Research Article

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On θ-generalized demimetric mappings and monotone operators in Hadamard spaces

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Abstract: Our main interest in this article is to introduce and study the class of θ-generalized demimetric mappings in Hadamard spaces. Also, a Halpern-type proximal point algorithm comprising this class of mappings and resolvents of monotone operators is proposed, and we prove that it converges strongly to a fixed point of a θ-generalized demimetric mapping and a common zero of a finite family of monotone operators in a Hadamard space. Furthermore, we apply the obtained results to solve a finite family of convex minimization problems, variational inequality problems and convex feasibility problems in Hadamard spaces.

Keywords: θ-generalized demimetric mappings, demimetric mappings, monotone operators, monotone inclusion problems, Hadamard spaces

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1 Introduction

Let X be a metric space and C be a nonempty subset of X. We denote $F(T)$ to be the set of fixed points of a nonlinear mapping $T: C \to C$, that is, $F(T) = \{v \in C: v = Tv\}$. The mapping T is called:

(i) nonexpansive, if

$$d(Tu, Tv) \leq d(u, v) \quad \text{for all} \quad u, v \in C,$$

(ii) quasi-nonexpansive, if $F(T) \neq \emptyset$ and

$$d(Tu, v) \leq d(u, v) \quad \text{for all} \quad v \in F(T),\ u \in C,$$

(iii) k-strictly pseudocontractive, if there exists $k \in [0, 1)$ such that

$$d^2(Tu, Tv) \leq d^2(u, v) + k[d(u, Tv) + d(u, Tv)]^2 \quad \text{for all} \quad u, v \in C,$$

(iv) nonspreading, if

$$2d^2(Tu, Tv) \leq d^2(Tu, v) + d^2(Tv, u) \quad \text{for all} \quad u, v \in C,$$

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(v) hybrid, if

$$3d^2(Tu, Tv) \leq d^2(u, v) + d^2(Tu, v) + d^2(Tv, u) \quad \text{for all} \quad u, v \in C,$$

(vi) generalized hybrid, if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha d^2(Tu, Tv) + (1-\alpha)d^2(u, Tv) \leq \beta d^2(Tu, v) + (1-\beta)d^2(u, v) \quad \text{for all} \quad u, v \in C. \tag{1}$$

The classes of both nonexpansive mappings and nonspreading mappings with $F(T) \neq \emptyset$ are contained in the class of quasi-nonexpansive mappings.

Recently, Takahashi [1] introduced the class of k-demimetric mappings in a real Hilbert space, which he defined as follows.

Let C be a nonempty subset of a real Hilbert space H. A mapping $T: C \to H$ is called k-demimetric, if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$ such that

$$\langle u - v, u - Tu \rangle \geq \frac{1-k}{2}\|u - Tu\|^2 \quad \text{for all} \quad u \in C, v \in F(T).$$

Demimetric mappings are of central importance in optimization since they contain many common types of operators emanating from optimization. For instance, the class of k-demimetric mappings with $k \in (-\infty, 1)$ is known to cover the class of θ-generalized hybrid mappings, the metric projections and the resolvents of maximal monotone operators (which are known as useful tools for solving optimization problems) in Hilbert spaces (see [1,2] and references therein). Thus, many authors have studied this class of mappings in both Hilbert and Banach spaces (see [1,3–5]). This was recently extended to Hadamard spaces by Aremu et al. [2]. They defined demimetric mappings in a Hadamard space as follows: let C be a nonempty subset of a CAT(0) space X. A mapping $T: C \to X$ is called k-demimetric if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$ such that

$$\langle \overrightarrow{uv}, \overrightarrow{uTu} \rangle \geq \frac{1-k}{2}d^2(u, Tu) \tag{2}$$

for all $u \in C$, $v \in F(T)$.

Furthermore, they gave an example of a demimetric mapping and established some basic results for this class of mappings. Moreover, they proved a strong convergence theorem for approximating a fixed point of demimetric mappings and a solution to a minimization problem in Hadamard spaces.

In 2018, Kawasaki and Takahashi [6] generalized the class of demimetric mappings as follows: let C be a nonempty subset of a smooth Banach space E and $\theta \neq 0$. A mapping $T: C \to E$ with $F(T) \neq \emptyset$ is said to be θ-generalized demimetric (see also [7]), if

$$\theta \langle u - v, J(u - Tu) \rangle \geq \|u - Tu\|^2 \tag{3}$$

for all $u \in C$ and $v \in F(T)$, where J is a duality mapping on E.

Fixed point problems of nonlinear mappings have been a very attractive area of research in nonlinear analysis that has enjoyed a prosperous development due to its extensive applications in diverse mathematical fields such as inverse problems, signal processing, game theory and fuzzy theory (see [8–20] and references therein). The pioneer work of this study in Hadamard spaces is due to Kirk [21,22]. Later, Dhompongsa and Panyanak [23], Khan and Abbas [24], Chang et al. [25] and other researchers (see [26–32]) continued to obtain interesting results in this direction.

