{E} . Suppose $t_1, t_2 \in \mathcal{J}_i$, $i = 1, 2, \dots, m$ so that $0 < t_1 < t_2 < T$ and let $x \in \mathbf{E}$, then

$$\begin{aligned} |\mathcal{J}x(t_2) - \mathcal{J}x(t_1)| &\leq \frac{1}{\Gamma(\mathfrak{r})} \int_{t_i}^{t_2} (t_2-\xi)^{\mathfrak{r}-1} |\tilde{x}(\xi)| d\xi + \frac{1}{\Gamma(\mathfrak{r})} \int_{t_i}^{t_1} (t_1-\xi)^{\mathfrak{r}-1} |\tilde{x}(\xi)| d\xi \\ &\quad + \sum_{0 < t_i < t_2 - t_1} \left[\frac{1}{\Gamma(\mathfrak{r})} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\mathfrak{r}-1} |\tilde{x}(\xi)| d\xi + |I_i(x(t_i))| \right] \\ &\leq \frac{1}{\Gamma(\mathfrak{r})} \int_{t_i}^{t_1} [(t_2-\xi)^{\mathfrak{r}-1} - (t_1-\xi)^{\mathfrak{r}-1}] |\tilde{x}(\xi)| d\xi + \frac{1}{\Gamma(\mathfrak{r})} \int_{t_1}^{t_2} (t_2-\xi)^{\mathfrak{r}-1} |\tilde{x}(\xi)| d\xi \\ &\quad + \sum_{0 < t_i < t_2 - t_1} \left[\frac{1}{\Gamma(\mathfrak{r})} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\mathfrak{r}-1} |\tilde{x}(\xi)| d\xi + |I_i(x(t_i))| \right]. \end{aligned} \quad (3.12)$$

Obviously, the right-hand side of inequality (3.12) tends to zero as $t_1 \rightarrow t_2$.

Therefore,

$$|\mathcal{J}x(t_2) - \mathcal{J}x(t_1)| \rightarrow 0 \text{ as } t_1 \rightarrow t_2.$$

Similarly, we can show for $t \in \mathcal{J}_0$. Thus, \mathcal{T} is equicontinuous and therefore completely continuous. Finally, we consider a set $\Omega \subset \mathcal{M}$, which is defined as

$$\Omega = \{x \in \mathcal{M} : x = \aleph \mathcal{T}x, 0 < \aleph < 1\}.$$

We need to prove that set Ω is bounded. Suppose $x \in \Omega$, so that

$$x(t) = \aleph \mathcal{T}x(t), \text{ where } 0 < \aleph < 1.$$

Then for each $t \in \mathcal{J}_i$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned} |x(t)| &= \left| \frac{\aleph}{\Gamma(\mathfrak{r})} \int_{t_i}^t (t-\xi)^{\mathfrak{r}-1} \tilde{x}(\xi) d\xi + \aleph \sum_{i=1}^m \left[\frac{1}{\Gamma(\mathfrak{r})} \int_{t_{i-1}}^{t_i} (t-\xi)^{\mathfrak{r}-1} \tilde{x}(\xi) d\xi + I_i(x(t_i)) \right] + \aleph h(x) \right| \\ &\leq \frac{1}{\Gamma(\mathfrak{r})} \int_{t_i}^t (t-\xi)^{\mathfrak{r}-1} |\tilde{x}(\xi)| d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\mathfrak{r})} \int_{t_{i-1}}^{t_i} (t-\xi)^{\mathfrak{r}-1} |\tilde{x}(\xi)| d\xi + |I_i(x(t_i))| \right] + |h(x)| \\ &\leq \frac{(1+m)t^{\mathfrak{r}}}{\Gamma(\mathfrak{r}+1)} \|\tilde{x}\|_{\mathcal{M}} + m(\mathcal{K}_{\mathcal{G}} + \mathcal{L}) + \lambda. \end{aligned} \quad (3.13)$$

Taking norm on both sides, we have $\|x\|_{\mathcal{M}} \leq \mathcal{Q}$.

Also, for $t \in \mathcal{J}_0$, we can show that $\|x\|_{\mathcal{M}} \leq \mathcal{Q}$.

Hence, Ω is bounded. Thus, by Schaefer's fixed point theorem, \mathcal{T} has at least one fixed point. So the considered problem (1.1) has at least one solution in \mathcal{M} . \square

The next result is based on Theorem 2.6 and concerned with the uniqueness of solution for (1.1).

Theorem 3.4. *If the hypothesis $[A_1]$ – $[A_3]$ along with the inequality*

$$\left[\left(\frac{(1+m)\Gamma^{\mathfrak{r}}}{\Gamma(\mathfrak{r}+1)} \right) \left(\frac{M_1}{1-N_1-N_2 \frac{\Gamma^{\sigma}}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^{\sigma}}{\sigma\Gamma(\delta)}}{1-N_1-N_2 \frac{\Gamma^{\sigma}}{\sigma\Gamma(\delta)}} \right) + mA_{I_i} + A_h \right] < 1 \quad (3.14)$$

are satisfied, then problem (1.1) has a unique solution.

Proof. Suppose $x, \bar{x} \in \mathcal{M}$ and for $t \in \mathcal{J}_i$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned} |(\mathcal{T}x)(t) - (\mathcal{T}\bar{x})(t)| &\leq \frac{1}{\Gamma(\mathfrak{r})} \int_{t_i}^t (t-\xi)^{\mathfrak{r}-1} |\tilde{x}(\xi) - \bar{\tilde{x}}(\xi)| d\xi + \sum_{i=1}^m \frac{1}{\Gamma(\mathfrak{r})} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\mathfrak{r}-1} |\tilde{x}(\xi) - \bar{\tilde{x}}(\xi)| d\xi \\ &\quad + \sum_{i=1}^m |I(x(t_i)) - I(\bar{x}(t_i))| + |h(x) - h(\bar{x})|, \end{aligned} \quad (3.15)$$

where $\tilde{x}, \bar{\tilde{x}} \in \mathcal{M}$ are given by

$$\tilde{x}(t) = \mathcal{F}(t, x(t), \tilde{x}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), \tilde{x}(\xi)) d\xi$$

and

$$\bar{\tilde{x}}(t) = \mathcal{F}(t, \bar{x}(t), \bar{\tilde{x}}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, \bar{x}(\xi), \bar{\tilde{x}}(\xi)) d\xi.$$

Using $[A_1]$, we have

$$\begin{aligned} |\tilde{x}(t) - \bar{x}(t)| &= \left| \mathcal{F}(t, x(t), \tilde{x}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), \tilde{x}(\xi)) d\xi - \mathcal{F}(t, \bar{x}(t), \bar{x}(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, \bar{x}(\xi), \bar{x}(\xi)) d\xi \right| \\ &\leq |\mathcal{F}(t, x(t), \tilde{x}(t)) - \mathcal{F}(t, \bar{x}(t), \bar{x}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}(\xi, x(\xi), \tilde{x}(\xi)) - \mathcal{G}(\xi, \bar{x}(\xi), \bar{x}(\xi))| d\xi \\ &\leq M_1 |x(t) - \bar{x}(t)| + N_1 |\tilde{x}(t) - \bar{x}(t)| + \frac{t^\sigma}{\sigma \Gamma(\delta)} \left(M_2 |x(t) - \bar{x}(t)| + N_2 |\tilde{x}(t) - \bar{x}(t)| \right). \end{aligned}$$

Thus

$$|\tilde{x}(t) - \bar{x}(t)| \leq \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} \right) |x(t) - \bar{x}(t)|. \quad (3.16)$$

Thus, using hypothesis $[A_1]$ – $[A_3]$ and (3.16), inequality (3.15) implies

$$|(\mathcal{T}x)(t) - (\mathcal{T}\bar{x})(t)| \leq \left[\left(\frac{(1+m)t^\tau}{\Gamma(\tau+1)} \right) \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} \right) + mA_{I_i} + A_h \right] |x(t) - \bar{x}(t)|.$$

Now taking norm on both sides, we have

$$\|\mathcal{T}x - \mathcal{T}\bar{x}\|_{\mathcal{M}} \leq \left[\left(\frac{(1+m)\Gamma^\tau}{\Gamma(\tau+1)} \right) \left(\frac{M_1}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}} \right) + mA_{I_i} + A_h \right] \|x - \bar{x}\|_{\mathcal{M}}. \quad (3.17)$$

Similarly, for $x, \bar{x} \in \mathcal{M}$ and $t \in \mathcal{J}_0$, we get

$$\|\mathcal{T}x - \mathcal{T}\bar{x}\|_{\mathcal{M}} \leq \left[\frac{\Gamma^\tau}{\Gamma(\tau+1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}} \right) + A_h \right] \|x - \bar{x}\|_{\mathcal{M}}. \quad (3.18)$$

Since,

$$\begin{aligned} &\left[\frac{\Gamma^\tau}{\Gamma(\tau+1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}} \right) + A_h \right] \\ &\leq \left[\left(\frac{(1+m)\Gamma^\tau}{\Gamma(\tau+1)} \right) \left(\frac{M_1}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma \Gamma(\delta)}} \right) + mA_{I_i} + A_h \right] < 1. \end{aligned}$$

Hence, \mathcal{T} is a contraction. Which implies \mathcal{T} has a unique fixed point, so the problem (1.1) has a unique solution. \square

After this, we consider a CS of nonlinear implicit FDEs with impulsive conditions of (1.2).

Theorem 3.5. *The system:*

$$\begin{cases} {}^c D^\tau x(t) = \alpha(t), \quad \forall t \in \mathcal{J}, \\ {}^c D^\sigma y(t) = \beta(t), \quad \forall t \in \mathcal{J}, \\ x(0) = h(x), \quad \Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \dots, m, \\ y(0) = g(y), \quad \Delta y(t_j) = I_j(y(t_j)), \quad j = 1, 2, \dots, n, \end{cases}$$

has a solution (x, y) if and only if

$$x(t) = \begin{cases} \frac{1}{\Gamma(\mathfrak{r})} \int_0^t (t-\xi)^{\mathfrak{r}-1} \alpha(\xi) d\xi + h(x), & t \in \mathcal{J}_0, \\ \frac{1}{\Gamma(\mathfrak{r})} \int_{t_i}^t (t-\xi)^{\mathfrak{r}-1} \alpha(\xi) d\xi \\ + \sum_{i=1}^m \left[\frac{1}{\Gamma(\mathfrak{r})} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\mathfrak{r}-1} \alpha(\xi) d\xi + I_i(x(t_i)) \right] + h(x), & t \in \mathcal{J}_i, i = 1, 2, \dots, m \end{cases}$$

and

$$y(t) = \begin{cases} \frac{1}{\Gamma(\mathfrak{s})} \int_0^t (t-\xi)^{\mathfrak{s}-1} \beta(\xi) d\xi + g(y), & t \in \mathcal{J}_0, \\ \frac{1}{\Gamma(\mathfrak{s})} \int_{t_j}^t (t-\xi)^{\mathfrak{s}-1} \beta(\xi) d\xi \\ + \sum_{j=1}^n \left[\frac{1}{\Gamma(\mathfrak{s})} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{\mathfrak{s}-1} \beta(\xi) d\xi + I_j(y(t_j)) \right] + g(y), & t \in \mathcal{J}_j, j = 1, 2, \dots, n, \end{cases}$$

where

$$\alpha(t) = \mathcal{F}(t, y(t), {}^c D^{\mathfrak{r}} x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), {}^c D^{\mathfrak{r}} x(\xi)) d\xi$$

and

$$\beta(t) = \mathcal{F}'(t, x(t), {}^c D^{\mathfrak{s}} y(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), {}^c D^{\mathfrak{s}} y(\xi)) d\xi.$$

Proof. The proof is similar as given in Theorem 3.1. \square

For $t_i \in \mathcal{J}$ so that $t_1 < t_2 < \dots < t_m$ and $\mathcal{J}' = \mathcal{J} - \{t_1, t_2, \dots, t_m\}$ define the space $\mathcal{X} = \left\{ x : \mathcal{J} \rightarrow \mathbb{R} \mid x \in \mathcal{C}(\mathcal{J}'), \text{ right limit } x(t_i^+) \text{ and left limit } x(t_i^-) \text{ exist and } x(t_i^-) - x(t_i^+) = \Delta x(t_i), 1 < i \leq m \right\}$.

Then, clearly $(\mathcal{X}, \|\cdot\|)$ is a Banach space with norm $\|x\| = \max_{t \in \mathcal{J}} |x|$.

Similarly, for $t_j \in \mathcal{J}$ so that $t_1 < t_2 < \dots < t_n$ and $\mathcal{J}' = \mathcal{J} - \{t_1, t_2, \dots, t_n\}$ define the space $\mathcal{Y} = \left\{ y : \mathcal{J} \rightarrow \mathbb{R} \mid y \in \mathcal{C}(\mathcal{J}'), \text{ right limit } y(t_j^+) \text{ left limit } y(t_j^-) \text{ exist and } y(t_j^-) - y(t_j^+) = \Delta y(t_j), 1 < j \leq n \right\}$, which is a Banach space under the norm $\|y\| = \max_{t \in \mathcal{J}} |y|$.

Consequently, the product space $\mathcal{X} \times \mathcal{Y}$ is a Banach space under the norm $\|(x, y)\| = \|x\| + \|y\|$ and $\|(x, y)\| = \max\{\|x\|, \|y\|\}$.

Theorem 3.6. Let $\mathcal{F}, \mathcal{G}, \mathcal{F}', \mathcal{G}'$ be continuous functions, then $(x, y) \in \mathcal{X} \times \mathcal{Y}$ is a solution of problem (1.2) if and only if

$$x(t) = \frac{1}{\Gamma(\mathfrak{r})} \int_{t_i}^t (t-\xi)^{\mathfrak{r}-1} \alpha(\xi) d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\mathfrak{r})} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\mathfrak{r}-1} \alpha(\xi) d\xi + I_i(x(t_i)) \right] + h(x), i = 1, 2, \dots, m$$

and (3.19)

$$y(t) = \frac{1}{\Gamma(\mathfrak{s})} \int_{t_j}^t (t-\xi)^{\mathfrak{s}-1} \beta(\xi) d\xi + \sum_{j=1}^n \left[\frac{1}{\Gamma(\mathfrak{s})} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{\mathfrak{s}-1} \beta(\xi) d\xi + I_j(y(t_j)) \right] + g(y), \quad j = 1, 2, \dots, n.$$

Proof. If (x, y) is a solution of the system (1.2), then it is a solution of (3.19). Conversely, let (x, y) is a solution of (3.19), then

$$\begin{cases} {}^c D^r x(t) = \mathcal{F}(t, y(t), {}^c D^r x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), {}^c D^r x(\xi)) d\xi, & t \in \mathcal{J}, \quad t \neq t_i \text{ for } i = 1, 2, \dots, m, \\ {}^c D^s y(t) = \mathcal{F}'(t, x(t), {}^c D^s y(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), {}^c D^s y(\xi)) d\xi, & t \in \mathcal{J}, \quad t \neq t_j \text{ for } j = 1, 2, \dots, n, \\ x(0) = h(x), \quad \Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \dots, m, \\ y(0) = g(y), \quad \Delta y(t_j) = I_j(y(t_j)), \quad j = 1, 2, \dots, n. \end{cases}$$

Thus (x, y) is a solution of (1.2). □

For convenience, we use the following notations:

$$\begin{aligned} \tilde{x}(t) &= \mathcal{F}(t, y(t), {}^c D^r x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), {}^c D^r x(\xi)) d\xi \\ &= \mathcal{F}(t, y(t), \tilde{x}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), \tilde{x}(\xi)) d\xi, \\ \tilde{y}(t) &= \mathcal{F}'(t, x(t), {}^c D^s y(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), {}^c D^s y(\xi)) d\xi \\ &= \mathcal{F}'(t, x(t), \tilde{y}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), \tilde{y}(\xi)) d\xi. \end{aligned}$$

The system (1.2) can be transformed into a fixed point problem.

