



## Research Article

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# Scalar linear impulsive Riemann-Liouville fractional differential equations with constant delay-explicit solutions and finite time stability

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**Abstract:** Riemann-Liouville fractional differential equations with a constant delay and impulses are studied in this article. The following two cases are considered: the case when the lower limit of the fractional derivative is fixed on the whole interval of consideration and the case when the lower limit of the fractional derivative is changed at any point of impulse. The initial conditions as well as impulsive conditions are defined in an appropriate way for both cases. The explicit solutions are obtained and applied to the study of finite time stability.

**Keywords:** Riemann-Liouville fractional derivative, constant delay, impulses, finite time stability

**MSC 2010:** 34A08, 34A37

## 1 Introduction

Fractional differential equations have been applied as more adequate models of real-world problems in engineering, physics, finance, etc. ([1,2]). One of the main qualitative problems is connected with finite time stability (FTS). FTS of linear fractional delay differential equations with controls was studied with the help of an inequality of Gronwall type in [3–5]. For other related contribution, one can refer to [6–8].

The question about Riemann-Liouville (RL) fractional differential equations is still at the initial stage of investigations (see, e.g., [9–11]). Li and Wang introduced the concept of a delayed Mittag-Leffler-type matrix function, and then they presented the finite-time stability results by virtue of a delayed Mittag-Leffler-type matrix in [12–14]. They study the case when the lower limit of the RL fractional derivative coincides with the left side end of the initial interval. It is not only different than the idea of the initial value problem (IVP) for delay equations but also it requires strong conditions for the initial function.

In this article, we study IVPs of systems of RL fractional differential equations with a constant delay and impulses at the fixed initially given points  $0 = t_0 < t_1 < t_2 < \dots < t_N < t_{N+1} = T$ . We study the following two cases: when the lower limit of the fractional derivative is fixed on the whole interval of consideration, i.e.,

$${}^{\text{RL}}D_t^q x(t) = Ax(t) + Bx(t - \tau) + F(t, x(t)) \quad \text{for } t \in \bigcup_{k=0}^N (t_k, t_{k+1}], \quad (1)$$

and the case when the lower limit of the fractional derivative is changed at any point of impulse, i.e.,

$${}^{\text{RL}}D_t^q x(t) = Ax(t) + Bx(t - \tau) + F(t, x(t)) \quad \text{for } t \in (t_k, t_{k+1}], k = 0, 1, \dots, N, \quad (2)$$

where  $A, B$  are constants,  $\tau > 0$  is a constant delay and  $T < \infty$ .

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Similar to the case of the ordinary derivative, the differential equation is given to the right of the initial time interval. It requires the lower bound of the RL fractional derivative to coincide with the right side end of the initial interval (usually this point is zero). Note that in this case any solution of an IVP with RL fractional derivatives is not continuous at the initial point. That is why RL fractional delay differential equations are convenient for the modeling process with impulsive types of initial conditions. This type of process can be found in physics, chemistry, engineering, biology, and economics. To determine the law of the initial impulsive reaction we need to add to the usual initial condition (e.g.,  $x(t) = \phi(t)$  on the initial interval  $[-\tau, 0]$ ,  $\tau > 0$  is the delay) a fractional condition. This conclusion is based on the results obtained in [1] and [8] concerning the physical interpretation of the RL fractional derivatives and initial conditions which include derivatives of the same kind. Based on the above, we set up appropriate IVPs for RL linear fractional differential equations with a lower limit of the RL derivative equal to the right side point of the initial interval, i.e., we study the initial conditions of the type

$$\begin{aligned} x(t) &= g(t) \quad \text{for } t \in [-\tau, 0], \\ {}_0 I_t^{1-q} x(t)|_{t=0} &= \lim_{t \rightarrow 0^+} \frac{1}{\Gamma(1-q)} \int_0^t \frac{x(s)}{(t-s)^q} ds = g(0). \end{aligned} \quad (3)$$

Similarly, the impulsive conditions are given by

$${}_{t_k} I_t^{1-q} x(t)|_{t=t_k} = D_k x(t_k - 0), \quad k = 1, 2, \dots, N, \quad (4)$$

where  $D_k$ ,  $k = 1, 2, \dots, N$  are constants.