On the other hand, monotone inclusion problem (MIP) is one of the most important problems in nonlinear and convex analyses because of its importance in optimization and other related mathematical problems. The MIP is defined as:

$$\text{Find } x \in D(A) \text{ such that } \mathbf{0} \in Ax, \tag{4}$$

where $A: X \to 2^{X^*}$ is the monotone and $D(A) = \{x \in X: A(x) \neq \emptyset\}$ is the domain of A. We denote the solution set of problem (4) by $A^{-1}(\mathbf{0})$, which is known to be closed and convex.

The interest in the study of MIPs stems from the fact that many optimization and related mathematical problems such as variational inequality problems (VIPs), minimization problems and convex feasibility problems can be posed as MIPs (see [10,31,33–36]). A well-known method for approximating solutions of MIPs is the proximal point algorithm (PPA), introduced by Martinet [37] in Hilbert spaces and studied by Rockafellar [38] in the same space as follows: let $\{x_n\}$ be given iteratively by

$$\begin{cases} x_0 \in H, \\ x_{n+1} = J_{\lambda_n}^A x_n, \quad n \geq 0, \end{cases} \tag{5}$$

where $J_{\lambda_n}^A = (I + \lambda_n A)^{-1}$ is the resolvent A and $\{\lambda_n\}$ is a sequence of positive real numbers. Rockafellar [38] established that Algorithm (5) converges weakly to a solution of MIP (4). Later, Bačák [8] studied the PPA in Hadamard spaces and established the Δ-convergence of it when A is the subdifferential of a convex, proper and lower semicontinuous function.

Recently, Khatibzadeh and Ranjbar [36] studied the PPA in Hadamard spaces when A is monotone. In 2017, Ranjbar and Khatibzadeh [31] studied both the Mann-type and Halpern-type PPA in Hadamard spaces for approximating solutions of MIP:

$$\begin{cases} x_0 \in X \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) J_{\lambda_n} x_n, \quad n \geq 0 \end{cases} \tag{6}$$

and

$$\begin{cases} x_0 \in X \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) J_{\lambda_n} x_n, \quad n \geq 0, \end{cases} \tag{7}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$. They obtained Δ-convergence result and strong convergence result using (6) and (7), respectively. Many other authors have also studied MIP in Hadamard spaces (see e.g. [10,34,39]).

Motivated by the aforementioned results and the current interest in this research direction, we introduce and study the class of θ-generalized demimetric mappings in Hadamard spaces. Also, a Halpern-type PPA comprising this class of mappings and resolvents of monotone operators is proposed, and we prove that it converges strongly to a fixed point of a θ-generalized demimetric mapping and a common zero of a finite family of monotone operators in a Hadamard space. Finally, we apply the obtained results to solve a finite family of convex minimization problems, VIPs and convex feasibility problems in Hadamard spaces.

2 Preliminaries

We now recall some basic and useful results that will be required for our study. We shall simply write X for a metric space (X, d).

Let X be a metric space and $x, y \in X$. A geodesic path joining u to v is an isometry $r: [0, d(u, v)] \to X$ such that $r(0) = u$, $r(d(u, v)) = v$ and $d(r(t), r(t')) = |t - t'|$ for all $t, t' \in [0, d(u, v)]$. The image of a geodesic path is called the geodesic segment, which is usually denoted as $[u, v]$ whenever it is unique.

A metric space X is called a geodesic space if every two points of X are joined by a geodesic. A geodesic metric space X is uniquely geodesic if every two points of X are joined by exactly one geodesic. Let $u, v \in X$ and $t \in [0, 1]$, then we represent $tu \oplus (1 - t)v$ as the unique point $w \in [u, v]$ joining u to v such that

$$d(u, w) = (1 - t)d(u, v) \text{ and } d(w, v) = td(u, v). \tag{8}$$

A geodesic triangle $\Delta(u_1, u_2, u_3)$ in a geodesic metric space X consists of three points u_1, u_2, u_3 in X (which are also called vertices of Δ) and a geodesic segment between each pair of these points (which are also called edges of Δ). For any geodesic triangle $\Delta(u_1, u_2, u_3)$, there is a comparison triangle $\bar{\Delta}(u_1, u_2, u_3) := \Delta(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d(u_i, u_j) = d_{\mathbb{R}^2}(\bar{u}_i, \bar{u}_j)$ for $i, j \in \{1, 2, 3\}$. Let Δ be a geodesic triangle in X and $\bar{\Delta}$ be its comparison triangle, then Δ is said to satisfy the CAT(0) inequality if for all points $u, v \in \Delta$ and $\bar{u}, \bar{v} \in \bar{\Delta}$, $d(u, v) \leq d_{\mathbb{R}^2}(\bar{u}, \bar{v})$. A geodesic space X is called a CAT(0) space if all geodesic triangles satisfy the CAT(0) inequality. Complete CAT(0) spaces are often referred to as Hadamard spaces. CAT(0) (Hadamard) spaces are examples of uniquely geodesic metric spaces. Examples of CAT(0) spaces can be found in [40].