Define an operators $\mathcal{T}_r, \mathcal{T}_s : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Y}$ by

$$\begin{aligned} \mathcal{T}_r(x, y)(t) &= \begin{cases} \frac{1}{\Gamma(\mathfrak{r})} \int_0^t (t-\xi)^{\mathfrak{r}-1} \tilde{x}(\xi) d\xi + h(x), & t \in \mathcal{J}_0, \\ \frac{1}{\Gamma(\mathfrak{r})} \int_{t_i}^t (t-\xi)^{\mathfrak{r}-1} \tilde{x}(\xi) d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\mathfrak{r})} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\mathfrak{r}-1} \tilde{x}(\xi) d\xi + I_i(x(t_i)) \right] + h(x), & t \in \mathcal{J}_i, \quad i = 1, 2, \dots, m, \end{cases} \\ \mathcal{T}_s(x, y)(t) &= \begin{cases} \frac{1}{\Gamma(\mathfrak{s})} \int_0^t (t-\xi)^{\mathfrak{s}-1} \tilde{y}(\xi) d\xi + g(y), & t \in \mathcal{J}_0, \\ \frac{1}{\Gamma(\mathfrak{s})} \int_{t_j}^t (t-\xi)^{\mathfrak{s}-1} \tilde{y}(\xi) d\xi + \sum_{j=1}^n \left[\frac{1}{\Gamma(\mathfrak{s})} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{\mathfrak{s}-1} \tilde{y}(\xi) d\xi + I_j(y(t_j)) \right] + g(y), & t \in \mathcal{J}_j, \quad j = 1, 2, \dots, n \end{cases} \end{aligned}$$

and $\mathcal{T}(x, y)(t) = (\mathcal{T}_r(x, y)(t), \mathcal{T}_s(x, y)(t))$.

In the sequel we need the following hypotheses.

Assume that

- $[H_1]$ there exist constants $M_1 > 0$ and $0 < N_1 < 1$ so that for each $t \in \mathcal{J}$ and for all $u, \bar{u} \in \mathcal{X}$ and $w, \bar{w} \in \mathbb{R}$, we have

$$|\mathcal{F}(t, u, w) - \mathcal{F}(t, \bar{u}, \bar{w})| \leq M_1|u - \bar{u}| + N_1|w - \bar{w}|;$$

Similarly, there exist constants $M_2 > 0$ and $0 < N_2 < 1$ so that for each $t \in \mathcal{J}$ and for all $u, \bar{u} \in \mathcal{X}$ and $w, \bar{w} \in \mathbb{R}$, we have

$$|\mathcal{G}(t, u, w) - \mathcal{G}(t, \bar{u}, \bar{w})| \leq M_2|u - \bar{u}| + N_2|w - \bar{w}|;$$

- $[H_2]$ there exist constants $M'_1 > 0$ and $0 < N'_1 < 1$ so that for each $t \in \mathcal{J}$ and for all $u, \bar{u} \in \mathcal{Y}$ and $w, \bar{w} \in \mathbb{R}$, we have

$$|\mathcal{F}'(t, u, w) - \mathcal{F}'(t, \bar{u}, \bar{w})| \leq M'_1|u - \bar{u}| + N'_1|w - \bar{w}|;$$

Similarly, there exist constants $M'_2 > 0$ and $0 < N'_2 < 1$ so that for each $t \in \mathcal{J}$ and for all $u, \bar{u} \in \mathcal{Y}$ and $w, \bar{w} \in \mathbb{R}$, we have

$$|\mathcal{G}'(t, u, w) - \mathcal{G}'(t, \bar{u}, \bar{w})| \leq M'_2|u - \bar{u}| + N'_2|w - \bar{w}|;$$

- $[H_3]$ for any $u, \bar{u}, w, \bar{w} \in \mathcal{X} \times \mathcal{Y}$, there exist constants $A_{I_i}, A_{I_j} > 0$ so that

$$|I_i(u(t_i)) - I_i(\bar{u}(t_i))| \leq A_{I_i}|u(t_i) - \bar{u}(t_i)|, |I_i(w(t_i)) - I_i(\bar{w}(t_i))| \leq A_{I_i}|w(t_i) - \bar{w}(t_i)|;$$

and

$$|I_j(u(t_j)) - I_j(\bar{u}(t_j))| \leq A_{I_j}|u(t_j) - \bar{u}(t_j)|, |I_j(w(t_j)) - I_j(\bar{w}(t_j))| \leq A_{I_j}|w(t_j) - \bar{w}(t_j)|;$$

- $[H_4]$ for any $u, w \in \mathbb{R}$, there exist constants $A_h, A_g > 0$ so that

$$|h(u) - h(w)| \leq A_h|u - w|;$$

and

$$|g(u) - g(w)| \leq A_g|u - w|;$$

- $[H_5]$ there exist $a_1, b_1, c_1 \in \mathcal{X}$ so that

$$|\mathcal{F}(t, u(t), w(t))| \leq a_1(t) + b_1(t)|u(t)| + c_1(t)|w(t)|$$

with $a_1^* = \sup_{t \in \mathcal{J}} a_1(t)$, $b_1^* = \sup_{t \in \mathcal{J}} b_1(t)$ and $c_1^* = \sup_{t \in \mathcal{J}} c_1(t) < 1$;

Similarly, there exist $a_2, b_2, c_2 \in \mathcal{X}$ so that

$$|\mathcal{G}(t, u(t), w(t))| \leq a_2(t) + b_2(t)|u(t)| + c_2(t)|w(t)|$$

with $a_2^* = \sup_{t \in \mathcal{J}} a_2(t)$, $b_2^* = \sup_{t \in \mathcal{J}} b_2(t)$ and $c_2^* = \sup_{t \in \mathcal{J}} c_2(t) < 1$;

- $[H_6]$ there exist $l_1, m_1, n_1 \in \mathcal{Y}$ so that

$$|\mathcal{F}'(t, u(t), w(t))| \leq l_1(t) + m_1(t)|u(t)| + n_1(t)|w(t)|$$

with $l_1^* = \sup_{t \in \mathcal{J}} l_1(t)$, $m_1^* = \sup_{t \in \mathcal{J}} m_1(t)$ and $n_1^* = \sup_{t \in \mathcal{J}} n_1(t) < 1$;

Similarly, there exist $l_2, m_2, n_2 \in \mathcal{Y}$ so that

$$|\mathcal{G}'(t, u(t), w(t))| \leq l_2(t) + m_2(t)|u(t)| + n_2(t)|w(t)|$$

with $l_2^* = \sup_{t \in \mathcal{J}} l_2(t)$, $m_2^* = \sup_{t \in \mathcal{J}} m_2(t)$ and $n_2^* = \sup_{t \in \mathcal{J}} n_2(t) < 1$;

- $[H_7]$ there exist constants $\lambda, \lambda' > 0$ so that

$$|h(u)| \leq \lambda, \forall u \in \mathcal{X};$$

and

$$|g(u)| \leq \lambda', \forall u \in \mathcal{Y};$$

- $[H_8]$ for each $u \in \mathbb{R}$, the function $I_i : \mathbb{R} \rightarrow \mathbb{R}$; $i = 1, 2, \dots, m$ are assumed to be continuous and for constants \mathcal{K} , $\mathcal{L} > 0$, the inequality $|I_i(u(t_i))| \leq \mathcal{K}|u(t)| + \mathcal{L}$, holds.
Similarly, for each $w \in \mathbb{R}$, the function $I_j : \mathbb{R} \rightarrow \mathbb{R}$; $j = 1, 2, \dots, n$ are assumed to be continuous and for constants \mathcal{K}' , $\mathcal{L}' > 0$, the inequality $|I_j(w(t_j))| \leq \mathcal{K}'|w(t)| + \mathcal{L}'$, holds.
- $[H_9]$ Define

$$\Delta_1 = \left[\left(\frac{(1+m)\Gamma^v}{\Gamma(v+1)} \right) \left(\frac{M_1}{1-N_1-N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}}{1-N_1-N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} \right) + mA_{I_i} + A_h \right] < 1$$

and

$$\Delta_2 = \left[\left(\frac{(1+n)\Gamma^s}{\Gamma(s+1)} \right) \left(\frac{M'_1}{1-N'_1-N'_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{M'_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}}{1-N'_1-N'_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} \right) + nA_{I_j} + A_g \right] < 1.$$

Theorem 3.7. In addition to the hypothesis $[H_3]$ – $[H_8]$, if $A_h + mA_{I_i} < 1$ and $A_g + nA_{I_j} < 1$, then system (1.2) has at least one solution.

Proof. Construct a closed ball $\mathcal{B} \subset \mathcal{X} \times \mathcal{Y}$ so that $\mathcal{B} = \{(x, y) \in \mathcal{X} \times \mathcal{Y} : \|(x, y)\| \leq \mathbf{R}\}$. Split the operator \mathcal{T} into two parts as $\mathcal{T} = \mathbb{F} + \mathbb{G}$ with $\mathbb{F} = (\mathbb{F}_v, \mathbb{F}_s)$ and $\mathbb{G} = (\mathbb{G}_v, \mathbb{G}_s)$, where

$$\begin{aligned} \mathbb{F}_v(x, y)(t) &= \frac{1}{\Gamma(v)} \int_{t_i}^t (t-\xi)^{v-1} \tilde{x}(\xi) d\xi + \sum_{i=1}^m \frac{1}{\Gamma(v)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{v-1} \tilde{x}(\xi) d\xi, \quad i = 1, 2, \dots, m, \\ \mathbb{F}_s(x, y)(t) &= \frac{1}{\Gamma(s)} \int_{t_j}^t (t-\xi)^{s-1} \tilde{y}(\xi) d\xi + \sum_{j=1}^n \frac{1}{\Gamma(s)} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{s-1} \tilde{y}(\xi) d\xi, \quad j = 1, 2, \dots, n, \\ \mathbb{G}_v(x)(t) &= h(x) + \sum_{i=1}^m I_i(x(t_i)), \\ \mathbb{G}_s(y)(t) &= g(y) + \sum_{j=1}^n I_j(y(t_j)). \end{aligned}$$

Clearly, $\mathcal{T}_v = \mathbb{F}_v + \mathbb{G}_v$, $\mathcal{T}_s = \mathbb{F}_s + \mathbb{G}_s$.

Now we show that $\mathcal{T}(x, y)(t) = \mathbb{F}(x, y)(t) + \mathbb{G}(x, y)(t) \in \mathcal{B}$, $\forall (x, y) \in \mathcal{B}$.

For any $(x, y) \in \mathcal{B}$, consider

$$|\mathcal{T}_v x(t)| \leq \frac{1}{\Gamma(v)} \int_{t_i}^t (t-\xi)^{v-1} |\tilde{x}(\xi)| d\xi + \sum_{i=1}^m \frac{1}{\Gamma(v)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{v-1} |\tilde{x}(\xi)| d\xi + \sum_{i=1}^m |I_i(x(t_i))| + |h(x)|. \quad (3.20)$$

By $[H_5]$, for $t \in \mathcal{J}_i$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned} |\tilde{x}(t)| &\leq |\mathcal{T}(t, x(t), \tilde{x}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}(\xi, x(\xi), \tilde{x}(\xi))| d\xi \\ &\leq a_1(t) + b_1(t)|x(t)| + c_1(t)|\tilde{x}(t)| + \frac{t^\sigma}{\sigma\Gamma(\delta)} \left(a_2(t) + b_2(t)|x(t)| + c_2(t)|\tilde{x}(t)| \right) \\ &\leq a_1^* + b_1^* \|x\|_{\mathcal{X}} + c_1^* \|\tilde{x}\|_{\mathcal{X}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \left(a_2^* + b_2^* \|x\|_{\mathcal{X}} + c_2^* \|\tilde{x}\|_{\mathcal{X}} \right). \end{aligned}$$

Therefore, we get

$$\|\tilde{x}(t)\| \leq \|\tilde{x}\|_{\mathcal{X}} \leq \frac{a_1^* + b_1^* \|x\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \frac{a_2^* + b_2^* \|x\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} = \bar{h}. \quad (3.21)$$

By using (3.21), $[H_7]$ and $[H_8]$, (3.20) becomes

$$\begin{aligned} |\mathcal{T}_{\tau}x(t)| &\leq \frac{\hbar}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} d\xi + \frac{\hbar}{\Gamma(\tau)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} d\xi + m(\mathcal{K}|x| + \mathcal{L}) + \lambda \\ &\leq \frac{\hbar(1+m)t^{\tau}}{\Gamma(\tau+1)} + m(\mathcal{K}_{\varphi} + \mathcal{L}) + \lambda = \eta. \end{aligned}$$

Thus

$$\|\mathcal{T}_{\tau}x\|_{\mathcal{X}} \leq \eta.$$

Similarly, for $t \in \mathcal{J}_0$ we can verify that

$$\|\mathcal{T}_{\tau}x\|_{\mathcal{X}} \leq \eta.$$

In the similar manner, we have

$$|\mathcal{T}_{\tau}y(t)| \leq \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} |\tilde{x}(\xi)| d\xi + \sum_{i=1}^m \frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} |\tilde{x}(\xi)| d\xi + \sum_{i=1}^m |I_i(y(t_i))| + |h(y)|. \quad (3.22)$$

By $[H_5]$, for $t \in \mathcal{J}_i$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned} |\tilde{x}(t)| &\leq |\mathcal{F}(t, x(t), \tilde{x}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}(\xi, x(\xi), \tilde{x}(\xi))| d\xi \\ &\leq a_1(t) + b_1(t)|x(t)| + c_1(t)|\tilde{x}(t)| + \frac{t^{\sigma}}{\sigma\Gamma(\delta)} \left(a_2(t) + b_2(t)|x(t)| + c_2(t)|\tilde{x}(t)| \right) \\ &\leq a_1^* + b_1^*\|x\|_{\mathcal{X}} + c_1^*\|\tilde{x}\|_{\mathcal{X}} + \frac{T^{\sigma}}{\sigma\Gamma(\delta)} \left(a_2^* + b_2^*\|x\|_{\mathcal{X}} + c_2^*\|\tilde{x}\|_{\mathcal{X}} \right). \end{aligned}$$

Therefore, we get

$$|\tilde{x}(t)| \leq \|\tilde{x}\|_{\mathcal{X}} \leq \frac{a_1^* + b_1^*\|x\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{T^{\sigma}}{\sigma\Gamma(\delta)}} + \frac{T^{\sigma}}{\sigma\Gamma(\delta)} \frac{a_2^* + b_2^*\|x\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{T^{\sigma}}{\sigma\Gamma(\delta)}} = \hbar. \quad (3.23)$$

By using (3.23) and $[H_4]$ – $[H_6]$, (3.22) becomes

$$\begin{aligned} |\mathcal{T}_{\tau}y(t)| &\leq \frac{\hbar}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} d\xi + \frac{\hbar}{\Gamma(\tau)} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} d\xi + m(\mathcal{K}|x| + \mathcal{L}) + \lambda \\ &\leq \frac{\hbar(1+m)t^{\tau}}{\Gamma(\tau+1)} + m(\mathcal{K}_{\varphi} + \mathcal{L}) + \lambda = \eta. \end{aligned}$$

Thus

$$\|\mathcal{T}_{\tau}y\|_{\mathcal{X}} \leq \eta.$$

Similarly, for $t \in \mathcal{J}_0$ we can verify that

$$\|\mathcal{T}_{\tau}y\|_{\mathcal{X}} \leq \eta.$$

Hence

$$\|\mathcal{T}_{\tau}(x, y)\|_{\mathcal{X}} \leq \eta.$$

Now, for any $(x, y) \in \mathcal{B}$, consider

$$|\mathcal{T}_{\mathfrak{s}}x(t)| \leq \frac{1}{\Gamma(\mathfrak{s})} \int_{t_j}^t (t-\xi)^{\mathfrak{s}-1} |\tilde{y}(\xi)| d\xi + \sum_{j=1}^n \frac{1}{\Gamma(\mathfrak{s})} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{\mathfrak{s}-1} |\tilde{y}(\xi)| d\xi + \sum_{j=1}^n |I_j(x(t_j))| + |g(x)|. \quad (3.24)$$