In this article, we study scalar linear RL fractional differential equations with a constant delay and impulses. The main contributions of the study are as follows:

1. Two types of fractional equations are studied:
  - the equation in which the lower bound of the fractional derivative is fixed at the initial time point;
  - the equation in which the lower limit of the fractional derivative is changing at each point of impulse.
2. The impulsive conditions are set up in both the aforementioned cases. It is connected with the presence of RL fractional derivative and the delay.
3. Explicit formulas of the solutions are obtained in both the aforementioned cases.
4. The obtained explicit formulas for the solutions are applied to study the FTS.

The rest of this article is organized as follows. In Section 2, we outline some basic notations and results from fractional calculus. In Section 3, the main two types of the interpretation of the presence in impulses in RL fractional differential equations are presented. In Section 4, explicit solutions of IVPs (1), (3) and (4) as well as of IVPs (2), (3) and (4) are given. In Section 5, the formulas for the exact solutions are applied to the study of the FTS of both IVPs.

## 2 Preliminary notes on fractional derivatives and equations

Let  $0 \leq t_0 < T < \infty$ . In this article, we will use the following definitions for fractional derivatives and integrals:

– *RL fractional integral* of order  $q \in (0, 1)$  [15,16]

$${}_{t_0} I_t^q m(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t \frac{m(s)}{(t-s)^{1-q}} ds, \quad t \in [t_0, T],$$

where  $\Gamma(\cdot)$  is the Gamma function when the integral exists.

This is called by some authors the left RL fractional integral of order  $q$  (because we integrate to  $t$  from the left).

Note sometimes the notation  ${}_{t_0} D_t^{-q} m(t) = {}_{t_0} I_t^q m(t)$  is used.

– RL fractional derivative of order  $q \in (0, 1)$  [15,16]

$${}^{\text{RL}}D_t^q m(t) = \frac{d}{dt} ({}_{t_0}I_t^{1-q} m(t)) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^t (t-s)^{-q} m(s) ds, \quad t \in [t_0, T], \tag{5}$$

where  $m(t)$  is measurable on  $[t_0, T]$ .

This is also called the left RL fractional derivative. We will call the point  $t_0$  a lower limit of the RL fractional derivative.

We will give fractional integrals and RL fractional derivatives of some elementary functions which will be used later.

**Proposition 1.** For  $t > t_0$  and  $\beta > 0$  the following equalities are true:

$$\begin{aligned} {}^{\text{RL}}D_t^q (t - t_0)^\beta &= \frac{\Gamma(1 + \beta)}{\Gamma(1 + \beta - q)} (t - t_0)^{\beta-q}, \\ {}_{t_0}I_t^{1-q} (t - t_0)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(1 + \beta - q)} (t - t_0)^{\beta-q}, \\ {}_{t_0}I_t^{1-q} (t - t_0)^{q-1} &= \Gamma(q), \\ {}^{\text{RL}}D_t^q (t - t_0)^{q-1} &= 0. \end{aligned}$$

The definitions of the initial condition for fractional differential equations with RL-derivatives are based on the following result.

**Lemma 1.** [17, Lemma 3.2] Let  $q \in (0, 1)$ ,  $0 \leq t_0 < T \leq \infty$  and  $m(t)$  be a Lebesgue measurable function on  $[t_0, T]$ .

(a) If there exists a.e. a limit  $\lim_{t \rightarrow t_0+} [(t - t_0)^{q-1} m(t)] = c$ , then there also exists a limit

$${}_{t_0}I_t^{1-q} m(t)|_{t=t_0} := \lim_{t \rightarrow t_0+} {}_{t_0}I_t^{1-q} m(t) = c\Gamma(q).$$

(b) If there exists a.e. a limit  ${}_{t_0}I_t^{1-q} m(t)|_{t=t_0} = b$  and if there exists the limit  $\lim_{t \rightarrow t_0+} [(t - t_0)^{1-q} m(t)]$ , then

$$\lim_{t \rightarrow t_0+} [(t - t_0)^{1-q} m(t)] = \frac{b}{\Gamma(q)}.$$

Let  $0 \leq a < T \leq \infty$  and consider the scalar RL fractional differential equation:

$${}^{\text{RL}}D_t^q x(t) = F(t, x(t)), \quad t \in (a, T]. \tag{6}$$

Note that according to Lemma 1 and [17] the initial conditions to (6) could be one of the following forms:

– integral form (see (3.1.6) [17])

$${}_a I_t^{1-q} x(t)|_{t=a} = B; \tag{7}$$

– weighted Cauchy-type problem (see (3.1.7) [17])

$$\lim_{t \rightarrow a+} ((t - a)^{1-q} x(t)) = C. \tag{8}$$

**Remark 1.** According to Lemma 1, if the function  $x(t)$  satisfies the initial conditions (8), then  $x(t)$  also satisfies condition (7) with  $B = C\Gamma(q)$ .

**Remark 2.** According to Lemma 1, it is enough to study one of the initial conditions (7) or (8). Following this result we will study only the initial condition of type (7).

Let  $E_{p,q}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jp + q)}$  be the Mittag-Leffler function with two parameters (see, e.g., [16]). In the case of a scalar linear RL fractional differential equation, we have the following result.

**Proposition 2.** [17, Example 4.1] *The solution of the Cauchy-type problem*

$${}^{\text{RL}}D_t^q x(t) = \lambda x(t) + f(t), \quad {}_a I_t^{1-q} x(t)|_{t=a} = b$$

*has the following form (formula 4.1.14 [17])*

$$x(t) = \frac{b}{(t-a)^{1-q}} E_{q,q}(\lambda(t-a)^q) + \int_a^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s) ds. \quad (9)$$

### 3 Interpretations of the impulses in the RL fractional equations

Let an increasing sequence of non-negative points  $\{t_i\}_{i=0}^{N+1}$  be given with  $t_0 = 0$ ,  $T = t_{N+1}$ .

**Remark 3.** The points  $t_k$ ,  $k = 1, 2, \dots, N$ , are called points of impulses.

The interpretation of the impulse in differential equations at a point  $\tau$  is that there is an instantaneous jump of the solution  $x(t)$ , which is determined by the value  $x(\tau + 0) = \lim_{\varepsilon \rightarrow 0^+} x(\tau + \varepsilon)$  depending significantly on the value of the solution  $x(\tau - 0) = \lim_{\varepsilon \rightarrow 0^+} x(\tau - \varepsilon)$  before the jump. The presence of the RL fractional derivative in the differential equation and the inequality  ${}_a I_b^q m(b) \neq {}_a I_c^q m(c) + {}_c I_b^q m(b)$  with  $a < c < b < \infty$  (which is not true for the ordinary case  $q = 1$ ) leads to two basic interpretations of the solution (see, e.g., [10,18]):

- *Fixed lower limit of the RL fractional derivative* – in this case the lower limit of the fractional derivative is kept equal to the initial time  $t_0$  on the whole interval of consideration. At points of impulses the amount of jump is taken into account.
- *Changed lower limit of the RL fractional derivative at each time of impulses* – this is based on the fact that the value of the solution is changed at each impulsive point and it is determined by the differential equation on each interval between two consecutive impulsive points. In this case, the impulsive time is considered as an initial time of the fractional differential equation. Then, the lower limit of the RL fractional derivative, being equal to the initial time, is changed at each impulsive time.

For some explanations about the presence of impulses in the fractional differential equation without any delays and Caputo fractional derivative we refer to [18,19]. In the case of the RL fractional derivative, impulses and no delays, a discussion about the interpretation of the solutions is given in [10].

Define the set  $PL_1^{\text{loc}}([0, T], \mathbb{R}) = \{u : [0, T] \rightarrow \mathbb{R} : u \in L_1^{\text{loc}}((t_k, t_{k+1}], \mathbb{R}), k = 0, 1, \dots, N\}$  with  $u(t_k) = u(t_k - 0) = \lim_{\varepsilon \rightarrow 0^+} u(t_k - \varepsilon)$ ,  $u(t_k + 0) = \lim_{\varepsilon \rightarrow 0^+} u(t_k + \varepsilon)$ .