Definition 2.1. [41] Let X be a CAT(0) space and denote the pair $(u, v) \in X \times X$ by $\overrightarrow{uv}$. Then, a mapping $\langle .,. \rangle: (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{uv}, \overrightarrow{wx} \rangle = \frac{1}{2}(d^2(u, x) + d^2(v, w) - d^2(u, w) - d^2(v, x)) \text{ for all } u, v, w, x \in X \tag{9}$$

is a quasilinearization mapping.

It is easy to verify that $\langle \overrightarrow{uv}, \overrightarrow{uv} \rangle = d^2(u, v)$, $\langle \overrightarrow{vu}, \overrightarrow{wx} \rangle = -\langle \overrightarrow{uv}, \overrightarrow{wx} \rangle$, $\langle \overrightarrow{uv}, \overrightarrow{wx} \rangle = \langle \overrightarrow{uy}, \overrightarrow{wx} \rangle + \langle \overrightarrow{yv}, \overrightarrow{wx} \rangle$ and $\langle \overrightarrow{uv}, \overrightarrow{wx} \rangle = \langle \overrightarrow{wx}, \overrightarrow{uv} \rangle$ for all $u, v, w, x, y \in X$.

Lim [42] introduced the notion of Δ-convergence in metric spaces, and Kirk and Panyanak [43] extended this concept to CAT(0) spaces. Recall that a bounded sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ (we write $\Delta - \lim_{n\to\infty} x_n = x$), where $A(\{x_n\}) := \{x \in X: r(x, \{x_n\}) = r(\{x_n\})\}$ is the asymptotic center of $\{x_n\}$, $r(\{x_n\}) := \inf\{x, \{x_n\}): x \in X\}$ is the asymptotic radius of $\{x_n\}$ and $r(\cdot, \{x_n\}): X \to [0, \infty)$ is a continuous functional defined by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$.

Kakavandi and Amini [44] introduced the concept of the dual space of a Hadamard space as follows: consider the map $\theta: \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by

$$\theta(t, u, v)(x) = t\langle \overrightarrow{uv}, \overrightarrow{ux} \rangle \quad (t \in \mathbb{R}, u, v, x \in X),$$

where $C(X, \mathbb{R})$ is the space of all continuous real valued functions on a Hadamard space X. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t, u, v), (s, w, y)) = L(\theta(t, u, v) - \theta(s, w, y)), \quad (t, s \in \mathbb{R}, u, v, w, y \in X).$$

The pseudometric space $(\mathbb{R} \times X \times X, D)$ is the subspace of the pseudometric space of all real valued Lipschitz functions $(\mathrm{Lip}(X, \mathbb{R}), L)$. We know from [44] that $D((t, u, v), (s, w, y)) = 0$ if and only if $t\langle \overrightarrow{uv}, \overrightarrow{xe} \rangle = s\langle \overrightarrow{wy}, \overrightarrow{xe} \rangle$ for all $x, e \in X$. Therefore, an equivalence relation on $\mathbb{R} \times X \times X$ is induced by D, where the equivalence class of (t, u, v) is defined by:

$$[\overrightarrow{tuv}] := \{\overrightarrow{swy}: D((t, u, v), (s, w, y)) = 0\}.$$

Thus, $X^* = \{[\overrightarrow{tuv}]: (t, u, v) \in \mathbb{R} \times X \times X\}$ endowed with $D([\overrightarrow{tuv}], [\overrightarrow{swy}]) := D((t, u, v), (s, w, y))$ is the dual space of X. In a real Hilbert space H, $t(v - u) \equiv [\overrightarrow{tuv}]$ for all $t \in \mathbb{R}$, $u, v \in H$ (see [44]). Note also that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{wx} \rangle = t\langle \overrightarrow{uv}, \overrightarrow{wx} \rangle, \quad (x^* = [\overrightarrow{tuv}] \in X^*, w, x \in X).$$

Using the same notation as in [45, p. 1649], we denote the zero element of X^* by $\mathbf{0}$. That is, the (unique) element of X^* such that the evaluation $\langle \mathbf{0}, \cdot \rangle$ vanishes for all elements in $X \times X$. In fact, $\mathbf{0} \equiv [\overrightarrow{txx}]$ for any $t \in \mathbb{R}$ and $x \in X$ (see [31,36]).