By $[H_6]$, for $t \in \mathcal{J}_j$, $j = 1, 2, \dots, n$, we have

$$\begin{aligned} |\tilde{y}(t)| &\leq |\mathcal{F}'(t, x(t), \tilde{y}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}'(\xi, x(\xi), \tilde{y}(\xi))| d\xi \\ &\leq l_1(t) + m_1(t)|x(t)| + n_1(t)|\tilde{y}(t)| + \frac{t^\sigma}{\sigma\Gamma(\delta)} \left(l_2(t) + m_2(t)|x(t)| + n_2(t)|\tilde{y}(t)| \right) \\ &\leq l_1^* + m_1^*\|x\|_{\mathcal{Y}} + n_1^*\|\tilde{y}\|_{\mathcal{Y}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \left(l_2^* + m_2^*\|x\|_{\mathcal{Y}} + n_2^*\|\tilde{y}\|_{\mathcal{Y}} \right). \end{aligned}$$

Therefore, we get

$$|\tilde{y}(t)| \leq \|\tilde{y}\|_{\mathcal{Y}} \leq \frac{l_1^* + m_1^*\|x\|_{\mathcal{Y}}}{1 - n_1^* - n_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \frac{l_2^* + m_2^*\|x\|_{\mathcal{Y}}}{1 - n_1^* - n_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} = \hbar. \quad (3.25)$$

By using (3.25), $[H_7]$ and $[H_8]$, (3.24) becomes

$$\begin{aligned} |\mathcal{T}_{\mathfrak{s}}x(t)| &\leq \frac{\hbar}{\Gamma(\mathfrak{s})} \int_{t_j}^t (t-\xi)^{\mathfrak{s}-1} d\xi + \frac{\hbar}{\Gamma(\mathfrak{s})} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (t_j-\xi)^{\mathfrak{s}-1} d\xi + n(\mathcal{K}'|x| + \mathcal{L}') + \lambda' \\ &\leq \frac{\hbar(1+n)t^{\mathfrak{s}}}{\Gamma(\mathfrak{s}+1)} + n(\mathcal{K}'_{\wp} + \mathcal{L}') + \lambda' = \eta^*. \end{aligned}$$

Thus

$$\|\mathcal{T}_{\mathfrak{s}}(x)\|_{\mathcal{Y}} \leq \eta^*.$$

In the similar manner, we have

$$|\mathcal{T}_{\mathfrak{s}}y(t)| \leq \frac{1}{\Gamma(\mathfrak{s})} \int_{t_j}^t (t-\xi)^{\mathfrak{s}-1} |\tilde{y}(\xi)| d\xi + \sum_{j=1}^n \frac{1}{\Gamma(\mathfrak{s})} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{\mathfrak{s}-1} |\tilde{y}(\xi)| d\xi + \sum_{j=1}^n |I_j(y(t_j))| + |g(y)|. \quad (3.26)$$

By $[H_6]$, for $t \in \mathcal{J}_j$, $j = 1, 2, \dots, n$, we have

$$\begin{aligned} |\tilde{y}(t)| &\leq |\mathcal{F}'(t, x(t), \tilde{y}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}'(\xi, x(\xi), \tilde{y}(\xi))| d\xi \\ &\leq l_1(t) + m_1(t)|x(t)| + n_1(t)|\tilde{y}(t)| + \frac{t^\sigma}{\sigma\Gamma(\delta)} \left(l_2(t) + m_2(t)|x(t)| + n_2(t)|\tilde{y}(t)| \right) \\ &\leq l_1^* + m_1^*\|x\|_{\mathcal{Y}} + n_1^*\|\tilde{y}\|_{\mathcal{Y}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \left(l_2^* + m_2^*\|x\|_{\mathcal{Y}} + n_2^*\|\tilde{y}\|_{\mathcal{Y}} \right). \end{aligned}$$

Therefore, we get

$$|\tilde{y}(t)| \leq \|\tilde{y}\|_{\mathcal{Y}} \leq \frac{l_1^* + m_1^*\|x\|_{\mathcal{Y}}}{1 - n_1^* - n_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \frac{l_2^* + m_2^*\|x\|_{\mathcal{Y}}}{1 - n_1^* - n_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} = \hbar. \quad (3.27)$$

By using (3.27), $[H_7]$ and $[H_8]$, (3.26) becomes

$$\begin{aligned} |\mathcal{T}_{\mathfrak{s}}y(t)| &\leq \frac{\hbar}{\Gamma(\mathfrak{s})} \int_{t_j}^t (t-\xi)^{\mathfrak{s}-1} d\xi + \frac{\hbar}{\Gamma(\mathfrak{s})} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (t_j-\xi)^{\mathfrak{s}-1} d\xi + n(\mathcal{K}'|x| + \mathcal{L}') + \lambda' \\ &\leq \frac{\hbar(1+n)t^{\mathfrak{s}}}{\Gamma(\mathfrak{s}+1)} + n(\mathcal{K}'_{\wp} + \mathcal{L}') + \lambda' = \eta^*. \end{aligned}$$

Thus

$$\|\mathcal{T}_{\mathfrak{s}}(y)\|_{\mathcal{Y}} \leq \eta^*.$$

Hence

$$\|\mathcal{T}_{\mathfrak{s}}(x, y)\|_{\mathcal{Y}} \leq \eta^*.$$

Thus

$$\|\mathcal{T}(x, y)\|_{\mathcal{X} \times \mathcal{Y}} \leq \|\mathcal{T}_r(x, y) + \mathcal{T}_s(x, y)\|_{\mathcal{X} \times \mathcal{Y}} \leq \eta + \eta^* = \mathbf{R}.$$

Which implies that $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{B}$.

Now we show that \mathbb{G} is contraction.

For any $(x, y), (\bar{x}, \bar{y}) \in \mathcal{B}$, we have

$$\begin{aligned} |\mathbb{G}_r(x) - \mathbb{G}_r(\bar{x})| &\leq |h(x) - h(\bar{x})| + \sum_{i=1}^m |I_i(x(t_i)) - I_i(\bar{x}(t_i))| \\ &\leq A_h |x - \bar{x}| + mA_{I_i} |x - \bar{x}| \end{aligned}$$

and

$$\begin{aligned} |\mathbb{G}_s(y) - \mathbb{G}_s(\bar{y})| &\leq |g(y) - g(\bar{y})| + \sum_{j=1}^n |I_j(y(t_j)) - I_j(\bar{y}(t_j))| \\ &\leq A_g |y - \bar{y}| + nA_{I_j} |y - \bar{y}|. \end{aligned}$$

The contraction of \mathbb{G} follows from the assumption that $A_h + mA_{I_i} < 1$ and $A_g + nA_{I_j} < 1$.

Next $\mathbb{F} = (\mathbb{F}_r + \mathbb{F}_s)$ is compact. The continuity of \mathbb{F} follows from the continuity of \mathcal{F} , \mathcal{G} , \mathcal{F}' , \mathcal{G}' . For $(x, y) \in \mathcal{B}$, we have

$$\begin{aligned} |\mathbb{F}_r x(t)| &= \left| \frac{1}{\Gamma(r)} \int_{t_i}^t (t - \xi)^{r-1} \tilde{x}(\xi) d\xi + \sum_{i=1}^m \frac{1}{\Gamma(r)} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{r-1} \tilde{x}(\xi) d\xi \right| \\ &\leq \frac{1}{\Gamma(r)} \int_{t_i}^t (t - \xi)^{r-1} |\tilde{x}(\xi)| d\xi + \sum_{i=1}^m \frac{1}{\Gamma(r)} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{r-1} |\tilde{x}(\xi)| d\xi. \end{aligned} \quad (3.28)$$

By $[H_5]$, for $t \in \mathcal{J}_i$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned} |\tilde{x}(t)| &\leq |\mathcal{F}(t, x(t), \tilde{x}(t))| + \int_0^t \frac{(t - \xi)^{\sigma-1}}{\Gamma(\sigma)} |\mathcal{G}(\xi, x(\xi), \tilde{x}(\xi))| d\xi \\ &\leq a_1(t) + b_1(t)|x(t)| + c_1(t)|\tilde{x}(t)| + \frac{t^\sigma}{\sigma\Gamma(\sigma)} \left(a_2(t) + b_2(t)|x(t)| + c_2(t)|\tilde{x}(t)| \right) \\ &\leq a_1^* + b_1^* \|x\|_{\mathcal{X}} + c_1^* \|\tilde{x}\|_{\mathcal{X}} + \frac{T^\sigma}{\sigma\Gamma(\sigma)} \left(a_2^* + b_2^* \|x\|_{\mathcal{X}} + c_2^* \|\tilde{x}\|_{\mathcal{X}} \right). \end{aligned}$$

Therefore, we get

$$|\tilde{x}(t)| \leq \|\tilde{x}\|_{\mathcal{X}} \leq \frac{a_1^* + b_1^* \|x\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{T^\sigma}{\sigma\Gamma(\sigma)}} + \frac{T^\sigma}{\sigma\Gamma(\sigma)} \frac{a_2^* + b_2^* \|x\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{T^\sigma}{\sigma\Gamma(\sigma)}} = \bar{h}. \quad (3.29)$$

Using (3.29) in (3.28) and after simplification, we get

$$\|\mathbb{F}_r x\|_{\mathcal{X}} \leq \eta_1.$$

In the similar manner, we have

$$\begin{aligned} |\mathbb{F}_s y(t)| &= \left| \frac{1}{\Gamma(r)} \int_{t_i}^t (t - \xi)^{r-1} \tilde{y}(\xi) d\xi + \sum_{i=1}^m \frac{1}{\Gamma(r)} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{r-1} \tilde{y}(\xi) d\xi \right| \\ &\leq \frac{1}{\Gamma(r)} \int_{t_i}^t (t - \xi)^{r-1} |\tilde{y}(\xi)| d\xi + \sum_{i=1}^m \frac{1}{\Gamma(r)} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{r-1} |\tilde{y}(\xi)| d\xi. \end{aligned} \quad (3.30)$$

By $[H_5]$, for $t \in \mathcal{J}_i$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned} |\tilde{x}(t)| &\leq |\mathcal{F}(t, x(t), \tilde{x}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}(\xi, x(\xi), \tilde{x}(\xi))| d\xi \\ &\leq a_1(t) + b_1(t)|x(t)| + c_1(t)|\tilde{x}(t)| + \frac{t^\sigma}{\sigma\Gamma(\delta)} \left(a_2(t) + b_2(t)|x(t)| + c_2(t)|\tilde{x}(t)| \right) \\ &\leq a_1^* + b_1^*\|x\|_{\mathcal{X}} + c_1^*\|\tilde{x}\|_{\mathcal{X}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \left(a_2^* + b_2^*\|x\|_{\mathcal{X}} + c_2^*\|\tilde{x}\|_{\mathcal{X}} \right). \end{aligned}$$

Therefore, we get

$$\|\tilde{x}(t)\|_{\mathcal{X}} \leq \|\tilde{x}\|_{\mathcal{X}} \leq \frac{a_1^* + b_1^*\|x\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \frac{a_2^* + b_2^*\|x\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} = h. \quad (3.31)$$

Using (3.31) in (3.30) and after simplification, we get

$$\|\mathbb{F}_{\tau}y\|_{\mathcal{X}} \leq \eta_1.$$

Hence

$$\|\mathbb{F}_{\tau}(x, y)\|_{\mathcal{X}} \leq \eta_1.$$

Now for any $(x, y) \in \mathcal{B}$, we have

$$\begin{aligned} |\mathbb{F}_{\mathfrak{s}}x(t)| &= \left| \frac{1}{\Gamma(\mathfrak{s})} \int_{t_j}^t (t-\xi)^{\mathfrak{s}-1} \tilde{y}(\xi) d\xi + \sum_{j=1}^n \frac{1}{\Gamma(\mathfrak{s})} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{\mathfrak{s}-1} \tilde{y}(\xi) d\xi \right| \\ &\leq \frac{1}{\Gamma(\mathfrak{s})} \int_{t_j}^t (t-\xi)^{\mathfrak{s}-1} |\tilde{y}(\xi)| d\xi + \sum_{j=1}^n \frac{1}{\Gamma(\mathfrak{s})} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{\mathfrak{s}-1} |\tilde{y}(\xi)| d\xi. \end{aligned} \quad (3.32)$$

By $[H_6]$, for $t \in \mathcal{J}_j$, $j = 1, 2, \dots, n$, we have

$$\begin{aligned} |\tilde{y}(t)| &\leq |\mathcal{F}'(t, x(t), \tilde{y}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}'(\xi, x(\xi), \tilde{y}(\xi))| d\xi \\ &\leq l_1(t) + m_1(t)|x(t)| + n_1(t)|\tilde{y}(t)| + \frac{t^\sigma}{\sigma\Gamma(\delta)} \left(l_2(t) + m_2(t)|x(t)| + n_2(t)|\tilde{y}(t)| \right) \\ &\leq l_1^* + m_1^*\|x\|_{\mathcal{Y}} + n_1^*\|\tilde{y}\|_{\mathcal{Y}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \left(l_2^* + m_2^*\|x\|_{\mathcal{Y}} + n_2^*\|\tilde{y}\|_{\mathcal{Y}} \right). \end{aligned}$$

Therefore, we get

$$\|\tilde{y}(t)\|_{\mathcal{Y}} \leq \|\tilde{y}\|_{\mathcal{Y}} \leq \frac{l_1^* + m_1^*\|x\|_{\mathcal{Y}}}{1 - n_1^* - n_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \frac{l_2^* + m_2^*\|x\|_{\mathcal{Y}}}{1 - n_1^* - n_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} = h. \quad (3.33)$$

Using (3.33) in (3.32) and after simplification, we get

$$\|\mathbb{F}_{\mathfrak{s}}x\| \leq \eta_2.$$

In the similar manner, we have

$$\begin{aligned} |\mathbb{F}_{\mathfrak{s}}y(t)| &= \left| \frac{1}{\Gamma(\mathfrak{s})} \int_{t_j}^t (t-\xi)^{\mathfrak{s}-1} \tilde{y}(\xi) d\xi + \sum_{j=1}^n \frac{1}{\Gamma(\mathfrak{s})} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{\mathfrak{s}-1} \tilde{y}(\xi) d\xi \right| \\ &\leq \frac{1}{\Gamma(\mathfrak{s})} \int_{t_j}^t (t-\xi)^{\mathfrak{s}-1} |\tilde{y}(\xi)| d\xi + \sum_{j=1}^n \frac{1}{\Gamma(\mathfrak{s})} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{\mathfrak{s}-1} |\tilde{y}(\xi)| d\xi. \end{aligned} \quad (3.34)$$

By $[H_6]$, for $t \in \mathcal{J}_j$, $j = 1, 2, \dots, n$, we have

$$\begin{aligned} |\tilde{y}(t)| &\leq |\mathcal{F}'(t, x(t), \tilde{y}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}'(\xi, x(\xi), \tilde{y}(\xi))| d\xi \\ &\leq l_1(t) + m_1(t)|x(t)| + n_1(t)|\tilde{y}(t)| + \frac{t^\sigma}{\sigma\Gamma(\delta)} \left(l_2(t) + m_2(t)|x(t)| + n_2(t)|\tilde{y}(t)| \right) \\ &\leq l_1^* + m_1^*\|x\|_{\mathcal{Y}} + n_1^*\|\tilde{y}\|_{\mathcal{Y}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \left(l_2^* + m_2^*\|x\|_{\mathcal{Y}} + n_2^*\|\tilde{y}\|_{\mathcal{Y}} \right). \end{aligned}$$

Therefore, we get

$$|\tilde{y}(t)| \leq \|\tilde{y}\|_{\mathcal{Y}} \leq \frac{l_1^* + m_1^*\|x\|_{\mathcal{Y}}}{1 - n_1^* - n_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \frac{l_2^* + m_2^*\|x\|_{\mathcal{Y}}}{1 - n_1^* - n_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} = \hbar. \quad (3.35)$$

Using (3.35) in (3.34) and after simplification, we get

$$\|\mathbb{F}_s y\|_{\mathcal{Y}} \leq \eta_2.$$

Hence

$$\|\mathbb{F}_s(x, y)\|_{\mathcal{Y}} \leq \eta_2.$$

Thus

$$\|\mathbb{F}(x, y)\|_{\mathcal{X} \times \mathcal{Y}} \leq \|\mathbb{F}_t(x, y) + \mathbb{F}_s(x, y)\|_{\mathcal{X} \times \mathcal{Y}} \leq \eta_1 + \eta_2 = \mathbf{R}_1.$$

Which implies that \mathbb{F} is uniformly bounded on \mathcal{B} .