## 4 Explicit formulas for the solutions

In this section, we will study the case  $F(t, x) \equiv F(t)$ .

### 4.1 Fixed lower limit of the RL fractional derivative at the initial time

Consider the IVP for the scalar linear RL fractional differential equations with a fixed lower bound of the RL fractional derivative at the given initial time and impulses (1), (3) and (4), where  $q \in (0, 1)$ ,  $A, B$  are real constants,  $F \in C([0, T], \mathbb{R})$ ,  $g : [-\tau, 0] \rightarrow \mathbb{R}$  be an integrable function.

**Theorem 1.** *The IVP for the linear scalar RL fractional differential equation with impulses (1), (3) and (4) has an exact solution  $x \in \text{PL}_1^{\text{loc}}([t_0, T], \mathbb{R})$  given by*

$$x(t) = D_k x(t_k - 0) \frac{E_{q,q}(A(t - t_k)^q)}{(t - t_k)^{1-q}} + \int_{t_k}^t \frac{E_{q,q}(A(t - s)^q)}{(t - s)^{1-q}} \left( Bx(s - \tau) + F(s) + \sum_{j=1}^k h_j(s) \right) ds, \tag{10}$$

for  $t \in (t_k, t_{k+1}]$ ,  $k = 0, 1, 2, \dots, N$ ,

where  $D_0 = 1$ ,  $t_0 = 0$ ,

$$h_k(t) = \frac{q}{\Gamma(1 - q)} \int_{t_{k-1}}^{t_k} \{ D_{k-1} x(t_{k-1} - 0) \frac{E_{q,q}(A(s - t_{k-1})^q)}{(s - t_{k-1})^{1-q}(t - s)^{1+q}} + \int_{t_{k-1}}^s \frac{E_{q,q}(A(s - \xi)^q)}{(s - \xi)^{1-q}(t - s)^{1+q}} \left[ Bx(\xi - \tau) + F(\xi) + \sum_{j=1}^{k-1} h_j(\xi) \right] d\xi \} ds, \tag{11}$$

$t \in (t_k, T]$ ,  $k = 1, 2, \dots, N$ .

**Proof.** We apply induction to prove the claim. Let  $t \in (0, t_1]$ . Then, from Proposition 2 and Eq. (1) we have

$$x(t) = g(0) \frac{E_{q,q}(At^q)}{t^{1-q}} + \int_0^t \frac{E_{q,q}(A(t - s)^q)}{(t - s)^{1-q}} (Bx(s - \tau) + F(s)) ds, \tag{12}$$

i.e., equality (10) holds for  $k = 0$ .

Let  $t \in (t_1, t_2]$ . From definition (5) of the RL fractional derivative and Eq. (12) we get

$$\begin{aligned} {}^{\text{RL}}D_t^q x(t) &= \frac{1}{\Gamma(1 - q)} \frac{d}{dt} \int_0^{t_1} (t - s)^{-q} x(s) ds + \frac{1}{\Gamma(1 - q)} \frac{d}{dt} \int_{t_1}^t (t - s)^{-q} x(s) ds \\ &= \frac{1}{\Gamma(1 - q)} \frac{d}{dt} \int_0^{t_1} (t - s)^{-q} \left( g(0) \frac{E_{q,q}(As^q)}{s^{1-q}} + \int_0^s \frac{E_{q,q}(A(s - \xi)^q)}{(s - \xi)^{1-q}} F(\xi) d\xi \right. \\ &\quad \left. + B \int_0^s \frac{E_{q,q}(A(s - \xi)^q)}{(s - \xi)^{1-q}} x(\xi - \tau) d\xi \right) ds + \frac{1}{\Gamma(1 - q)} \frac{d}{dt} \int_{t_1}^t (t - s)^{-q} x(s) ds \\ &= - \frac{q}{\Gamma(1 - q)} \int_0^{t_1} \left\{ g(0) \frac{E_{q,q}(As^q)}{(t - s)^{1+q} s^{1-q}} \right. \\ &\quad \left. + \int_0^s \frac{E_{q,q}(A(s - \xi)^q)}{(t - s)^{1+q} (s - \xi)^{1-q}} [F(\xi) + Bx(\xi - \tau)] d\xi \right\} ds + {}^{\text{RL}}D_{t_1}^q x(t) \\ &= - h_1(t) + {}^{\text{RL}}D_{t_1}^q x(t). \end{aligned} \tag{13}$$