Let X^* be the dual space of a Hadamard space X. An operator $A: X \to 2^{X^*}$ is monotone if and only if

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geq 0, \quad \text{for all } x, y \in D(A), \quad x^* \in Ax, \quad y^* \in Ay.$$

A monotone operator A is called a maximal monotone operator if the graph $G(A)$ of A defined by

$$G(A) := \{(x, x^*) \in X \times X^*: x^* \in A(x)\}$$

is not properly contained in the graph of any other monotone operator. The resolvent of A of order $\lambda > 0$ is the mapping $J_\lambda^A: X \to 2^X$ defined by

$$J_\lambda^A := \left\{z \in X \middle| \left[\frac{1}{\lambda}\overrightarrow{zx}\right] \in Az\right\}. \tag{10}$$

The operator A satisfies the range condition if for every $\lambda > 0$, $D(J_\lambda^A) = X$.

Definition 2.2. Let C be a nonempty closed and convex subset of a Hadamard space X. A mapping $T: C \to C$ is said to be Δ-demiclosed, if for any bounded sequence $\{x_n\}$ in X such that $\Delta - \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x = Tx$.

Definition 2.3. Let C be a nonempty closed and convex subset of a CAT(0) space X. The metric projection is a mapping $P_C: X \to C$ which assigns to each $x \in X$, the unique point $P_C x \in C$ such that

$$d(x, P_C x) = \inf\{d(x, y): y \in C\}.$$

Recall that a mapping T is firmly nonexpansive [36], if

$$d^2(Tu, Tv) \leq \langle \overrightarrow{TuTv}, \overrightarrow{uv}\rangle \text{ for all } u, v \in X. \tag{11}$$

It is well known that firmly nonexpansive mappings are nonexpansive (see [10]).

Lemma 2.4. *Let X be a* CAT(0) *space, $u, v, w \in X$* and $t \in [0, 1]$. *Then,*
(a) $d(tu \oplus (1 - t)v, w) \leq td(u, w) + (1 - t)d(v, w)$ *(see* [23]).
(b) $d^2(tu \oplus (1 - t)v, w) \leq td^2(u, w) + (1 - t)d^2(v, w) - t(1 - t)d^2(u, v)$ *(see* [23]).
(c) $d^2(tu \oplus (1 - t)v, w) \leq t^2 d^2(u, w) + (1 - t)^2 d^2(v, w) + 2t(1 - t)\langle \overrightarrow{uw}, \overrightarrow{vw}\rangle$ *(see* [46]).

Lemma 2.5. [23] *Every bounded sequence in a Hadamard space always has a Δ-convergence subsequence.*

Lemma 2.6. [47] *Let X be a Hadamard space, $\{x_n\}$ be a sequence in X and $u \in X$. Then, $\{x_n\}$ Δ-converges to u if and only if* $\limsup_{n\to\infty} \langle \overrightarrow{ux_n}, \overrightarrow{uw}\rangle \leq 0$ *for all* $w \in C$.

Lemma 2.7. [48] *Let X be a Hadamard space and $T: X \to X$ be a nonexpansive mapping. Then, T is Δ-demiclosed.*

Lemma 2.8. [49] *Let $\{a_n\}$ be a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,$$

where $\{b_n\}$ is a sequence of real numbers bounded from above and $\{\alpha_n\} \subset [0, 1]$ satisfies $\sum_{n=1}^\infty \alpha_n = \infty$. Then, it holds that

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n.$$

Lemma 2.9. [36] *Let X be a CAT(0) space and J_λ^A be the resolvent of A with order λ. Then,*
(a) *For any $\lambda > 0$,* $R(J_\lambda^A) \subset D(A)$ and $F(J_\lambda^A) = A^{-1}(0)$, *where $R(J_\lambda^A)$ is the range of J_λ^A.*
(b) *If A is monotone, then J_λ^A is a single-valued and firmly nonexpansive mapping.*
(c) *If A is monotone and $0 < \lambda \leq \mu$, then $d^2(J_\lambda^A x, J_\mu^A x) \leq \frac{\mu - \lambda}{\mu + \lambda} d^2(x, J_\mu^A x)$, which implies that $d(x, J_\lambda^A x) \leq 2d(x, J_\mu^A x)$.*

Lemma 2.10. [34] *Let X be a CAT(0) space and $A: X \to 2^X$ be monotone. Then,*

$$d^2(u, J_\lambda^A v) + d^2(J_\lambda^A v, v) \leq d^2(u, v), \tag{12}$$

for all $u \in F(J_\lambda^A)$, $v \in X$ and $\lambda > 0$.