Take a bounded subset \mathcal{C} of \mathcal{B} and $(x, y) \in \mathcal{C}$. Then for $t_1, t_2 \in \mathcal{J}_i$, $i = 1, 2, \dots, m$ with $0 \leq t_1 \leq t_2 \leq 1$, we have

$$|\mathbb{F}_t x(t_1) - \mathbb{F}_t x(t_2)| \leq \frac{1}{\Gamma(\tau)} \int_{t_i}^{t_2} [(t_2 - \xi)^{\tau-1} - (t_1 - \xi)^{\tau-1}] |\tilde{x}(\xi)| d\xi + \frac{1}{\Gamma(\tau)} \int_{t_2}^{t_1} (t_1 - \xi)^{\tau-1} |\tilde{x}(\xi)| d\xi. \quad (3.36)$$

By $[H_5]$, for $t \in \mathcal{J}_i$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned} |\tilde{x}(t)| &\leq |\mathcal{F}(t, x(t), \tilde{x}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}(\xi, x(\xi), \tilde{x}(\xi))| d\xi \\ &\leq a_1(t) + b_1(t)|x(t)| + c_1(t)|\tilde{x}(t)| + \frac{t^\sigma}{\sigma\Gamma(\delta)} \left(a_2(t) + b_2(t)|x(t)| + c_2(t)|\tilde{x}(t)| \right) \\ &\leq a_1^* + b_1^*\|x\|_{\mathcal{X}} + c_1^*\|\tilde{x}\|_{\mathcal{X}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \left(a_2^* + b_2^*\|x\|_{\mathcal{X}} + c_2^*\|\tilde{x}\|_{\mathcal{X}} \right). \end{aligned}$$

Therefore, we get

$$|\tilde{x}(t)| \leq \|\tilde{x}\|_{\mathcal{X}} \leq \frac{a_1^* + b_1^*\|x\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \frac{a_2^* + b_2^*\|x\|_{\mathcal{X}}}{1 - c_1^* - c_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} = \hbar. \quad (3.37)$$

Using (3.37) in (3.36) we see that the right-hand side of (3.36) tends to zero as $t_1 \rightarrow t_2$.

Thus

$$|\mathbb{F}_t x(t_1) - \mathbb{F}_t x(t_2)| \rightarrow 0 \text{ as } t_1 \rightarrow t_2.$$

Similarly,

$$|\mathbb{F}_t y(t_1) - \mathbb{F}_t y(t_2)| \rightarrow 0 \text{ as } t_1 \rightarrow t_2.$$

Now for any $t_1, t_2 \in \mathcal{J}_j$, $j = 1, 2, \dots, n$ with $0 \leq t_1 \leq t_2 \leq 1$, we have

$$|\mathbb{F}_s x(t_1) - \mathbb{F}_s x(t_2)| \leq \frac{1}{\Gamma(\varsigma)} \int_{t_j}^{t_2} [(t_2 - \xi)^{\varsigma-1} - (t_1 - \xi)^{\varsigma-1}] |\tilde{y}(\xi)| d\xi + \frac{1}{\Gamma(\varsigma)} \int_{t_2}^{t_1} (t_1 - \xi)^{\varsigma-1} |\tilde{y}(\xi)| d\xi. \quad (3.38)$$

By $[H_6]$, for $t \in \mathcal{J}_j$, $j = 1, 2, \dots, n$, we have

$$\begin{aligned} |\tilde{y}(t)| &\leq |\mathcal{F}'(t, x(t), \tilde{y}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}'(\xi, x(\xi), \tilde{y}(\xi))| d\xi \\ &\leq l_1(t) + m_1(t)|x(t)| + n_1(t)|\tilde{y}(t)| + \frac{t^\sigma}{\sigma\Gamma(\delta)} \left(l_2(t) + m_2(t)|x(t)| + n_2(t)|\tilde{y}(t)| \right) \\ &\leq l_1^* + m_1^* \|x\|_y + n_1^* \|\tilde{y}\|_y + \frac{T^\sigma}{\sigma\Gamma(\delta)} \left(l_2^* + m_2^* \|x\|_y + n_2^* \|\tilde{y}\|_y \right). \end{aligned}$$

Therefore, we get

$$\|\tilde{y}(t)\| \leq \|\tilde{y}\|_y \leq \frac{l_1^* + m_1^* \|x\|_y}{1 - n_1^* - n_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{T^\sigma}{\sigma\Gamma(\delta)} \frac{l_2^* + m_2^* \|x\|_y}{1 - n_1^* - n_2^* \frac{T^\sigma}{\sigma\Gamma(\delta)}} = \bar{h}. \quad (3.39)$$

Using (3.39) in (3.38), we see that the right-hand side of (3.38) tends to zero as $t_1 \rightarrow t_2$.

Similarly,

$$|\mathbb{F}_s y(t_1) - \mathbb{F}_s y(t_2)| \rightarrow 0 \text{ as } t_1 \rightarrow t_2.$$

Thus

$$|\mathbb{F}(x, y)(t_1) - \mathbb{F}(x, y)(t_2)| \rightarrow 0 \text{ as } t_1 \rightarrow t_2.$$

Hence, \mathbb{F} is equicontinuous and by the Arzela–Ascoli theorem, \mathbb{F} is compact. By Theorem 2.7, system (1.2) has at least one solution. \square

Theorem 3.8. *If $\Delta = \max(\Delta_1, \Delta_2) < 1$, then under the hypothesis $[H_1]–[H_9]$, system (1.2) has a unique solution.*

Proof. Suppose $x, \bar{x} \in \mathcal{X}$ and for $t \in \mathcal{J}_i$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned} |\mathcal{T}_\tau x(t) - \mathcal{T}_\tau \bar{x}(t)| &\leq \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} |\tilde{x}(\xi) - \bar{x}(\xi)| d\xi + \sum_{i=1}^m \frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{\tau-1} |\tilde{x}(\xi) - \bar{x}(\xi)| d\xi \\ &\quad + \sum_{i=1}^m |I_i(x(t_i)) - I_i(\bar{x}(t_i))| + |h(x) - h(\bar{x})|, \end{aligned} \quad (3.40)$$

where

$$\tilde{x}(t) = \mathcal{F}(t, y(t), \tilde{x}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), \tilde{x}(\xi)) d\xi$$

and

$$\bar{\tilde{x}}(t) = \mathcal{F}(t, \bar{y}(t), \bar{\tilde{x}}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, \bar{y}(\xi), \bar{\tilde{x}}(\xi)) d\xi.$$

Using $[H_1]$, we have

$$\begin{aligned} |\tilde{x}(t) - \bar{\tilde{x}}(t)| &= \left| \mathcal{F}(t, y(t), \tilde{x}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), \tilde{x}(\xi)) d\xi - \mathcal{F}(t, \bar{y}(t), \bar{\tilde{x}}(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, \bar{y}(\xi), \bar{\tilde{x}}(\xi)) d\xi \right| \\ &\leq |\mathcal{F}(t, y(t), \tilde{x}(t)) - \mathcal{F}(t, \bar{y}(t), \bar{\tilde{x}}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}(\xi, y(\xi), \tilde{x}(\xi)) - \mathcal{G}(\xi, \bar{y}(\xi), \bar{\tilde{x}}(\xi))| d\xi \\ &\leq M_1 |y(t) - \bar{y}(t)| + N_1 |\tilde{x}(t) - \bar{\tilde{x}}(t)| + \frac{t^\sigma}{\sigma\Gamma(\delta)} \left(M_2 |y(t) - \bar{y}(t)| + N_2 |\tilde{x}(t) - \bar{\tilde{x}}(t)| \right). \end{aligned}$$

Thus

$$|\tilde{x}(t) - \bar{x}(t)| \leq \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}} \right) |y(t) - \bar{y}(t)|.$$

Using hypothesis $[H_1]$, $[H_3]$ and $[H_4]$, inequality (3.40) implies

$$\begin{aligned} |\mathcal{T}_\tau x(t) - \mathcal{T}_\tau \bar{x}(t)| &\leq \left[\left(\frac{(1+m)t^\tau}{\Gamma(\tau+1)} \right) \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}} \right) \right] |y(t) - \bar{y}(t)| \\ &\quad + (mA_{I_i} + A_h) |x(t) - \bar{x}(t)|. \end{aligned}$$

Taking norm on both sides, we have

$$\begin{aligned} \|\mathcal{T}_\tau x - \mathcal{T}_\tau \bar{x}\|_{\mathcal{X}} &\leq \left[\left(\frac{(1+m)\Gamma^\tau}{\Gamma(\tau+1)} \right) \left(\frac{M_1}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} \right) \right] \|y - \bar{y}\|_{\mathcal{X}} \\ &\quad + (mA_{I_i} + A_h) \|x - \bar{x}\|_{\mathcal{X}}. \end{aligned} \quad (3.41)$$

Similarly, for x , $\bar{x} \in \mathcal{X}$ and $t \in \mathcal{J}_0$, we get

$$\|\mathcal{T}_\tau x - \mathcal{T}_\tau \bar{x}\|_{\mathcal{X}} \leq \left[\frac{\Gamma^\tau}{\Gamma(\tau+1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} \right) \right] \|y - \bar{y}\|_{\mathcal{X}} + A_h \|x - \bar{x}\|_{\mathcal{X}}.$$

In the same manner, we can obtain

$$\|\mathcal{T}_\tau y - \mathcal{T}_\tau \bar{y}\|_{\mathcal{X}} \leq \left[\left(\frac{(1+m)\Gamma^\tau}{\Gamma(\tau+1)} \right) \left(\frac{M_1}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} \right) \right] \|y - \bar{y}\|_{\mathcal{X}} + (mA_{I_i} + A_h) \|x - \bar{x}\|_{\mathcal{X}}. \quad (3.42)$$

Similarly, for y , $\bar{y} \in \mathcal{X}$ and $t \in \mathcal{J}_0$, we get

$$\|\mathcal{T}_\tau y - \mathcal{T}_\tau \bar{y}\|_{\mathcal{X}} \leq \left[\frac{\Gamma^\tau}{\Gamma(\tau+1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} \right) \right] \|y - \bar{y}\|_{\mathcal{X}} + A_h \|x - \bar{x}\|_{\mathcal{X}}.$$

So from (3.41) and (3.42), we get

$$\|\mathcal{T}_\tau(x, y) - \mathcal{T}_\tau(\bar{x}, \bar{y})\|_{\mathcal{X}} \leq \Delta_1 \|(x, y) - (\bar{x}, \bar{y})\|_{\mathcal{X}}. \quad (3.43)$$

Now, suppose x , $\bar{x} \in \mathcal{X}$ and for $t \in \mathcal{J}_j$, $j = 1, 2, \dots, n$, we have

$$\begin{aligned} |\mathcal{T}_s x(t) - \mathcal{T}_s \bar{x}(t)| &\leq \frac{1}{\Gamma(s)} \int_{t_j}^t (t - \xi)^{s-1} |\tilde{y}(\xi) - \bar{\tilde{y}}(\xi)| d\xi + \sum_{j=1}^n \frac{1}{\Gamma(s)} \int_{t_{j-1}}^{t_j} (t_j - \xi)^{s-1} |\tilde{y}(\xi) - \bar{\tilde{y}}(\xi)| d\xi \\ &\quad + \sum_{j=1}^n |I_j(y(t_j)) - I_j(\bar{y}(t_j))| + |g(y) - g(\bar{y})|, \end{aligned} \quad (3.44)$$

where

$$\tilde{y}(t) = \mathcal{F}'(t, x(t), \tilde{y}(t)) + \int_0^t \frac{(t - \xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), \tilde{y}(\xi)) d\xi$$

and

$$\bar{\tilde{y}}(t) = \mathcal{F}'(t, \bar{x}(t), \bar{\tilde{y}}(t)) + \int_0^t \frac{(t - \xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, \bar{x}(\xi), \bar{\tilde{y}}(\xi)) d\xi.$$

Using $[H_2]$, we have

$$\begin{aligned} |\tilde{y}(t) - \bar{y}(t)| &= \left| \mathcal{F}'(t, x(t), \tilde{y}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), \tilde{y}(\xi)) d\xi - \mathcal{F}'(t, \bar{x}(t), \bar{y}(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, \bar{x}(\xi), \bar{y}(\xi)) d\xi \right| \\ &\leq |\mathcal{F}'(t, x(t), \tilde{y}(t)) - \mathcal{F}'(t, \bar{x}(t), \bar{y}(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}'(\xi, x(\xi), \tilde{y}(\xi)) - \mathcal{G}'(\xi, \bar{x}(\xi), \bar{y}(\xi))| d\xi \\ &\leq M'_1 |x(t) - \bar{x}(t)| + N'_1 |\tilde{y}(t) - \bar{y}(t)| + \frac{t^\sigma}{\sigma \Gamma(\delta)} \left(M'_2 |x(t) - \bar{x}(t)| + N'_2 |\tilde{y}(t) - \bar{y}(t)| \right). \end{aligned}$$

Thus

$$|\tilde{y}(t) - \bar{y}(t)| \leq \left(\frac{M'_1}{1 - N'_1 - N'_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{M'_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}}{1 - N'_1 - N'_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} \right) |x(t) - \bar{x}(t)|.$$

By using hypothesis $[H_2]$, $[H_3]$ and $[H_4]$, inequality (3.44) implies

$$\begin{aligned} |\mathcal{T}_s x(t) - \mathcal{T}_s \bar{x}(t)| &\leq \left[\left(\frac{(1+n)t^s}{\Gamma(s+1)} \right) \left(\frac{M'_1}{1 - N'_1 - N'_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{M'_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}}{1 - N'_1 - N'_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} \right) \right] |x(t) - \bar{x}(t)| \\ &\quad + (nA_{I_j} + A_g) |y(t) - \bar{y}(t)|. \end{aligned}$$

Taking norm on both sides, we have

$$\|\mathcal{T}_s x - \mathcal{T}_s \bar{x}\|_{\mathcal{Y}} \leq \left[\left(\frac{(1+n)T^s}{\Gamma(s+1)} \right) \left(\frac{M'_1}{1 - N'_1 - N'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} + \frac{M'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}}{1 - N'_1 - N'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} \right) \right] \|x - \bar{x}\|_{\mathcal{Y}} + (nA_{I_j} + A_g) \|y - \bar{y}\|_{\mathcal{Y}}. \quad (3.45)$$

Similarly, for $x, \bar{x} \in \mathcal{Y}$ and $t \in \mathcal{J}_0$, we get

$$\|\mathcal{T}_s x - \mathcal{T}_s \bar{x}\|_{\mathcal{Y}} \leq \left[\frac{T^s}{\Gamma(s+1)} \left(\frac{M'_1}{1 - N'_1 - N'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} + \frac{M'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}}{1 - N'_1 - N'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} \right) \right] \|x - \bar{x}\|_{\mathcal{Y}} + A_g \|y - \bar{y}\|_{\mathcal{Y}}.$$

In the same manner, we can obtain

$$\begin{aligned} \|\mathcal{T}_s y - \mathcal{T}_s \bar{y}\|_{\mathcal{Y}} &\leq \left[\left(\frac{(1+n)T^s}{\Gamma(s+1)} \right) \left(\frac{M'_1}{1 - N'_1 - N'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} + \frac{M'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}}{1 - N'_1 - N'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} \right) \right] \|y - \bar{y}\|_{\mathcal{Y}} \\ &\quad + (nA_{I_j} + A_g) \|x - \bar{x}\|_{\mathcal{Y}}. \end{aligned} \quad (3.46)$$

Similarly, for $y, \bar{y} \in \mathcal{Y}$ and $t \in \mathcal{J}_0$, we get

$$\begin{aligned} \|\mathcal{T}_s y - \mathcal{T}_s \bar{y}\|_{\mathcal{Y}} &\leq \left[\frac{T^s}{\Gamma(s+1)} \left(\frac{M'_1}{1 - N'_1 - N'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} + \frac{M'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}}{1 - N'_1 - N'_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} \right) \right] \|y - \bar{y}\|_{\mathcal{Y}} \\ &\quad + A_g \|x - \bar{x}\|_{\mathcal{Y}}. \end{aligned}$$

So from (3.45) and (3.46), we get

$$\|\mathcal{T}_s(x, y) - \mathcal{T}_s(\bar{x}, \bar{y})\|_{\mathcal{Y}} \leq \Delta_2 \|(x, y) - (\bar{x}, \bar{y})\|_{\mathcal{Y}}.$$

Hence, it follows that

$$\|\mathcal{T}(x, y) - \mathcal{T}(\bar{x}, \bar{y})\|_{\mathcal{X} \times \mathcal{Y}} \leq \max(\Delta_1, \Delta_2) (\|x - \bar{x}\|_{\mathcal{X} \times \mathcal{Y}} + \|y - \bar{y}\|_{\mathcal{X} \times \mathcal{Y}}).$$

Which implies that \mathcal{T} is contraction, hence it has a unique fixed point. \square

4 Ulam stability results

In this section, we investigate HU stability and its various kinds for problem (1.1). The following definitions are adopted from [7].