From Eqs. (1), (13) and (3), we get the following IVP

$${}^{\text{RL}}D_t^q x(t) = Ax(t) + Bx(t - \tau) + F(t) + h_1(t), \quad t \in (t_1, t_2], \quad {}^{\text{RL}}D_{t_1}^{1-q} x(t)|_{t=t_1} = D_1 x(t_1 - 0). \tag{14}$$

According to Proposition 2 and Eq. (14) for  $a = t_1$ ,  $f(t) = Bx(t - \tau) + F(t) + h_1(t)$  for  $t \in [t_1, t_2]$ , and  $b = B_1$  we obtain

$$x(t) = D_1 x(t_1 - 0) \frac{E_{q,q}(A(t - t_1)^q)}{(t - t_1)^{1-q}} + \int_{t_1}^t \frac{E_{q,q}(A(t - s)^q)}{(t - s)^{1-q}} (F(s) + Bx(s - \tau) + h_1(s)) ds, \quad t \in (t_1, t_2],$$

i.e., Eq. (10) holds for  $k = 1$ .

Continuing this process to the intervals  $(t_k, t_{k+1}]$ ,  $k = 2, \dots, T$  we prove the claim. □

## 4.2 Changed lower limit of the RL fractional derivative at each time of impulses

Consider the IVP for the scalar linear RL fractional differential Eqs. (2), (3) and (4), where  $q \in (0, 1)$ ,  $A, B$  are real constants,  $f \in C([0, T], \mathbb{R})$ ,  $g : [-\tau, 0] \rightarrow \mathbb{R}$  be an integrable function.

**Theorem 2.** *The IVP for the linear scalar RL fractional differential equation with impulses (1), (3) and (4) has an exact solution  $x \in PL_1^{\text{loc}}([t_0, T], \mathbb{R})$  given by*

$$x(t) = D_k x(t_k - 0) \frac{E_{q,q}(A(t - t_k)^q)}{(t - t_k)^{1-q}} + \int_{t_k}^t \frac{E_{q,q}(A(t - s)^q)}{(t - s)^{1-q}} (Bx(s - \tau) + F(s)) ds, \quad (15)$$

for  $t \in (t_k, t_{k+1}]$ ,  $k = 0, 1, 2, \dots, N$ ,

where  $D_0 = 1$ ,  $t_0 = 0$ .

**Proof.** The proof follows from induction and Proposition 2 applied to each interval  $(t_k, t_{k+1}]$ ,  $k = 0, 1, 2, \dots, N$ .  $\square$

## 5 Finite time stability

In this section, we use the obtained above exact solutions to study the FTS of the IVP for linear RL fractional differential equations with a constant delay and impulses.

In our further considerations, we will assume that  $F(t, 0) \equiv 0$ , i.e., the linear RL fractional differential equations with zero initial condition  $g(t) \equiv 0$ , will have a zero solution.

Note that because of the singularities of  $t^{q-1}$  at 0 and  $(t - t_k)^{q-1}$  at  $t_k$  we could prove the FTS on intervals which do not contain the initial time 0 as well as the impulsive points  $t_k$ .

**Definition 1.** The zero solution of IVPs (1), (3) and (4), respectively, and IVPs (2), (3) and (4) is a finite time stable if there exists a positive number  $\Lambda$  such that for any real positive number  $\delta$  and  $\varepsilon \in (0, \Lambda]$  there exists a positive number  $K$  depending on  $\delta$  and  $\varepsilon$  such that the inequality  $\max_{t \in [-\tau, 0]} |g(t)| < \delta$  implies  $|x(t)| < K$  for  $t \in \bigcup_{k=0}^N [t_k + \varepsilon, t_{k+1}]$ , where  $x(t)$  is the corresponding solution of IVPs (1), (3) and (4), respectively, and IVPs (2), (3) and (4).