Remark 2.11. We observe that inequality (12) is a property of any firmly nonexpansive mapping. That is, if T is a firmly nonexpansive mapping, then from (9) and (11), we obtain

$$d^2(u, Tv) + d^2(Tv, v) \le d^2(u, v) \quad \text{for all} \;\; u \in F(T) \;\; \text{and} \;\; v \in X.$$

Lemma 2.12. [34] *Let $X^\star$ be the dual space of a Hadamard space X and $T\colon X \to X$ be a nonexpansive mapping for each $i = 1, 2, \ldots, N$. Let J_λ^i be the resolvent of monotone operators A_i of order $\lambda > 0$. Then,*

$$F(T \circ J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1) = F(T) \cap F(J_\lambda^N) \cap F(J_\lambda^{N-1}) \cap \cdots \cap F(J_\lambda^2) \cap F(J_\lambda^1).$$

Lemma 2.13. [50] *Let X be a Hadamard space, for any $t \in [0, 1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$, we have*

$$\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \le t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle.$$

3 Main results

Following the idea of (2) and (3), we give the following definition.

Definition 3.1. Let C be a nonempty subset of a CAT(0) space X and $T\colon C \subseteq X \to X$ be a nonlinear mapping. Then, T is called θ-generalized demimetric, if $F(T) \neq \emptyset$ and there exists $\theta \neq 0$ such that

$$\theta\langle \overrightarrow{uv}, \overrightarrow{uTu} \rangle \ge d^2(u, Tu), \tag{13}$$

for all $u \in C$ and $v \in F(T)$.

The following is an example of a θ-generalized demimetric mapping but not a k-demimetric mapping.

Example 3.2. Define $T\colon [0, 1] \to \mathbb{R}$ by $Tx = x + x^2$. Then, T is θ-generalized demimetric with $\theta = -1$ but not a k-demimetric mapping.

Proof. Observe that $F(T) = \{0\}$. Thus, for any $x \in [0, 1]$, we have

$$\langle x - 0, x - Tx \rangle = -\langle x, x^2 \rangle = -|x||x^2| \le -|x^2|^2 = -|x - Tx|^2,$$

which implies that $-\langle x - 0, x - Tx \rangle \ge |x - Tx|^2$. Therefore, T is a (-1)-generalized demimetric mapping.

To show that T is not a k-demimetric mapping, let $x = 1$. Then, we see that

$$\langle x - 0, x - Tx \rangle = \langle 1, -1 \rangle = -1. \tag{14}$$

On the other hand,

$$\frac{1-k}{2}|x - Tx|^2 = \frac{1-k}{2}|1|^2 = \frac{1-k}{2}. \tag{15}$$

Now, since $-1 < \frac{1-k}{2}$ for all $k \in (-\infty, 1)$, we obtain from (14) and (15) that $\langle x - 0, x - Tx \rangle < \frac{1-k}{2}|x - Tx|^2$ for all $k \in (-\infty, 1)$. Therefore, T is not a k-demimetric mapping. □

Remark 3.3. Other examples of θ-generalized demimetric mappings in CAT(0) spaces include the following.
1. If $T\colon X \to X$ is a k-strictly pseudocontractive mapping with $k \in [0, 1)$ and $F(T) \neq \emptyset$, then T is $\left(\frac{2}{1-k}\right)$-generalized demimetric mapping. The proof is similar to that in [2].

2. If $F(T) \neq \emptyset$ and $T\colon X \to X$ is a generalized hybrid mapping, then T is 2-generalized demimetric. Indeed, for $u \in F(T)$ and $v \in X$, we obtain from (1) that
$$d^2(u, Tv) \leq d^2(u, v). \tag{16}$$
Also, from (9), we have that
$$2\langle \overrightarrow{vu}, \overrightarrow{vTv} \rangle = d^2(v, Tv) + d^2(u, v) - d^2(u, Tv),$$
which implies from (16) that
$$2\langle \overrightarrow{vu}, \overrightarrow{vTv} \rangle \geq d^2(v, Tv) + d^2(u, v) - d^2(u, v) = d^2(v, Tv).$$
Hence, T is a 2-generalized demimetric mapping. Therefore, nonexpansive, nonspreading and hybrid mappings are examples of θ-generalized demimetric mappings.
3. If $T\colon X \to X$ is a k-demicontractive mapping, then T is a $\left(\frac{2}{1-k}\right)$-generalized demimetric.

We now study some properties of θ-generalized demimetric mappings in Hadamard spaces.