For $x \in \mathcal{M}$, $\epsilon_\tau > 0$, $\phi_\tau \leq 0$ and a nondecreasing function $\psi_\tau \in C(\mathcal{J}, \mathbb{R}_+)$, the following set of inequalities satisfy:

$$\begin{cases} |{}^c D^\tau x(t) - \mathcal{F}(t, x(t), {}^c D^\tau x(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^\tau x(\xi)) d\xi| \leq \epsilon_\tau, & t \in \mathcal{J}_i, i = 1, 2, \dots, m, \\ |\Delta x(t_i) - I_i(x(t_i))| \leq \epsilon_\tau, & i = 1, 2, \dots, m, \end{cases} \quad (4.1)$$

$$\begin{cases} |{}^c D^\tau x(t) - \mathcal{F}(t, x(t), {}^c D^\tau x(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^\tau x(\xi)) d\xi| \leq \psi_\tau(t), & t \in \mathcal{J}_i, i = 1, 2, \dots, m, \\ |\Delta x(t_i) - I_i(x(t_i))| \leq \phi_\tau, & i = 1, 2, \dots, m \end{cases} \quad (4.2)$$

and

$$\begin{cases} |{}^c D^\tau x(t) - \mathcal{F}(t, x(t), {}^c D^\tau x(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^\tau x(\xi)) d\xi| \leq \epsilon_\tau \psi_\tau(t), & t \in \mathcal{J}_i, i = 1, 2, \dots, m, \\ |\Delta x(t_i) - I_i(x(t_i))| \leq \epsilon_\tau \phi_\tau, & i = 1, 2, \dots, m. \end{cases} \quad (4.3)$$

Definition 4.1. The problem (1.1) is said to be HU stable if \exists a real number $\mathcal{C}_{\mathcal{F}, \mathcal{G}} > 0$, so that for each $\epsilon_\tau > 0$ and any solution $x \in \mathcal{M}$ of the inequality (4.1), \exists a unique solution $x^* \in \mathcal{M}$ of problem (1.1), so that

$$|x(t) - x^*(t)| \leq \mathcal{C}_{\mathcal{F}, \mathcal{G}} \epsilon_\tau, \quad \forall t \in \mathcal{J}.$$

Definition 4.2. The problem (1.1) is said to be generalized HU stable if \exists a function $\mathcal{G} \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, with $\mathcal{G}(0) = 0$, so that for each $\epsilon_\tau > 0$ and any solution $x \in \mathcal{M}$ of the inequality (4.1), \exists a unique solution $x^* \in \mathcal{M}$ of problem (1.1), so that

$$|x(t) - x^*(t)| \leq \mathcal{C}_{\mathcal{F}, \mathcal{G}} \mathcal{G}(\epsilon_\tau), \quad \forall t \in \mathcal{J}.$$

Definition 4.3. The problem (1.1) is said to be HU–Rassias stable with respect to (ϕ_τ, ψ_τ) , if \exists a real number $\mathcal{C}_{\mathcal{F}, \mathcal{G}} > 0$, so that for each $\epsilon_\tau > 0$ and any solution $x \in \mathcal{M}$ of the inequality (4.3), \exists a unique solution $x^* \in \mathcal{M}$ of problem (1.1), so that

$$|x(t) - x^*(t)| \leq \mathcal{C}_{\mathcal{F}, \mathcal{G}} \epsilon_\tau (\phi_\tau + \psi_\tau(t)), \quad \forall t \in \mathcal{J}.$$

Definition 4.4. The problem (1.1) is said to be generalized HU–Rassias stable with respect to (ϕ_τ, ψ_τ) , if \exists a real number $\mathcal{C}_{\mathcal{F}, \mathcal{G}} > 0$, so that for each $\epsilon_\tau > 0$ and any solution $x \in \mathcal{M}$ of the inequality (4.2), \exists a unique solution $x^* \in \mathcal{M}$ of problem (1.1), so that

$$|x(t) - x^*(t)| \leq \mathcal{C}_{\mathcal{F}, \mathcal{G}} (\phi_\tau + \psi_\tau(t)), \quad \forall t \in \mathcal{J}.$$

Remark 4.5. Definition 4.1 implies Definition 4.2 and Definition 4.3 implies Definition 4.4.

Remark 4.6. A function $x \in \mathcal{M}$ is a solution of the inequality (4.1) if \exists a function $\Phi \in \mathcal{M}$ and a sequence Φ_i (which depends on x) so that

i) $|\Phi(t)| \leq \epsilon_\tau$, $|\Phi_i| \leq \epsilon_\tau$, $\forall t \in \mathcal{J}$, $i = 1, 2, \dots, m$;

ii) ${}^c D^\tau x(t) = \mathcal{F}(t, x(t), {}^c D^\tau x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^\tau x(\xi)) d\xi + \Phi(t)$, $\forall t \in \mathcal{J}$;

and

$$\text{iii) } \Delta x(t_i) = I_i(x(t_i)) + \Phi_i, \forall t \in \mathcal{J}, i = 1, 2, \dots, m.$$

Remark 4.7. A function $x \in \mathcal{M}$ is a solution of the inequality (4.2) if \exists a function $\Phi \in \mathcal{M}$ and a sequence Φ_i (which depends on x) so that

$$\text{i) } |\Phi(t)| \leq \psi_\tau(t), |\Phi_i| \leq \phi_\tau, \forall t \in \mathcal{J}, i = 1, 2, \dots, m;$$

$$\text{ii) } {}^c D^\tau x(t) = \mathcal{F}(t, x(t), {}^c D^\tau x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^\tau x(\xi)) d\xi + \Phi(t), \forall t \in \mathcal{J};$$

and

$$\text{iii) } \Delta x(t_i) = I_i(x(t_i)) + \Phi_i, \forall t \in \mathcal{J}, i = 1, 2, \dots, m.$$

Remark 4.8. A function $x \in \mathcal{M}$ is a solution of the inequality (4.3) if \exists a function $\Phi \in \mathcal{M}$ and a sequence Φ_i (which depends on x) so that

$$\text{i) } |\Phi(t)| \leq \epsilon_\tau \psi_\tau(t), |\Phi_i| \leq \epsilon_\tau \phi_\tau, \forall t \in \mathcal{J}, i = 1, 2, \dots, m;$$

$$\text{ii) } {}^c D^\tau x(t) = \mathcal{F}(t, x(t), {}^c D^\tau x(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^\tau x(\xi)) d\xi + \Phi(t), \forall t \in \mathcal{J};$$

and

$$\text{iii) } \Delta x(t_i) = I_i(x(t_i)) + \Phi_i, \forall t \in \mathcal{J}, i = 1, 2, \dots, m.$$

Definition 4.9. A function $x \in \mathcal{J}$ is a solution of the problem (1.1) if x satisfies (1.1) with its conditions on \mathcal{J} .

Theorem 4.10. If the hypothesis $[A_1] - [A_3]$ and inequality (3.14) are satisfied, then problem (1.1) is HU stable and consequently, it is generalized HU stable.

Proof. Let $x \in \mathcal{M}$ be any solution of the inequality (4.1) and let $x^* \in \mathcal{M}$ be the solution of the following problem:

$$\begin{cases} {}^c D^\tau x^*(t) = \mathcal{F}(t, x^*(t), {}^c D^\tau x^*(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x^*(\xi), {}^c D^\tau x^*(\xi)) d\xi, \forall t \in \mathcal{J}, t \neq t_i \text{ for } i = 1, 2, \dots, m, \\ x^*(0) = h(x^*), \\ \Delta x^*(t_i) = I_i(x^*(t_i)), i = 1, 2, \dots, m. \end{cases}$$

By Theorem 3.1, for $t \in \mathcal{J}_i$, we have

$$x^*(t) = \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} \bar{x}(\xi) d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} \bar{x}(\xi) d\xi + I_i(x^*(t_i)) \right] + h(x^*), i = 1, 2, \dots, m,$$

where $\bar{x} \in C(\mathcal{J}, \mathbb{R})$ is given by

$$\bar{x}(t) = \mathcal{F}(t, \bar{x}(t), \bar{x}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, \bar{x}(\xi), \bar{x}(\xi)) d\xi.$$

Since x is a solution of the inequality (4.1), hence by Remark 4.6, we have

$$\begin{cases} {}^c D^\tau x(t) = \mathcal{F}(t, x(t), {}^c D^\tau x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^\tau x(\xi)) d\xi + \Phi(t), \forall t \in \mathcal{J}, t \neq t_i \text{ for } i = 1, 2, \dots, m, \\ x(0) = h(x), \\ \Delta x(t_i) = I_i(x(t_i)) + \Phi_i(t), i = 1, 2, \dots, m. \end{cases} \quad (4.4)$$

Clearly, the solution of (4.4) will be

$$x(t) = \begin{cases} \frac{1}{\Gamma(\tau)} \int_0^t (t-\xi)^{\tau-1} \tilde{x}(\xi) d\xi + \frac{1}{\Gamma(\tau)} \int_0^t (t-\xi)^{\tau-1} \Phi(\xi) d\xi + h(x), & t \in \mathcal{J}_0, \\ \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} \tilde{x}(\xi) d\xi + \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} \Phi(\xi) d\xi + \sum_{i=1}^m \frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} \tilde{x}(\xi) d\xi \\ + \sum_{i=1}^m \frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} \Phi(\xi) d\xi + \sum_{i=1}^m I_i(x(t_i)) + \sum_{i=1}^m \Phi_i + h(x), & t \in \mathcal{J}_i, \quad i = 1, 2, \dots, m, \end{cases}$$

where $\tilde{x} \in C(\mathcal{J}, \mathbb{R})$ is given by

$$\tilde{x}(t) = \mathcal{F}(t, x(t), \tilde{x}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), \tilde{x}(\xi)) d\xi.$$

Therefore, for each $t \in \mathcal{J}_i$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned} |x(t) - x^*(t)| &\leq \frac{1}{\Gamma(r)} \int_{t_i}^t (t-\xi)^{r-1} |\tilde{x}(\xi) - \tilde{x}(\xi)| d\xi + \frac{1}{\Gamma(r)} \int_{t_i}^t (t-\xi)^{r-1} |\Phi(\xi)| d\xi \\ &+ \sum_{i=1}^m \frac{1}{\Gamma(r)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{r-1} |\tilde{x}(\xi) - \tilde{x}(\xi)| d\xi + \sum_{i=1}^m \frac{1}{\Gamma(r)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{r-1} |\Phi(\xi)| d\xi \\ &+ \sum_{i=1}^m |I_i(x(t_i)) - I_i(x^*(t_i))| + \sum_{i=1}^m |\Phi_i| + |h(x) - h(x^*)|. \end{aligned} \quad (4.5)$$

By $[A_2]$, we obtain

$$|\tilde{x}(t) - \tilde{x}(t)| \leq \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} \right) |x(t) - x^*(t)|.$$

Hence, using $[A_1]$ – $[A_3]$ and part (i) of Remark 4.6, inequality (4.5) implies that

$$\begin{aligned} |x(t) - x^*(t)| &\leq \frac{t^\tau}{\Gamma(\tau+1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} \right) |x(t) - x^*(t)| + \frac{\epsilon_\tau t^\tau}{\Gamma(\tau+1)} \\ &+ \frac{mt^\tau}{\Gamma(\tau+1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} \right) |x(t) - x^*(t)| + \frac{m\epsilon_\tau t^\tau}{\Gamma(\tau+1)} \\ &+ mA_{I_i} |x(t) - x^*(t)| + m\epsilon_\tau + A_h |x(t) - x^*(t)|. \end{aligned}$$

By taking norm and simplification, we get

$$\begin{aligned} \|x - x^*\|_{\mathcal{M}} &\leq \epsilon_\tau \left(\frac{(1+m)T^\tau}{\Gamma(\tau+1)} + m \right) + \left[\frac{(1+m)T^\tau}{\Gamma(\tau+1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} \right) \right. \\ &\quad \left. + A_h + mA_{I_i} \right] \|x - x^*\|_{\mathcal{M}}. \end{aligned}$$

From which, we obtain

$$\|x - x^*\|_{\mathcal{M}} \leq \frac{\epsilon_\tau \left(\frac{(1+m)T^\tau}{\Gamma(\tau+1)} + m \right)}{1 - \left[\frac{(1+m)T^\tau}{\Gamma(\tau+1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} \right) + A_h + mA_{I_i} \right]}. \quad (4.6)$$

Similarly, for $t \in \mathcal{J}_0$, we have

$$\|x - x^*\|_{\mathcal{M}} \leq \frac{\epsilon_{\tau} \left(\frac{T^{\tau}}{\Gamma(\tau+1)} \right)}{1 - \left[\frac{T^{\tau}}{\Gamma(\tau+1)} \left(\frac{M_1}{1-N_1-N_2} \frac{T^{\sigma}}{\sigma \Gamma(\delta)} + \frac{M_2}{1-N_1-N_2} \frac{T^{\sigma}}{\sigma \Gamma(\delta)} \right) + A_h \right]}. \quad (4.7)$$

Combining (4.6) and (4.7), for $t \in \mathcal{J}$, we have

$$\begin{aligned} \|x - x^*\|_{\mathcal{M}} \leq & \left[\frac{\left(\frac{(1+m)T^{\tau}}{\Gamma(\tau+1)} + m \right)}{1 - \left[\frac{(1+m)T^{\tau}}{\Gamma(\tau+1)} \left(\frac{M_1}{1-N_1-N_2} \frac{T^{\sigma}}{\sigma \Gamma(\delta)} + \frac{M_2}{1-N_1-N_2} \frac{T^{\sigma}}{\sigma \Gamma(\delta)} \right) + A_h + mA_{I_i} \right]} \right. \\ & \left. + \frac{\left(\frac{T^{\tau}}{\Gamma(\tau+1)} \right)}{1 - \left[\frac{T^{\tau}}{\Gamma(\tau+1)} \left(\frac{M_1}{1-N_1-N_2} \frac{T^{\sigma}}{\sigma \Gamma(\delta)} + \frac{M_2}{1-N_1-N_2} \frac{T^{\sigma}}{\sigma \Gamma(\delta)} \right) + A_h \right]} \right] \epsilon_{\tau}. \end{aligned}$$

Thus

$$\|x - x^*\|_{\mathcal{M}} \leq \mathbf{C}_1 \epsilon_{\tau},$$

where

$$\begin{aligned} \mathbf{C}_1 = & \frac{\left(\frac{(1+m)T^{\tau}}{\Gamma(\tau+1)} + m \right)}{1 - \left[\frac{(1+m)T^{\tau}}{\Gamma(\tau+1)} \left(\frac{M_1}{1-N_1-N_2} \frac{T^{\sigma}}{\sigma \Gamma(\delta)} + \frac{M_2}{1-N_1-N_2} \frac{T^{\sigma}}{\sigma \Gamma(\delta)} \right) + A_h + mA_{I_i} \right]} \\ & + \frac{\left(\frac{T^{\tau}}{\Gamma(\tau+1)} \right)}{1 - \left[\frac{T^{\tau}}{\Gamma(\tau+1)} \left(\frac{M_1}{1-N_1-N_2} \frac{T^{\sigma}}{\sigma \Gamma(\delta)} + \frac{M_2}{1-N_1-N_2} \frac{T^{\sigma}}{\sigma \Gamma(\delta)} \right) + A_h \right]}. \end{aligned}$$