**Remark 4.** Let  $x^*(t)$  is a nontrivial solution of IVPs (1), (3) and (4). Then, if we substitute  $y = x - x^*(t)$  we get

$${}^{\text{RL}}D_t^q y(t) = Ay(t) + By(t - \tau) + G(t, y) \quad \text{for } t \in \bigcup_{k=0}^N (t_k, t_{k+1}], \quad (16)$$

$$y(t) = 0 \quad \text{for } t \in [-\tau, 0], \quad {}_0I_t^{1-q} y(t)|_{t=0} = 0, \quad (17)$$

$${}_{t_k}I_t^{1-q} y(t)|_{t=t_k} = D_k y(t_k - 0), \quad k = 1, 2, \dots, N, \quad (18)$$

where  $G(t, y) = F(t, y + x^*(t)) - F(t, x^*(t))$  and  $G(t, 0) \equiv 0$ .

Then, the FTS of zero solution of IVP (16)–(18) is equivalent to the FTS of the solution  $x^*(t)$  of IVPs (1), (3) and (4).

### 5.1 Fixed lower limit of the RL fractional derivative

We will obtain sufficient conditions for FTS of the RL fractional differential Eq. (1) with a constant delay and impulses.

**Theorem 3.** *Let the following conditions be satisfied:*

1. *The function  $F \in C([0, T] \times \mathbb{R}, \mathbb{R})$  is bounded, i.e.,  $\sup_{(t,x) \in [0, T] \times \mathbb{R}} |F(t, x)| \leq K$ .*
2. *The function  $g \in C([-\tau, 0], \mathbb{R})$ ,  $|g(0)| < \infty$ .*
3.  *$t_{k+1} - t_k \leq \tau$  for  $k = 0, 1, \dots, N$ .*

*Then, the zero solution of IVPs (1), (3) and (4) is a finite time stable.*

**Proof.** Let  $\delta$  be arbitrary positive number and the initial function  $g : \max_{t \in [-\tau, 0]} |g(t)| < \delta$ . According to Theorem 1, IVPs (1), (3) and (4) have a solution  $x(t)$  given by (10).

Denote  $M = \sup_{t \in [0, T]} |E_{q,q}(At^q)|$ .

Choose a positive number  $\Lambda : \Lambda < \min\{\min_{i=1,2, \dots, N} \sqrt[q]{|D_i| M}, \min_{k=0,1,2, \dots, N} (t_{k+1} - t_k)\}$ . Let  $\varepsilon \in (0, \Lambda)$  be an arbitrary number.

Let  $t \in [\varepsilon, t_1]$ . Then, according to Theorem 1 and formula (10) we have

$$\begin{aligned}
 |x(t)| &\leq \left| \delta \frac{E_{q,q}(At^q)}{t^{1-q}} + \int_0^t \frac{E_{q,q}(A(t-s)^q)}{(t-s)^{1-q}} (|B| |g(s-\tau)| + |F(s, x(s))|) ds \right| \\
 &\leq \delta \frac{M}{\varepsilon^{1-q}} + \int_0^t \frac{M}{(t-s)^{1-q}} (|B|\delta + |F(s)|) ds = \frac{M\delta}{\varepsilon^{1-q}} + \frac{\tau^q}{q} (|B|\delta + K) = P_0.
 \end{aligned}
 \tag{19}$$

Let  $t \in [t_1 + \varepsilon, t_2]$ . Then, according to Theorem 1, formulas (10), (11), inequality (19) and  $t - s > \varepsilon$  for  $s \in [0, t_1]$  we have

$$|h_1(t)| \leq \frac{M\tau^q}{\Gamma(1-q)\varepsilon^{1+q}} \delta + \frac{M\tau^{1+q}}{\Gamma(1-q)\varepsilon^{1+q}(1+q)} (B\delta + K) = Q_1$$

and

$$\begin{aligned}
 |x(t)| &= \left| D_1 x(t_1 - 0) \frac{E_{q,q}(A(t-t_1)^q)}{(t-t_1)^{1-q}} + \int_{t_1}^t \frac{E_{q,q}(A(t-s)^q)}{(t-s)^{1-q}} (Bx(s-\tau) + F(s, x(s)) + h_1(s)) ds \right| \\
 &\leq |D_1| P_0 \frac{M}{\varepsilon^{1-q}} + \frac{M\tau^q}{q} [|B|P_0 + K + Q_1] \\
 &= P_0 \left( \frac{|D_1|M}{\varepsilon^{1-q}} + \frac{|B|M\tau^q}{q} \right) + \frac{M\tau^q}{q} [K + Q_1] = P_1, \quad t \in [t_1 + \varepsilon, t_2].
 \end{aligned}
 \tag{20}$$