Proposition 3.4. *Let X be a Hadamard space and $T\colon X \to X$ be a θ-generalized demimetric mapping with $\theta > 0$. Then, T is a $\left(1 - \frac{2}{\theta}\right)$-demimetric.*

Proof. It follows from the definition of demimetric mapping and θ-generalized demimetric mapping. □

Proposition 3.5. *Let X be a Hadamard space and $T\colon X \to X$ be a θ-generalized demimetric mapping with $\theta \neq 0$. Then, $F(T)$ is closed and convex.*

Proof. We first show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$ converges to x^*. Then, from the definition of θ-generalized demimetric mappings, we have
$$\theta\langle \overrightarrow{x^*x_n}, \overrightarrow{x^*Tx^*} \rangle \geq d^2(x^*, Tx^*), \tag{17}$$
which implies from the Cauchy Schwartz inequality that
$$\theta d(x^*, x_n) d(x^*, Tx^*) \geq d^2(x^*, Tx^*).$$
Taking limits, we have that $0 \geq d^2(x^*, Tx^*)$, which implies that $x^* = Tx^*$.

Thus, $F(T)$ is closed. Next, we show that $F(T)$ is convex. For this, let $u, v \in F(T)$. Then, it suffices to show that $(tu \oplus (1-t)v) \in F(T)$, for $t \in [0,1]$. Let $w = tu \oplus (1-t)v$, $t \in [0,1]$, then we obtain from Lemma 2.13 that
$$\begin{aligned} d^2(w, Tw) &= \langle \overrightarrow{wTw}, \overrightarrow{wTw} \rangle \\ &= \langle \overrightarrow{(tu \oplus (1-t)v)Tw}, \overrightarrow{wTw} \rangle \\ &\leq t\langle \overrightarrow{uTw}, \overrightarrow{wTw} \rangle + (1-t)\langle \overrightarrow{vTw}, \overrightarrow{wTw} \rangle \\ &= t[\langle \overrightarrow{uw}, \overrightarrow{wTw} \rangle + \langle \overrightarrow{wTw}, \overrightarrow{wTw} \rangle] + (1-t)[\langle \overrightarrow{vw}, \overrightarrow{wTw} \rangle + \langle \overrightarrow{wTw}, \overrightarrow{wTw} \rangle] \\ &\leq \frac{-t}{\theta} d^2(w, Tw) + t d^2(w, Tw) - \frac{(1-t)}{\theta} d^2(w, Tw) + (1-t) d^2(w, Tw) \\ &= \frac{-1}{\theta} d^2(w, Tw) + d^2(w, Tw), \end{aligned}$$
which implies that $\frac{1}{\theta} d^2(w, Tw) \leq 0$. Since $\theta \neq 0$, we obtain that $w \in F(T)$ as required. □

Lemma 3.6. *Let X be a CAT(0) space and $T\colon X \to X$ be a θ-generalized demimetric mapping with $\theta \neq 0$. Suppose that $S_\lambda u = \lambda u \oplus (1-\lambda)Tu$ with $\theta \leq \frac{2}{1-\lambda}$* and *$\lambda \in (0,1)$, then S_λ is quasi-nonexpansive and $F(S_\lambda) = F(T)$.*

Proof. Let $u \in X$ and $w \in F(T)$, then since T is θ-generalized demimetric, we obtain by Lemma 2.13 that

$$\begin{aligned}\langle \overrightarrow{wu}, \overrightarrow{uS_\lambda u}\rangle &= \langle \overrightarrow{(\lambda u \oplus (1-\lambda)Tu)u}, \overrightarrow{uw}\rangle \le \lambda\langle \overrightarrow{uu}, \overrightarrow{uw}\rangle + (1-\lambda)\langle \overrightarrow{Tuu}, \overrightarrow{uw}\rangle \\ &= (1-\lambda)\langle \overrightarrow{Tuu}, \overrightarrow{uw}\rangle \le \frac{-(1-\lambda)^2}{\theta(1-\lambda)} d^2(u, Tu).\end{aligned} \tag{18}$$

Now, from (8), we obtain that $d^2(u, S_\lambda u) = (1-\lambda)^2 d^2(u, Tu)$. Thus, we obtain from (18) that

$$\langle \overrightarrow{wu}, \overrightarrow{uS_\lambda u}\rangle \le \frac{-1}{\theta(1-\lambda)} d^2(u, S_\lambda u),$$

which implies that

$$\langle \overrightarrow{uw}, \overrightarrow{uS_\lambda u}\rangle \ge \frac{1}{\theta(1-\lambda)} d^2(u, S_\lambda u) \ge \frac{1}{2} d^2(u, S_\lambda u).$$

Thus, we obtain that

$$d^2(u, S_\lambda u) + d^2(w, u) - d^2(w, S_\lambda u) \ge d^2(u, S_\lambda u),$$

which implies that

$$d^2(w, S_\lambda u) \le d^2(w, u).$$

Hence, S_λ is quasi-nonexpansive.