Therefore, problem (1.1) is HU stable. Further, if we set $\mathcal{G}(\epsilon_{\tau}) = C(\epsilon_{\tau})$; $\mathcal{G}(0) = 0$, then the problem (1.1) becomes generalized HU stable. \square

Assume that

– $[A_7] \exists$ a nondecreasing function $\psi_{\tau} \in \mathcal{M}$ and a constant $\varrho_{\psi_{\tau}} > 0$ so that for each $t \in \mathcal{J}$:

$$I^{\varrho} \psi_{\tau}(t) \leq \varrho_{\psi_{\tau}} \psi_{\tau}(t).$$

Theorem 4.11. *If the hypothesis $[A_1]$ – $[A_3]$, $[A_7]$ and the inequality (3.14) are satisfied, then problem (1.1) is HU–Rassias stable with respect to $(\phi_{\tau}, \psi_{\tau})$ and consequently, it is generalized HU–Rassias stable.*

Proof. Let $x \in \mathcal{M}$ be any solution of the inequality (4.3) and let $x^* \in \mathcal{M}$ be the solution of the problem:

$$\begin{cases} {}^c D^{\tau} x^*(t) = \mathcal{F}(t, x^*(t), {}^c D^{\tau} x^*(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x^*(\xi), {}^c D^{\tau} x^*(\xi)) d\xi, \quad \forall t \in \mathcal{J}, \quad t \neq t_i \text{ for } i = 1, 2, \dots, m, \\ x^*(0) = h(x^*), \\ \Delta x^*(t_i) = I_i(x^*(t_i)), \quad i = 1, 2, \dots, m. \end{cases}$$

From the proof of Theorem 4.10, we have

$$|x(t) - x^*(t)| \leq \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} |\bar{x}(\xi) - \tilde{x}(\xi)| d\xi + \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} |\Phi(\xi)| d\xi$$

$$\begin{aligned}
& + \sum_{i=1}^m \frac{1}{\Gamma(\mathfrak{r})} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{\mathfrak{r}-1} |\bar{\tilde{x}}(\xi) - \tilde{x}(\xi)| d\xi + \sum_{i=1}^m \frac{1}{\Gamma(\mathfrak{r})} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{\mathfrak{r}-1} |\Phi(\xi)| d\xi \\
& + \sum_{i=1}^m |I_i(x(t_i)) - I_i(x^*(t_i))| + \sum_{i=1}^m |\Phi_i| + |h(x) - h(x^*)|.
\end{aligned}$$

By $[A_1]$, we obtain

$$|\bar{\tilde{x}}(t) - \tilde{x}(t)| \leq \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}} \right) |x(t) - x^*(t)|.$$

Hence, using $[A_1]$ – $[A_3]$ and part (i) of Remark 4.8, above inequality, we have

$$\begin{aligned}
|x(t) - x^*(t)| & \leq \frac{t^\mathfrak{r}}{\Gamma(\mathfrak{r} + 1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}} \right) |x(t) - x^*(t)| + \frac{\psi_\mathfrak{r}(t)\epsilon_\mathfrak{r}t^\mathfrak{r}}{\Gamma(\mathfrak{r} + 1)} \\
& + \frac{mt^\mathfrak{r}}{\Gamma(\mathfrak{r} + 1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma\Gamma(\delta)}} \right) |x(t) - x^*(t)| + \frac{m\psi_\mathfrak{r}(t)\epsilon_\mathfrak{r}t^\mathfrak{r}}{\Gamma(\mathfrak{r} + 1)} \\
& + mA_{I_i}|x(t) - x^*(t)| + m\epsilon_\mathfrak{r} + A_h|x(t) - x^*(t)|.
\end{aligned}$$

Using $[A_7]$, we get

$$\begin{aligned}
\|x - x^*\|_{\mathcal{M}} & \leq \epsilon_\mathfrak{r}(\varrho_{\psi_\mathfrak{r}}\psi_\mathfrak{r}(t)(1 + \mathcal{K}) + \mathcal{L}\phi_\mathfrak{r}) + \left[\frac{(1+m)T^\mathfrak{r}}{\Gamma(\mathfrak{r} + 1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} \right) \right. \\
& \quad \left. + A_h + mA_{I_i} \right] \|x - x^*\|_{\mathcal{M}} \\
& \leq \epsilon_\mathfrak{r}((\psi_\mathfrak{r}(t) + \phi_\mathfrak{r})(\varrho_{\psi_\mathfrak{r}}(1 + \mathcal{K}) + \mathcal{L}\phi_\mathfrak{r}) + \left[\frac{(1+m)T^\mathfrak{r}}{\Gamma(\mathfrak{r} + 1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} \right) \right. \\
& \quad \left. + A_h + mA_{I_i} \right] \|x - x^*\|_{\mathcal{M}}.
\end{aligned}$$

Which yields

$$\|x - x^*\|_{\mathcal{M}} \leq \frac{\epsilon_\mathfrak{r}(\psi_\mathfrak{r}(t) + \phi_\mathfrak{r})(\varrho_{\psi_\mathfrak{r}}(1 + \mathcal{K}) + \mathcal{L}\phi_\mathfrak{r})}{1 - \left[\frac{(1+m)T^\mathfrak{r}}{\Gamma(\mathfrak{r} + 1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} \right) + A_h + mA_{I_i} \right]}. \quad (4.8)$$

Similarly, for $t \in \mathcal{J}_0$, we have

$$\|x - x^*\|_{\mathcal{M}} \leq \frac{\epsilon_\mathfrak{r}(\psi_\mathfrak{r}(t) + \phi_\mathfrak{r})\varrho_{\psi_\mathfrak{r}}}{1 - \left[\frac{T^\mathfrak{r}}{\Gamma(\mathfrak{r} + 1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} \right) + A_h \right]}. \quad (4.9)$$

Combining (4.8) and (4.9), for $t \in \mathcal{J}$, we have

$$\begin{aligned}
\|x - x^*\|_{\mathcal{M}} & \leq \epsilon_\mathfrak{r}(\psi_\mathfrak{r}(t) + \phi_\mathfrak{r}) \left[\frac{\varrho_{\psi_\mathfrak{r}}(1 + \mathcal{K}) + \mathcal{K}}{1 - \left[\frac{(1+m)T^\mathfrak{r}}{\Gamma(\mathfrak{r} + 1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} \right) + A_h + mA_{I_i} \right]} \right. \\
& \quad \left. + \frac{\varrho_{\psi_\mathfrak{r}}}{1 - \left[\frac{T^\mathfrak{r}}{\Gamma(\mathfrak{r} + 1)} \left(\frac{M_1}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} \right) + A_h \right]} \right].
\end{aligned}$$

Thus

$$\|x - x^*\|_{\mathcal{M}} \leq \mathbf{C}_2 \epsilon_\mathfrak{r}(\psi(t) + \phi),$$

where

$$\mathbf{C}_2 = \left[\frac{\varrho\psi(1+\mathcal{K}) + \mathcal{K}}{1 - \left[\frac{(1+m)\Gamma^\tau}{\Gamma(\tau+1)} \left(\frac{M_1}{1-N_1-N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}}{1-N_1-N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} \right) + A_h + mA_{I_i} \right]} + \frac{\varrho\psi}{1 - \left[\frac{\Gamma^\tau}{\Gamma(\tau+1)} \left(\frac{M_1}{1-N_1-N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}}{1-N_1-N_2 \frac{\Gamma^\sigma}{\sigma\Gamma(\delta)}} \right) + A_h \right]} \right].$$

Therefore, problem (1.1) is HU–Rassias stable. Similarly, we can show that it is generalized HU–Rassias stable. \square

Next, we study the stability results of the proposed system (1.2). The following definitions are adopted from [7].

Let $\epsilon_\tau, \epsilon_s > 0$, $\mathcal{F}, \mathcal{G}, \mathcal{F}', \mathcal{G}'$ be continuous functions and $\psi_\tau, \psi_s : \mathcal{J} \rightarrow \mathbb{R}^+$ are nondecreasing functions. Consider the following inequalities:

$$\left\{ \begin{array}{l} \left| {}^c D^\tau x(t) - \mathcal{F}(t, y(t), {}^c D^\tau x(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), {}^c D^\tau x(\xi)) d\xi \right| \leq \epsilon_\tau, \quad \forall t \in \mathcal{J}, \\ \left| {}^c D^s y(t) - \mathcal{F}'(t, x(t), {}^c D^s y(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), {}^c D^s y(\xi)) d\xi \right| \leq \epsilon_s, \quad \forall t \in \mathcal{J}, \\ |\Delta x(t_i) - I_i(x(t_i))| \leq \epsilon_\tau, \quad i = 1, 2, \dots, m, \\ |\Delta y(t_j) - I_j(y(t_j))| \leq \epsilon_s, \quad j = 1, 2, \dots, n, \end{array} \right. \quad (4.10)$$

$$\left\{ \begin{array}{l} \left| {}^c D^\tau x(t) - \mathcal{F}(t, y(t), {}^c D^\tau x(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), {}^c D^\tau x(\xi)) d\xi \right| \leq \psi_\tau(t), \quad \forall t \in \mathcal{J}, \\ \left| {}^c D^s y(t) - \mathcal{F}'(t, x(t), {}^c D^s y(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), {}^c D^s y(\xi)) d\xi \right| \leq \psi_s(t), \quad \forall t \in \mathcal{J}, \\ |\Delta x(t_i) - I_i(x(t_i))| \leq \phi_\tau, \quad i = 1, 2, \dots, m, \\ |\Delta y(t_j) - I_j(y(t_j))| \leq \phi_s, \quad j = 1, 2, \dots, n \end{array} \right. \quad (4.11)$$

and

$$\left\{ \begin{array}{l} \left| {}^c D^\tau x(t) - \mathcal{F}(t, y(t), {}^c D^\tau x(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), {}^c D^\tau x(\xi)) d\xi \right| \leq \epsilon_\tau \psi_\tau(t), \quad \forall t \in \mathcal{J}, \\ \left| {}^c D^s y(t) - \mathcal{F}'(t, x(t), {}^c D^s y(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), {}^c D^s y(\xi)) d\xi \right| \leq \epsilon_s \psi_s(t), \quad \forall t \in \mathcal{J}, \\ |\Delta x(t_i) - I_i(x(t_i))| \leq \epsilon_\tau \phi_\tau, \quad i = 1, 2, \dots, m, \\ |\Delta y(t_j) - I_j(y(t_j))| \leq \epsilon_s \phi_s, \quad j = 1, 2, \dots, n. \end{array} \right. \quad (4.12)$$

Definition 4.12. The problem (1.2) is said to be HU stable, if there is $\mathcal{C}_{\tau,s} = (\mathcal{C}_\tau, \mathcal{C}_s) > 0$ for some $\epsilon = (\epsilon_\tau, \epsilon_s)$ and for each solution $(x, y) \in \mathcal{X} \times \mathcal{Y}$ of (4.10), there is a solution $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ of problem (1.2) with

$$|(x, y)(t) - (x^*, y^*)(t)| \leq \mathcal{C}_{\tau,s} \epsilon, \quad \forall t \in \mathcal{J}. \quad (4.13)$$

Definition 4.13. The problem (1.2) is said to be generalized HU stable, if there is $\Theta \in C(\mathcal{J}, \mathbb{R})$ with $\Theta(0) = 0$, so that for each solution $(x, y) \in \mathcal{X} \times \mathcal{Y}$ of (4.10), there is a solution $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ of problem (1.2) with

$$|(x, y)(t) - (x^*, y^*)(t)| \leq \Theta(\epsilon), \quad \forall t \in \mathcal{J}. \quad (4.14)$$

Definition 4.14. The problem (1.2) is said to be HU–Rassias stable with respect to $\psi_{\tau,s} = (\psi_\tau, \psi_s) \in C^1(\mathcal{J}, \mathbb{R})$, if there is a constant $\mathcal{C}_{\psi_\tau, \psi_s} = (\mathcal{C}_{\psi_\tau}, \mathcal{C}_{\psi_s})$ so that for some $\epsilon = (\epsilon_\tau, \epsilon_s) > 0$ and for each solution $(x, y) \in \mathcal{X} \times \mathcal{Y}$ of (4.11), there is a solution $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ of problem (1.2) with

$$|(x, y)(t) - (x^*, y^*)(t)| \leq \mathcal{C}_{\psi_\tau, \psi_s} \epsilon, \quad \forall t \in \mathcal{J}. \quad (4.15)$$

Definition 4.15. The problem (1.2) is said to be generalized HU–Rassias stable with respect to $\psi_{\tau,s} = (\psi_\tau, \psi_s)$, if there is a constant $\mathcal{C}_{\psi_\tau, \psi_s} = (\mathcal{C}_{\psi_\tau}, \mathcal{C}_{\psi_s}) > 0$, so that for each solution $(x, y) \in \mathcal{X} \times \mathcal{Y}$ of (4.12), there is a solution $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ of problem (1.2) with

$$|(x, y)(t) - (x^*, y^*)(t)| \leq \mathcal{C}_{\psi_\tau, \psi_s} \psi_{\tau,s}, \quad \forall t \in \mathcal{J}. \quad (4.16)$$

Remark 4.16. Definition 4.12 implies Definition 4.13 and Definition 4.14 implies Definition 4.15.

Remark 4.17. We say that $(x, y) \in \mathcal{X} \times \mathcal{Y}$ is a solution of (4.10) if there are functions $\mu_{\mathcal{F}, \mathcal{G}}, \Lambda_{\mathcal{F}', \mathcal{G}'} \in \mathcal{X} \times \mathcal{Y}$ which depend upon (x, y) respectively, so that

i) $|\mu_{\mathcal{F}, \mathcal{G}}(t)| \leq \epsilon_\tau, |\Lambda_{\mathcal{F}', \mathcal{G}'}(t)| \leq \epsilon_s, \quad \forall t \in \mathcal{J};$

ii)

$${}^c D^\tau x(t) = \mathcal{F}(t, y(t), {}^c D^\tau x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), {}^c D^\tau x(\xi)) d\xi + \mu_{\mathcal{F}, \mathcal{G}}(t), \quad t \in \mathcal{J}_i;$$

and

$${}^c D^s y(t) = \mathcal{F}(t, x(t), {}^c D^s y(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, x(\xi), {}^c D^s y(\xi)) d\xi + \Lambda_{\mathcal{F}', \mathcal{G}'}(t), \quad t \in \mathcal{J}_j;$$

iii) $\Delta x(t_i) = I_i(x(t_i)) + \mu_i, \quad t \in \mathcal{J}_i, \quad i = 1, 2, \dots, m$ and

$\Delta y(t_j) = I_j(y(t_j)) + \Lambda_j, \quad t \in \mathcal{J}_j, \quad j = 1, 2, \dots, n.$

Theorem 4.18. Let $(x, y) \in \mathcal{X} \times \mathcal{Y}$ be the solution of the inequality (4.10), then we have

$$\begin{cases} |x(t) - x^*(t)| \leq \left[\frac{(1+m)t^\tau}{\Gamma(\tau+1)} + m \right] \epsilon_\tau, \\ |y(t) - y^*(t)| \leq \left[\frac{(1+n)t^s}{\Gamma(s+1)} + n \right] \epsilon_s. \end{cases}$$