From the choice of the numbers  $\Lambda, \varepsilon$  it follows that  $D_1 M > \Lambda^{1-q} > \varepsilon^{1-q}$ ,  $\frac{|D_1|M}{\varepsilon^{1-q}} > 1$ ,  $\frac{|D_1|M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} > 1$  and

$$P_1 > P_0 \left( |D_1| \frac{M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} \right) > P_0.$$

Let  $t \in [t_2 + \varepsilon, t_3]$ . Then, apply the inequality  $t - s > \varepsilon$  for  $s \in [t_1, t_2]$  we obtain

$$\begin{aligned}
 |h_2(t)| &= \left| \frac{q}{\Gamma(1-q)} \int_{t_1}^{t_2} \left\{ D_1 x(t_1 - 0) \frac{E_{q,q}(A(s-t_{k-1})^q)}{(s-t_1)^{1-q}(t-s)^{1+q}} \right. \right. \\
 &\quad \left. \left. + \int_{t_1}^s \frac{E_{q,q}(A(s-\xi)^q)}{(s-\xi)^{1-q}(t-s)^{1+q}} (Bx(\xi-\tau) + F(\xi, x(\xi)) + h_1(\xi)) d\xi \right\} ds \right| \\
 &\leq \frac{M\tau^q}{\Gamma(1-q)\varepsilon^{1+q}} |D_1| P_0 + \frac{M\tau^{1+q}}{(1+q)\Gamma(1-q)\varepsilon^{1+q}} (|B|P_0 + K + Q_1) = Q_2
 \end{aligned}
 \tag{21}$$

and

$$\begin{aligned}
 |x(t)| &= \left| D_2 x(t_2 - 0) \frac{E_{q,q}(A(t - t_k)^q)}{(t - t_2)^{1-q}} \right. \\
 &\quad \left. + \int_{t_2}^t \frac{E_{q,q}(A(t - s)^q)}{(t - s)^{1-q}} (Bx(s - \tau) + F(s, x(s)) + h_1(s) + h_2(s)) ds \right| \\
 &\leq |D_2| P_1 \frac{M}{\varepsilon^{1-q}} + \frac{M\tau^q}{q} (|B|P_1 + K + Q_1 + Q_2) \\
 &= P_1 \left( \frac{|D_2|M}{\varepsilon^{1-q}} + \frac{|B|M\tau^q}{q} \right) + \frac{M\tau^q}{q} (K + Q_1 + Q_2) = P_2, \quad t \in [t_2 + \varepsilon, t_3].
 \end{aligned} \tag{22}$$

From the choice of the numbers  $\Lambda, \varepsilon$  it follows that  $|D_2|M > \Lambda^{1-q} > \varepsilon^{1-q}$ ,  $\frac{|D_2|M}{\varepsilon^{1-q}} > 1$ ,  $\frac{|D_2|M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} > 1$  and

$$P_2 > P_1 \left( |D_1| \frac{M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} \right) > P_1.$$

Continue this process and obtain that

$$|x(t)| \leq P_N \quad \text{for } t \in \bigcup_{k=0}^N [t_k + \varepsilon, t_{k+1}],$$

where  $P_N$  is defined recursively by

$$P_N = P_{N-1} \left( \frac{|D_N|M}{\varepsilon^{1-q}} + \frac{|B|M\tau^q}{q} \right) + \frac{M\tau^q}{q} \left( K + \sum_{j=1}^N Q_j \right), \quad k = 1, 2, \dots, N$$

and  $P_N = \max_{i=1,2, \dots, N} P_i$ . □

### 5.2 Changed lower limit of the RL fractional derivative

We will obtain sufficient conditions for FTS of the RL fractional differential Eq. (2) with a constant delay and impulses.