Next, we show that $F(S_\lambda) = F(T)$. From (8), we obtain that

$$d(u, S_\lambda u) = (1-\lambda) d(u, Tu).$$

This implies that $S_\lambda u = u$ if and only if $Tu = u$. Therefore, $F(S_\lambda) = F(T)$. □

Theorem 3.7. *Let X be a Hadamard space and X^* be its dual space. Let $A_i\colon X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be a finite family of multivalued monotone mappings satisfying the range condition and $T\colon X \to X$ be a θ-generalized demimetric mapping with $\theta \neq 0$. Suppose that $\Gamma := F(T) \cap (\bigcap_{i=1}^N A_i^{-1}(\mathbf{0})) \neq \emptyset$ and for arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n), \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \quad n \ge 1, \end{cases} \tag{19}$$

where $S_\mu x := \mu x \oplus (1-\mu)Tx$ such that S_μ is Δ-demiclosed, with $\theta \le \frac{2}{1-\mu}$, $\mu \in (0,1)$, $\lambda \in (0,\infty)$ and $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ are in (0,1) satisfying the following:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,

(C2) $0 < a \le \beta_n, \gamma_n \le b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

Proof. We first show that $\{x_n\}$ is bounded.

Let $p \in \Gamma$, from (19), Lemmas 2.4 and 3.6, we have

$$\begin{aligned} d^2(z_n, p) &= d^2((1-\gamma_n)y_n \oplus \gamma_n S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n), p) \\ &\le (1-\gamma_n)d^2(y_n, p) + \gamma_n d^2(S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n), p) \\ &\quad - \gamma_n(1-\gamma_n)d^2(y_n, S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)) \\ &\le (1-\gamma_n)d^2(y_n, p) + \gamma_n d^2(y_n, p) - \gamma_n(1-\gamma_n)d^2(y_n, S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)) && (20) \\ &\le d^2(y_n, p). && (21) \end{aligned}$$

We also have from (19) and (8) that

$$d(x_{n+1}, y_n) = \beta_n d(z_n, y_n), \tag{22}$$

which implies that

$$d^2(z_n, y_n) = \frac{\alpha_n}{\beta_n}\left(\frac{d^2(x_{n+1}, y_n)}{\alpha_n \beta_n}\right). \tag{23}$$

From (19), (21), (23) and Lemma 2.4, we obtain

$$d^2(x_{n+1}, p) \leq (1-\beta_n)d^2(y_n, p) + \beta_n d^2(z_n, p) - \beta_n(1-\beta_n)d^2(y_n, z_n) \tag{24}$$

$$\leq d^2(y_n, p) - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n) \leq d^2(y_n, p). \tag{25}$$

Thus, we obtain from Lemma 2.4 that

$$\begin{aligned} d(x_{n+1}, p) &\leq d(y_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(u, p) \\ &\leq \max \quad \{d(x_n, p), d(u, p)\} \\ &\vdots \\ &\leq \max \quad \{d(x_1, p), d(u, p)\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Hence, $\{y_n\}$ and $\{z_n\}$ are also bounded.

Next, we show that

$$\lim_{n\to\infty} d(y_n, S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)) = 0.$$

From (19), (24) and Lemma 2.4, we have that

$$\begin{aligned} d^2(x_{n+1}, p) &\leq d^2(y_n, p) - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n) \\ &= d^2((1-\alpha_n)x_n \oplus \alpha_n u, p) - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n) \\ &\leq (1-\alpha_n)^2 d^2(x_n, p) + \alpha_n^2 d^2(u, p) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{x_n p}, \overrightarrow{up}\rangle - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n^2 d^2(u, p) - 2\alpha_n(1-\alpha_n)\langle\overrightarrow{x_n p}, \overrightarrow{pu}\rangle - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n) \\ &= (1-\alpha_n)d^2(x_n, p) + \alpha_n(-d_n), \end{aligned} \tag{26}$$

where

$$d_n = \left[2(1-\alpha_n)\langle\overrightarrow{x_n p}, \overrightarrow{pu}\rangle - \alpha_n d^2(u, p) + \frac{1}{\beta_n \alpha_n}(1-\beta_n)d^2(x_{n+1}, y_n)\right]. \tag{27}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, they are bounded below. Thus, $\{d_n\}$ is bounded below, which implies that $\{-d_n\}$ is bounded above.

Therefore, we obtain from Lemma 2.8 and Condition $C1$ of Theorem 3.7 that

$$\limsup_{n\to\infty} d^2(x_n, p) \leq \limsup_{n\to\infty}(-d_n) = -\liminf d_n, \tag{28}$$

which implies that $\liminf(d_n) \leq -\limsup d^2(x_n, p)$. Thus, we conclude that $\liminf\limits_{n\to\infty} d_n$ exists.