Proof. Let (x, y) be the solution of the inequality (4.10), then by Remark 4.17. (x, y) will also be the solution of

$$\begin{cases} {}^c D^\tau x(t) = \mathcal{F}(t, y(t), {}^c D^\tau x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), {}^c D^\tau x(\xi)) d\xi + \mu_{\mathcal{F}, \mathcal{G}}, \quad \forall t \in \mathcal{J}, \quad t \neq t_i \text{ for } i = 1, 2, \dots, m, \\ {}^c D^s y(t) = \mathcal{F}'(t, x(t), {}^c D^s y(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), {}^c D^s y(\xi)) d\xi + \Lambda_{\mathcal{F}', \mathcal{G}'}, \quad \forall t \in \mathcal{J}, \quad t \neq t_j \text{ for } j = 1, 2, \dots, n, \\ x(0) = h(x), \quad y(0) = g(y), \\ \Delta x(t_i) = I_i(x(t_i)) + \mu_i, \quad i = 1, 2, \dots, m, \\ \Delta y(t_j) = I_j(y(t_j)) + \Lambda_j, \quad j = 1, 2, \dots, n. \end{cases} \quad (4.17)$$

i.e.,

$$x(t) = \begin{cases} \frac{1}{\Gamma(\tau)} \int_0^t (t-\xi)^{\tau-1} \alpha(\xi) d\xi + \frac{1}{\Gamma(\tau)} \int_0^t (t-\xi)^{\tau-1} \mu(\xi) d\xi + h(x), & t \in \mathcal{J}_0, \\ \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} \alpha(\xi) d\xi + \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} \mu(\xi) d\xi + \sum_{i=1}^m \frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} \alpha(\xi) d\xi \\ + \sum_{i=1}^m \frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} \mu(\xi) d\xi + \sum_{i=1}^m I_i(x(t_i)) + \sum_{i=1}^m \mu_i + h(x), & t \in \mathcal{J}_i, i = 1, 2, \dots, m \end{cases} \quad (4.18)$$

and

$$y(t) = \begin{cases} \frac{1}{\Gamma(s)} \int_0^t (t-\xi)^{s-1} \beta(\xi) d\xi + \frac{1}{\Gamma(s)} \int_0^t (t-\xi)^{s-1} \Lambda(\xi) d\xi + g(y), & t \in \mathcal{J}_0, \\ \frac{1}{\Gamma(s)} \int_{t_j}^t (t-\xi)^{s-1} \beta(\xi) d\xi + \frac{1}{\Gamma(s)} \int_{t_j}^t (t-\xi)^{s-1} \Lambda(\xi) d\xi + \sum_{j=1}^n \frac{1}{\Gamma(s)} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{s-1} \beta(\xi) d\xi \\ + \sum_{j=1}^n \frac{1}{\Gamma(s)} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{s-1} \Lambda(\xi) d\xi + \sum_{j=1}^n I_j(y(t_j)) + \sum_{j=1}^n \Lambda_j + g(y), & t \in \mathcal{J}_j, j = 1, 2, \dots, n, \end{cases} \quad (4.19)$$

where

$$\alpha(t) = \mathcal{F}(t, y(t), {}^c D^\tau x(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), {}^c D^\tau x(\xi)) d\xi$$

and

$$\beta(t) = \mathcal{F}'(t, x(t), {}^c D^s y(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}'(\xi, x(\xi), {}^c D^s y(\xi)) d\xi.$$

From (4.18), we have

$$x(t) = \begin{cases} \frac{1}{\Gamma(\tau)} \int_0^t (t-\xi)^{\tau-1} \alpha(\xi) d\xi + \frac{1}{\Gamma(\tau)} \int_0^t (t-\xi)^{\tau-1} \mu(\xi) d\xi + h(x), & t \in \mathcal{J}_0, \\ \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} \alpha(\xi) d\xi + \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} \mu(\xi) d\xi + \sum_{i=1}^m \frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} \alpha(\xi) d\xi \\ + \sum_{i=1}^m \frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} \mu(\xi) d\xi + \sum_{i=1}^m I_i(x(t_i)) + \sum_{i=1}^m \mu_i + h(x), & t \in \mathcal{J}_i, i = 1, 2, \dots, m. \end{cases} \quad (4.20)$$

Thus (4.20) becomes

$$|x(t) - x^*(t)| \leq \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} |\mu(\xi)| d\xi + \sum_{i=1}^m \frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i-\xi)^{\tau-1} |\mu(\xi)| d\xi + \sum_{i=1}^m |\mu_i|,$$

where

$$x^*(t) = \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} \alpha(\xi) d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t-\xi)^{\tau-1} \alpha(\xi) d\xi + I_i(x^*(t_i)) \right] + h(x^*), \quad i = 1, 2, \dots, m.$$

Using (i) of Remark 4.17, we obtain

$$|x(t) - x^*(t)| \leq \left[\frac{(1+m)t^\tau}{\Gamma(\tau+1)} + m \right] \epsilon_\tau.$$

Repeating the similar procedure for (4.19) together with (i) from Remark 4.17, we have

$$|y(t) - y^*(t)| \leq \left[\frac{(1+n)t^s}{\Gamma(s+1)} + n \right] \epsilon_s,$$

where

$$y^*(t) = \frac{1}{\Gamma(s)} \int_{t_j}^t (t-\xi)^{s-1} \beta(\xi) d\xi + \sum_{j=1}^n \left[\frac{1}{\Gamma(s)} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{s-1} \beta(\xi) d\xi + I_j(y^*(t_j)) \right] + g(y^*), \quad j = 1, 2, \dots, n.$$

□

Theorem 4.19. *If the hypothesis $[H_1]$ – $[H_4]$, $[H_8]$ and $[H_9]$ satisfies with*

$$\Delta = 1 - Q_\tau Q_s > 0, \quad (4.21)$$

then system (1.2) is stable in the sense of HU.

Proof. Suppose $(x, y) \in \mathcal{X} \times \mathcal{Y}$ be the solution of the inequality (4.12) and $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ is the solution of the given system

$$\begin{cases} {}^c D^\tau x^*(t) = \mathcal{F}(t, y^*(t), {}^c D^\tau x^*(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)} \mathcal{G}(\xi, y^*(\xi), {}^c D^\tau x^*(\xi)) d\xi, \quad \forall t \in \mathcal{J}, \quad t \neq t_i \text{ for } i = 1, 2, \dots, m, \\ {}^c D^s y^*(t) = \mathcal{F}'(t, x^*(t), {}^c D^s y^*(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)} \mathcal{G}'(\xi, x^*(\xi), {}^c D^s y^*(\xi)) d\xi, \quad \forall t \in \mathcal{J}, \quad t \neq t_j \text{ for } j = 1, 2, \dots, n, \\ x^*(0) = h(x^*), \quad y^*(0) = g(y^*), \\ \Delta x^*(t_i) = I_i(x^*(t_i)), \quad i = 1, 2, \dots, m, \\ \Delta y^*(t_j) = I_j(y^*(t_j)), \quad j = 1, 2, \dots, n. \end{cases} \quad (4.22)$$

Then in view of Theorem 3.5, the solution of (4.22) is

$$x^*(t) = \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} \alpha(\xi) d\xi + \sum_{i=1}^m \left[\frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t-\xi)^{\tau-1} \alpha(\xi) d\xi + I_i(x^*(t_i)) \right] + h(x^*), \quad i = 1, 2, \dots, m$$

and

$$y^*(t) = \frac{1}{\Gamma(s)} \int_{t_j}^t (t-\xi)^{s-1} \beta(\xi) d\xi + \sum_{j=1}^n \left[\frac{1}{\Gamma(s)} \int_{t_{j-1}}^{t_j} (t_j-\xi)^{s-1} \beta(\xi) d\xi + I_j(y^*(t_j)) \right] + g(y^*), \quad j = 1, 2, \dots, n,$$

where

$$\alpha(t) = \mathcal{F}(t, y^*(t), {}^c D^\tau x^*(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)} \mathcal{G}(\xi, y^*(\xi), {}^c D^\tau x^*(\xi)) d\xi$$

and

$$\beta(t) = \mathcal{F}'(t, x^*(t), {}^c D^s y^*(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)} \mathcal{G}'(\xi, x^*(\xi), {}^c D^s y^*(\xi)) d\xi.$$

Consider

$$\begin{aligned}
 |x(t) - x^*(t)| &\leq |x(t) - q(t)| + |q(t) - x^*(t)| \\
 &\leq \left[\frac{(1+m)t^\tau}{\Gamma(\tau+1)} + m \right] \epsilon_\tau + \frac{1}{\Gamma(\tau)} \int_{t_i}^t (t-\xi)^{\tau-1} |\tilde{x}(\xi) - \tilde{x}^*(\xi)| d\xi + \sum_{i=1}^m \frac{1}{\Gamma(\tau)} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{\tau-1} |\tilde{x}(\xi) - \tilde{x}^*(\xi)| d\xi \\
 &\quad + \sum_{i=1}^m |I_i(x(t_i)) - I_i(x^*(t_i))| + |h(x) - h(x^*)|,
 \end{aligned} \tag{4.23}$$

where

$$\tilde{x}(t) = \mathcal{F}(t, y(t), \tilde{x}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), \tilde{x}(\xi)) d\xi$$

and

$$\tilde{x}^*(t) = \mathcal{F}(t, y^*(t), \tilde{x}^*(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y^*(\xi), \tilde{x}^*(\xi)) d\xi.$$

Using $[H_1]$, we have

$$\begin{aligned}
 |\tilde{x}(t) - \tilde{x}^*(t)| &= \left| \mathcal{F}(t, y(t), \tilde{x}(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y(\xi), \tilde{x}(\xi)) d\xi \right. \\
 &\quad \left. - \mathcal{F}(t, y^*(t), \tilde{x}^*(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} \mathcal{G}(\xi, y^*(\xi), \tilde{x}^*(\xi)) d\xi \right| \\
 &\leq |\mathcal{F}(t, y(t), \tilde{x}(t)) - \mathcal{F}(t, y^*(t), \tilde{x}^*(t))| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\delta)} |\mathcal{G}(\xi, y(\xi), \tilde{x}(\xi)) - \mathcal{G}(\xi, y^*(\xi), \tilde{x}^*(\xi))| d\xi \\
 &\leq M_1 |y(t) - y^*(t)| + N_1 |\tilde{x}(t) - \tilde{x}^*(t)| + \frac{t^\sigma}{\sigma \Gamma(\delta)} \left(M_2 |y(t) - y^*(t)| + N_2 |\tilde{x}(t) - \tilde{x}^*(t)| \right).
 \end{aligned}$$

Thus

$$|\tilde{x}(t) - \tilde{x}^*(t)| \leq \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} \right) |y(t) - y^*(t)|. \tag{4.24}$$

Using hypothesis (H_3) , (H_4) and (4.24) in (4.23), we have

$$\begin{aligned}
 |x(t) - x^*(t)| &\leq \left[\frac{(1+m)t^\tau}{\Gamma(\tau+1)} + m \right] \epsilon_\tau + \left[\left(\frac{(1+m)t^\tau}{\Gamma(\tau+1)} \right) \left(\frac{M_1}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{t^\sigma}{\sigma \Gamma(\delta)}} \right) \right] |y(t) - y^*(t)| \\
 &\quad + (mA_{I_i} + A_h) |x(t) - x^*(t)|.
 \end{aligned}$$

By taking norm and simplification, we get

$$\begin{aligned}
 \|x - x^*\|_{\mathcal{X}} &\leq \left[\frac{(1+m)T^\tau}{\Gamma(\tau+1)} + m \right] \epsilon_\tau + \left[\left(\frac{(1+m)T^\tau}{\Gamma(\tau+1)} \right) \left(\frac{M_1}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma \Gamma(\delta)}} \right) \right] \|y - y^*\|_{\mathcal{X}} \\
 &\quad + (mA_{I_i} + A_h) \|x - x^*\|_{\mathcal{X}}.
 \end{aligned} \tag{4.25}$$

For simplicity, we consider

$$S_\tau = \frac{\left[\frac{(1+m)T^\tau}{\Gamma(\tau+1)} + m \right]}{1 - (mA_{I_i} + A_h)},$$

$$Q_{\tau} = \frac{\left[\left(\frac{(1+m)\Gamma^{\tau}}{\Gamma(\tau+1)} \right) \left(\frac{M_1}{1-N_1-N_2 \frac{\Gamma^{\sigma}}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{\Gamma^{\sigma}}{\sigma\Gamma(\delta)}}{1-N_1-N_2 \frac{\Gamma^{\sigma}}{\sigma\Gamma(\delta)}} \right) \right]}{1 - (mA_{I_i} + A_h)},$$

(4.25) implies

$$\|x - x^*\|_{\mathcal{X}} \leq S_{\tau} \epsilon_{\tau} + Q_{\tau} \|y - y^*\|_{\mathcal{X}} \quad (4.26)$$

and similarly

$$\|y - y^*\|_{\mathcal{Y}} \leq S_{\mathfrak{s}} \epsilon_{\mathfrak{s}} + Q_{\mathfrak{s}} \|x - x^*\|_{\mathcal{Y}}, \quad (4.27)$$

where

$$S_{\mathfrak{s}} = \frac{\left[\frac{(1+n)\Gamma^{\mathfrak{s}}}{\Gamma(\mathfrak{s}+1)} + n \right]}{1 - (nA_{I_j} + A_g)},$$

$$Q_{\mathfrak{s}} = \frac{\left[\left(\frac{(1+n)\Gamma^{\mathfrak{s}}}{\Gamma(\mathfrak{s}+1)} \right) \left(\frac{M'_1}{1-N'_1-N'_2 \frac{\Gamma^{\sigma}}{\sigma\Gamma(\delta)}} + \frac{M'_2 \frac{\Gamma^{\sigma}}{\sigma\Gamma(\delta)}}{1-N'_1-N'_2 \frac{\Gamma^{\sigma}}{\sigma\Gamma(\delta)}} \right) \right]}{1 - (nA_{I_j} + A_g)},$$

$$\|x - x^*\|_{\mathcal{X}} - Q_{\tau} \|y - y^*\|_{\mathcal{X}} \leq S_{\tau} \epsilon_{\tau},$$

$$\|y - y^*\|_{\mathcal{Y}} - Q_{\mathfrak{s}} \|x - x^*\|_{\mathcal{Y}} \leq S_{\mathfrak{s}} \epsilon_{\mathfrak{s}},$$

$$\begin{bmatrix} 1 & -Q_{\tau} \\ -Q_{\mathfrak{s}} & 1 \end{bmatrix} \begin{bmatrix} \|x - x^*\|_{\mathcal{X} \times \mathcal{Y}} \\ \|y - y^*\|_{\mathcal{X} \times \mathcal{Y}} \end{bmatrix} \leq \begin{bmatrix} S_{\tau} \epsilon_{\tau} \\ S_{\mathfrak{s}} \epsilon_{\mathfrak{s}} \end{bmatrix}.$$

Solving the above inequality, we have

$$\begin{bmatrix} \|x - x^*\|_{\mathcal{X} \times \mathcal{Y}} \\ \|y - y^*\|_{\mathcal{X} \times \mathcal{Y}} \end{bmatrix} \leq \begin{bmatrix} \frac{1}{\Delta} & \frac{Q_{\tau}}{\Delta} \\ \frac{Q_{\mathfrak{s}}}{\Delta} & \frac{1}{\Delta} \end{bmatrix} \begin{bmatrix} S_{\tau} \epsilon_{\tau} \\ S_{\mathfrak{s}} \epsilon_{\mathfrak{s}} \end{bmatrix},$$

where

$$\Delta = 1 - Q_{\tau} Q_{\mathfrak{s}} > 0.$$

Further simplification gives

$$\|x - x^*\|_{\mathcal{X} \times \mathcal{Y}} \leq \frac{S_{\tau} \epsilon_{\tau}}{\Delta} + \frac{Q_{\tau} S_{\mathfrak{s}} \epsilon_{\mathfrak{s}}}{\Delta},$$

$$\|y - y^*\|_{\mathcal{X} \times \mathcal{Y}} \leq \frac{S_{\mathfrak{s}} \epsilon_{\mathfrak{s}}}{\Delta} + \frac{Q_{\mathfrak{s}} S_{\tau} \epsilon_{\tau}}{\Delta}.$$

From which we have

$$\|x - x^*\|_{\mathcal{X} \times \mathcal{Y}} + \|y - y^*\|_{\mathcal{X} \times \mathcal{Y}} \leq \frac{S_{\tau} \epsilon_{\tau}}{\Delta} + \frac{S_{\mathfrak{s}} \epsilon_{\mathfrak{s}}}{\Delta} + \frac{Q_{\tau} S_{\mathfrak{s}} \epsilon_{\mathfrak{s}}}{\Delta} + \frac{Q_{\mathfrak{s}} S_{\tau} \epsilon_{\tau}}{\Delta}. \quad (4.28)$$

Let $\max \{ \epsilon_{\tau}, \epsilon_{\mathfrak{s}} \} = \epsilon$, then from (4.28) we get

$$\|(x, y) - (x^*, y^*)\|_{\mathcal{X} \times \mathcal{Y}} \leq C_{\tau, \mathfrak{s}} \epsilon,$$

where

$$C_{\mathfrak{z}} = \left[\frac{S_{\tau}}{\Delta} + \frac{S_{\mathfrak{s}}}{\Delta} + \frac{Q_{\tau} S_{\mathfrak{s}}}{\Delta} + \frac{Q_{\mathfrak{s}} S_{\tau}}{\Delta} \right].$$

This completes the proof. \square

Remark 4.20. We set $\Theta(\epsilon) = C_{\mathfrak{z}} \epsilon$, $\Theta(0) = 0$ in (4.28). By Definition 4.13 the proposed system (1.2) is generalized HU stable.