**Theorem 4.** *Let the conditions of Theorem 3 be satisfied.*

*Then, the zero solution of IVPs (2), (3) and (4) is a finite time stable.*

**Proof.** Let  $\delta$  be an arbitrary positive number and the initial function  $g : \max_{t \in [-gt, 0]} |g(t)| < \delta$ . According to Theorem 2, IVPs (2), (3) and (4) have a solution  $x(t)$  given by (15).

Denote  $M = \sup_{t \in [0, T]} |E_{q,q}(At^q)|$ . Choose  $\varepsilon : \varepsilon < \min \{ \min_{i=1,2, \dots, N} \frac{1}{\sqrt[q]{|D_i|M}}, \tau \}$ .

Let  $t \in [\varepsilon, t_1]$ . Then, according to the choice of the initial function and the condition of Theorem 4 we have

$$|x(t)| \leq \delta \frac{M}{\varepsilon^{1-q}} + \int_0^t \frac{M}{(t-s)^{1-q}} (|B|\delta + K) ds \leq \delta \frac{M}{\varepsilon^{1-q}} + \frac{M\tau^q}{q} (|B|\delta + K) = P_0. \tag{23}$$

Let  $t \in [t_1 + \varepsilon, t_2]$ . Then, according to formula (15), inequality (23) and  $s - \tau \in (t_1 - \tau, t_1]$  for  $s \in (t_1, t]$  we have

$$\begin{aligned}
 |x(t)| &\leq \left[ \delta \frac{M}{\varepsilon^{1-q}} + \frac{M\tau^q}{q} (|B|\delta + K) \right] \left( |D_1| \frac{M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} \right) + \frac{M\tau^q}{q} K \\
 &= P_0 \left( |D_1| \frac{M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} \right) + \frac{M\tau^q}{q} K = P_1.
 \end{aligned} \tag{24}$$

From the choice of the constant  $\varepsilon$  it follows that  $|D_1|M > \varepsilon^{1-q}$ ,  $\frac{|D_1|M}{\varepsilon^{1-q}} > 1$ ,  $\frac{|D_1|M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} > 1$  and

$$P_1 = P_0 \left( |D_1| \frac{M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} \right) + \frac{M\tau^q}{q} K > P_0 \left( |D_1| \frac{M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} \right) \geq P_0.$$

Let  $t \in [t_1 + \varepsilon, t_2]$ . Then, according to formula (15), inequalities (23), (24) and  $s - \tau \in (t_1 - \tau, t_1]$  for  $s \in (t_1 - \tau, t_1]$  we have

$$\begin{aligned} |x(t)| &= \left| D_2 x(t_2 - 0) \frac{E_{q,q}(A(t - t_k)^q)}{\varepsilon^{1-q}} + \int_{t_2}^t \frac{E_{q,q}(A(t - s)^q)}{(t - s)^{1-q}} (Bx(s - \tau) + F(s, x(s))) ds \right| \\ &\leq P_1 \left( |D_2| \frac{M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} \right) + \frac{M\tau^q}{q} K = P_2. \end{aligned} \quad (25)$$

From the choice of the constants  $\varepsilon$  it follows that  $|D_2|M > \varepsilon^{1-q}$ ,  $\frac{|D_2|M}{\varepsilon^{1-q}} > 1$ ,  $\frac{|D_2|M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} > 1$  and

$$P_2 > P_1 \left( |D_2| \frac{M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} \right) \geq P_1.$$

Continuing the process we obtain that

$$|x(t)| \leq P_N, \quad t \in \bigcup_{k=0}^N [t_k + \varepsilon, t_{k+1}], \quad (26)$$

where the constant  $P_N$  is defined recursively by

$$P_N = P_{N-1} \left( |D_N| \frac{M}{\varepsilon^{1-q}} + \frac{M|B|\tau^q}{q} \right) + \frac{M\tau^q}{q} K$$

and  $P_N = \max \{P_k, k = 1, 2, \dots, N\}$ . □

**Remark 5.** In the case  $\tau = 0$ , i.e., the case of scalar linear delay RL fractional differential equations without any delay, most of the obtained results are reduced to the ones in [9].

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