In order to obtain the connections between the HU–Rassias stability concepts we introduce the following hypothesis.

- $[H_{10}]$ Let $\Omega_{\tau}, \Omega_{\varsigma} \in C(\mathcal{J}, \mathbb{R}^+)$ be an increasing functions. Then there exist $\Lambda_{\Omega_{\tau}}, \Lambda_{\Omega_{\varsigma}} > 0$ such that for each $t \in \mathcal{J}$ the integral inequalities:

$$I^{\tau} \Omega_{\tau}(t) \leq \Lambda_{\Omega_{\tau}} \Omega_{\tau}(t) \quad \text{and} \quad I^{\tau-1} \Omega_{\tau}(t) \leq \Lambda_{\Omega_{\tau}} \Omega_{\tau}(t)$$

and

$$I^{\varsigma} \Omega_{\varsigma}(t) \leq \Lambda_{\Omega_{\varsigma}} \Omega_{\varsigma}(t) \quad \text{and} \quad I^{\varsigma-1} \Omega_{\varsigma}(t) \leq \Lambda_{\Omega_{\varsigma}} \Omega_{\varsigma}(t),$$

holds.

Under the hypothesis $[H_1]$ – $[H_{10}]$, (4.21) with Theorem 4.18 and Theorem 4.19 system (1.2) will be HU–Rassias and generalized HU–Rassias stable.

5 Illustrative example

Example 5.1.

$$\begin{cases} {}^c D^{\frac{1}{2}} x(t) = \frac{1 + |x(t)| + \cos|{}^c D^{\frac{1}{2}} x(t)|}{104e^{t+5}(1 + |x(t)| + |{}^c D^{\frac{1}{2}} x(t)|)} + \int_0^1 \frac{(t-\xi)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} \frac{1 + |x(\xi)| + \sin|{}^c D^{\frac{1}{2}} x(\xi)|}{104e^{t+5}(1 + |x(\xi)| + |{}^c D^{\frac{1}{2}} x(\xi)|)} d\xi, \quad t \in [0, 1], \\ x(0) = h(x) = \frac{\cos|x(t)|}{18 + e^t}, \\ I_1(x(\frac{1}{2})) = \frac{|x(\frac{1}{2})|}{40 + |x(\frac{1}{2})|}, \end{cases} \quad (5.1)$$

where $\tau = \frac{1}{2}$, $\mathcal{J}_0 = [0, \frac{1}{2}]$, $\mathcal{J}_1 = (\frac{1}{2}, 1]$, $\sigma = \delta = \frac{5}{2}$. Set

$$\begin{aligned} \mathcal{F}(t, x, y) &= \frac{1 + |x(t)| + \cos|{}^c D^{\frac{1}{2}} x(t)|}{104e^{t+5}(1 + |x(t)| + |{}^c D^{\frac{1}{2}} x(t)|)}, \quad \forall t \in [0, 1], \\ \mathcal{G}(t, x, y) &= \frac{1 + |x(t)| + \sin|{}^c D^{\frac{1}{2}} x(t)|}{104e^{t+5}(1 + |x(t)| + |{}^c D^{\frac{1}{2}} x(t)|)}, \quad \forall t \in [0, 1]. \end{aligned}$$

Obviously, \mathcal{F} and \mathcal{G} are jointly continuous functions.

Now, for each $x, \bar{x} \in \mathcal{M}$ and $y, \bar{y} \in \mathbb{R}$ and $t \in [0, 1]$, we have

$$|\mathcal{F}(t, x, y) - \mathcal{F}(t, \bar{x}, \bar{y})| \leq \frac{1}{104e^5} (|x - \bar{x}| + |y - \bar{y}|)$$

and

$$|\mathcal{G}(t, x, y) - \mathcal{G}(t, \bar{x}, \bar{y})| \leq \frac{1}{104e^5} (|x - \bar{x}| + |y - \bar{y}|).$$

Which satisfies $[A_1]$ with $M_1 = M_2 = N_1 = N_2 = \frac{1}{104e^5}$. Set

$$I_1(x(\frac{1}{3})) = \frac{|x(\frac{1}{3})|}{40 + |x(\frac{1}{3})|}, \quad \forall x \in \mathcal{M}.$$

Then for $x, \bar{x} \in \mathcal{M}$, we have

$$\left| I_1(x(\frac{1}{3})) - I_1(\bar{x}(\frac{1}{3})) \right| = \left| \frac{|x(\frac{1}{3})|}{40 + |x(\frac{1}{3})|} - \frac{|\bar{x}(\frac{1}{3})|}{40 + |\bar{x}(\frac{1}{3})|} \right| \leq \frac{1}{35} |x - \bar{x}|.$$

Hence, with $A_{I_1} = \frac{1}{35}$, $[A_2]$ satisfies. Set

$$h(x) = \frac{\cos|x(t)|}{18 + e^t}.$$

Then for $x, \bar{x} \in \mathbb{R}$, we have

$$|h(x) - h(\bar{x})| = \left| \frac{\cos|x(t)|}{18 + e^t} - \frac{\cos|\bar{x}(t)|}{18 + e^t} \right| \leq \frac{1}{19} |x - \bar{x}|.$$

Hence, $[A_3]$ satisfies with $A_h = \frac{1}{19}$. Also

$$\left[\left(\frac{(1+m)T^\tau}{\Gamma(\tau+1)} \right) \left(\frac{M_1}{1-N_1-N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}}{1-N_1-N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} \right) + mA_{I_1} + A_h \right] \approx 0.08138 < 1$$

satisfies with $m = 1$, $T = 1$, $\sigma = \delta = \frac{5}{2}$, $\tau = \frac{1}{2}$, $M_1 = N_1 = M_2 = N_2 = \frac{1}{104e^5}$, $A_{I_1} = \frac{1}{35}$, $A_h = \frac{1}{19}$. Therefore, by Theorem 3.4, the problem (5.1) has a unique solution.

Now to confirm the existence of at least one solution, we need to check $[A_4]$ and $[A_6]$.

For each $t \in [0, 1]$, we have

$$|\mathcal{F}(t, x(t), {}^c D^{\frac{1}{2}} x(t))| \leq \frac{1}{104e^{t+5}} (1 + |u(t) + \cos t| {}^c D^{\frac{1}{2}} x(t)|).$$

Hence, $[A_4]$ satisfies with

$$a_1(t) = \frac{1}{38e^{t+5}}, \quad b_1(t) = c_1(t) = \frac{1}{104e^{t+5}}.$$

Also

$$|\mathcal{G}(t, x(t), {}^c D^{\frac{1}{2}} x(t))| \leq \frac{1}{104e^{t+5}} (1 + |u(t) + \sin t| {}^c D^{\frac{1}{2}} x(t)|).$$

Hence, $[A_4]$ satisfies with

$$a_2(t) = \frac{1}{38e^{t+5}}, \quad b_2(t) = c_2(t) = \frac{1}{104e^{t+5}}.$$

And for each $x \in \mathcal{M}$, we have

$$|I_i x(t)| \leq \frac{1}{35} |x(t)| + 1.$$

Hence, $[A_6]$ also satisfies with $\mathcal{K} = \frac{1}{35}$ and $\mathcal{L} = 1$.

Therefore, by Theorem 3.3, the problem (5.1) has at least one solution on the given interval.

Let for any $t \in [0, 1]$, we have $\psi(t) = |t|$ and $\phi = 1$. Then

$$I^{\frac{1}{2}} x(t) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-\xi)^{\frac{1}{2}-1} |\xi| d\xi \leq \frac{2t}{\sqrt{\pi}}.$$

Thus, $[A_7]$ satisfies with $\varrho_{\psi^{\frac{1}{2}}} = \frac{2}{\sqrt{\pi}}$. Therefore, by Theorem 4.10, the problem (5.1) is HU–Rassias stable and consequently, it is generalized HU–Rassias stable.

Example 5.2.

$$\begin{cases} {}^c D^{\frac{1}{2}} x(t) = \frac{1 + |y(t)| + \cos|{}^c D^{\frac{1}{2}} x(t)|}{104e^{t+5}(1 + |y(t)| + |{}^c D^{\frac{1}{2}} x(t)|)} + \int_0^t \frac{(t-\xi)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} \frac{1 + |y(\xi)| + \sin|{}^c D^{\frac{1}{2}} x(\xi)|}{104e^{t+5}(1 + |y(\xi)| + |{}^c D^{\frac{1}{2}} x(\xi)|)} d\xi, & t \in [0, 1], t \neq \frac{1}{3}, \\ {}^c D^{\frac{1}{2}} y(t) = \frac{2 + |x(t)| + \cos|{}^c D^{\frac{1}{2}} y(t)|}{70e^{t+2}(1 + |x(t)| + |{}^c D^{\frac{1}{2}} y(t)|)} + \int_0^t \frac{(t-\xi)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} \frac{|x(\xi)| + \cos|{}^c D^{\frac{1}{2}} y(\xi)|}{70e^{t+2}(1 + |x(\xi)| + |{}^c D^{\frac{1}{2}} y(\xi)|)} d\xi, & t \in [0, 1], t \neq \frac{1}{4}, \\ x(0) = h(x) = \frac{\cos|x(t)|}{18 + e^t}, \quad y(0) = g(y) = \frac{\sin|y(t)|}{18 + e^t}, \\ I_1(x(\frac{1}{3})) = \frac{|x(\frac{1}{3})|}{40 + |x(\frac{1}{3})|}, \quad I_1(y(\frac{1}{4})) = \frac{1}{50 + |y(\frac{1}{4})|}, \end{cases} \quad (5.2)$$

$t_i = \frac{1}{3}$ for $i = 1, 2, 3, \dots, 60$, and $t_j = \frac{1}{4}$ for $j = 1, 2, 3, \dots, 100$.

For any $x, \bar{x}, y, \bar{y} \in \mathbb{R}$, $t \in [0, 1]$, we obtain

$$|\mathcal{F}(t, x, y) - \mathcal{F}(t, \bar{x}, \bar{y})| \leq \frac{1}{104e^5} (|x - \bar{x}| + |y - \bar{y}|)$$

and

$$|\mathcal{G}(t, x, y) - \mathcal{G}(t, \bar{x}, \bar{y})| \leq \frac{1}{104e^5}(|x - \bar{x}| + |y - \bar{y}|).$$

Similarly, For any $x, \bar{x}, y, \bar{y} \in \mathbb{R}, t \in [0, 1]$, we obtain

$$|\mathcal{F}'(t, x, y) - \mathcal{F}'(t, \bar{x}, \bar{y})| \leq \frac{1}{70e^2}(|x - \bar{x}| + |y - \bar{y}|)$$

and

$$|\mathcal{G}'(t, x, y) - \mathcal{G}'(t, \bar{x}, \bar{y})| \leq \frac{1}{70e^2}(|x - \bar{x}| + |y - \bar{y}|).$$

Which satisfies $[H_1]$ with $M_1 = M_2 = N_1 = N_2 = \frac{1}{104e^5}$, $M'_1 = M'_2 = N'_1 = N'_2 = \frac{1}{70e^2}$. Set

$$I_i(x(\frac{1}{3})) = \frac{|x(\frac{1}{3})|}{40 + |x(\frac{1}{3})|}, \forall x \in \mathcal{X}.$$

Then for $x, \bar{x} \in \mathcal{X}$, we have

$$\left| I_i(x(\frac{1}{3})) - I_i(\bar{x}(\frac{1}{3})) \right| = \left| \frac{|x(\frac{1}{3})|}{40 + |x(\frac{1}{3})|} - \frac{|\bar{x}(\frac{1}{3})|}{40 + |\bar{x}(\frac{1}{3})|} \right| \leq \frac{1}{35}|x - \bar{x}|.$$

Hence, with $A_{I_i} = \frac{1}{35}$.

Similarly,

$$I_j(y(\frac{1}{4})) = \frac{|y(\frac{1}{4})|}{50 + |y|}, \forall y \in \mathcal{Y}.$$

Then for $y, \bar{y} \in \mathcal{Y}$, we have

$$\left| I_j(y(\frac{1}{4})) - I_j(\bar{y}(\frac{1}{4})) \right| = \left| \frac{|y(\frac{1}{4})|}{50 + |y|} - \frac{|\bar{y}(\frac{1}{4})|}{50 + |\bar{y}|} \right| \leq \frac{1}{50}|y - \bar{y}|.$$

Hence, with $A_{I_j} = \frac{1}{50}$, the $[H_3]$ satisfies. Set

$$h(x) = \frac{\cos|x(t)|}{18 + e^t}.$$

Then for $x, \bar{x} \in \mathbb{R}$, we have

$$|h(x) - h(\bar{x})| = \left| \frac{\cos|x(t)|}{18 + e^t} - \frac{\cos|\bar{x}(t)|}{18 + e^t} \right| \leq \frac{1}{19}|x - \bar{x}|.$$

Hence, $[H_4]$ satisfies with $A_h = \frac{1}{19}$.

Similarly, Set

$$g(y) = \frac{\sin|y(t)|}{100 + e^{2t}}.$$

Then for $y, \bar{y} \in \mathbb{R}$, we have

$$|g(y) - g(\bar{y})| = \left| \frac{\sin|y(t)|}{100 + e^{2t}} - \frac{\sin|\bar{y}(t)|}{100 + e^{2t}} \right| \leq \frac{1}{101}|y - \bar{y}|.$$

Hence, $[H_4]$ satisfies with $A_g = \frac{1}{101}$.

Also

$$\Delta_1 = \left[\left(\frac{(1+m)T^r}{\Gamma(r+1)} \right) \left(\frac{M_1}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{M_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}}{1 - N_1 - N_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} \right) + mA_{I_i} + A_h \right] \approx 0.08138 < 1$$

satisfies with $m = 1, T = 1, \sigma = \delta = \frac{5}{2}, r = \frac{1}{2}, M_1 = N_1 = M_2 = N_2 = \frac{1}{104e^5}, A_{I_i} = \frac{1}{35}, A_h = \frac{1}{19}$ and

$$\Delta_2 = \left[\left(\frac{(1+n)T^s}{\Gamma(s+1)} \right) \left(\frac{M'_1}{1 - N'_1 - N'_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} + \frac{M'_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}}{1 - N'_1 - N'_2 \frac{T^\sigma}{\sigma\Gamma(\delta)}} \right) + nA_{I_j} + A_g \right] \approx 0.03559 < 1$$

satisfies with $n = 1$, $T = 1$, $\sigma = \delta = \frac{5}{2}$, $s = \frac{1}{2}$, $M'_1 = N'_1 = M'_2 = N'_2 = \frac{1}{70e^2}$, $A_{I_j} = \frac{1}{50}$, $A_g = \frac{1}{101}$. Hence $\Delta = \max(\Delta_1, \Delta_2) < 1$ satisfies.

Therefore, by Theorem 3.8, the problem (5.2) has a unique solution. It is easy to check that

$$\Delta = 1 - Q_\tau Q_s \approx 1.00000 > 0$$

and (4.21) is verified. We conclude that problem (5.2) is HU stable, generalized HU stable, HU–Rassias stable and generalized HU–Rassias stable.

Conclusion

We have presented some existence and uniqueness results for an impulsive initial value problem of coupled fractional integrodifferential systems involving the Caputo type fractional derivative. The proof of the existence results is based on the nonlinear alternative of Schaefer's and Krasnoselskii's fixed point theorem, while the uniqueness of the solution is proved by applying the Banach contraction principle. We have also given the notion of Hyers–Ulam stability for our problem and have given sufficient conditions for EUS and Hyers–Ulam stability. This work provides a base to the study of EUS and different sorts of stabilities for the fractional integrodifferential equations with impulsive initial condition.

Acknowledgments: The third author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17. The research of the fourth author was supported by the Natural Science Foundation of Jiangxi Province (grant:20192BAB201011) and by the National Natural Science Foundation of China (grant:11861053).

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