Hence, we obtain from (27) and Condition $C1$ of Theorem 3.7 that

$$\liminf_{n\to\infty} d_n = \liminf_{n\to\infty}\left[2\langle\overrightarrow{x_n p}, \overrightarrow{pu}\rangle + \frac{1}{\beta_n \alpha_n}(1-\beta_n)d^2(x_{n+1}, y_n)\right].$$

Since $\{x_n\}$ is bounded, we obtain by Lemma 2.5 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta - \lim_{k\to\infty} x_{n_k} = z \in X$, and

$$\liminf_{n\to\infty} d_n = \lim_{k\to\infty}\left[2\langle \overrightarrow{x_{n_k}p}, \overrightarrow{pu}\rangle + \frac{1}{\beta_{n_k}\alpha_{n_k}}(1-\beta_{n_k})d^2(x_{n_k+1}, y_{n_k})\right], \tag{29}$$

for some subsequences $\{y_{n_k}\}$, $\{\beta_{n_k}\}$ and $\{\alpha_{n_k}\}$ of $\{y_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$, respectively.

Using the fact that $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d_n$ exists, we get that $\left\{\frac{1}{\beta_{n_k}\alpha_{n_k}}(1-\beta_{n_k})d^2(x_{n_k+1}, y_{n_k})\right\}$ is bounded. Also, by Condition $C2$, we obtain that $\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k}) \geq \frac{1}{\alpha_{n_k}\beta_{n_k}}(1-b) > 0$. Thus, $\left\{\frac{1}{\beta_{n_k}\alpha_{n_k}}d^2(x_{n_k+1}, y_{n_k})\right\}$ is bounded.

Again, from C1 and C2, we obtain that $0 < \frac{\alpha_{n_k}}{\beta_{n_k}} \leq \frac{\alpha_{n_k}}{a} \to 0$, as $k \to \infty$. Thus, $\frac{\alpha_{n_k}}{\beta_{n_k}} \to 0$ as $k \to \infty$.

Therefore, we obtain from (23) that

$$\lim_{k\to\infty} d(z_{n_k}, y_{n_k}) = 0. \tag{30}$$

From (22), (30) and Condition C2, we obtain that

$$\lim_{k\to\infty} d(x_{n_k+1}, y_{n_k}) = 0. \tag{31}$$

Also, from (21) and (30), we have

$$\begin{aligned}
&\gamma_{n_k}(1-\gamma_{n_k})d^2(y_{n_k}, S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_{n_k}))\\
&\quad \leq d^2(y_{n_k}, p) - d^2(z_{n_k}, p) \leq d^2(y_{n_k}, z_{n_k}) + 2d(y_{n_k}, z_{n_k})d(z_{n_k}, p) + d^2(z_{n_k}, p) - d^2(z_{n_k}, p) \to 0, \text{ as } k \to \infty.
\end{aligned}$$

Thus, from Condition $C2$, we have that

$$\lim_{k\to\infty}(y_{n_k}, S_\mu(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_{n_k})) = 0. \tag{32}$$

Next, we show that $\lim_{k\to\infty} d(v_{n_k}, S_\mu v_{n_k}) = 0$.

Let $v_{n_k} = \Phi_\lambda^N y_{n_k}$, where $\Phi_\lambda^N = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1$ with $\Phi_\lambda^0 = 1$. Since J_λ^N is firmly nonexpansive, we obtain from Remark 2.11 and (32) that

$$\begin{aligned}
d^2(v_{n_k}, \Phi_\lambda^{N-1}y_{n_k}) &\leq d^2(p, \Phi_\lambda^{N-1}y_{n_k}) - d^2(p, v_{n_k}) \leq d^2(p, y_{n_k}) - d^2(p, S_\mu v_{n_k})\\
&\leq d^2(p, S_\mu v_{n_k}) + 2d(p, S_\mu v_{n_k})d(S_\mu v_{n_k}, y_{n_k}) + d^2(S_\mu v_{n_k}, y_{n_k}) - d^2(p, S_\mu v_{n_k}) \to 0 \text{ as } k \to \infty.
\end{aligned} \tag{33}$$

Similarly, since J_λ^{N-1} is firmly nonexpansive, we obtain that

$$\begin{aligned}
d^2\left(\Phi_\lambda^{N-1}y_{n_k}, \Phi_\lambda^{N-2}y_{n_k}\right) &\leq d^2(p, \Phi_\lambda^{N-2}y_{n_k}) - d^2(p, \Phi_\lambda^{N-1}y_{n_k}) \leq d^2(p, y_{n_k}) - d^2(p, v_{n_k}) \leq d^2(p, y_{n_k}) - d^2(p, S_\mu v_{n_k})\\
&\leq d^2(p, S_\mu v_{n_k}) + 2d(p, S_\mu v_{n_k})d(S_\mu v_{n_k}, y_{n_k}) + d^2(S_\mu v_{n_k}, y_{n_k}) - d^2(p, S_\mu v_{n_k}) \to 0 \text{ as } k \to \infty.
\end{aligned} \tag{34}$$

In the same manner, we can show that

$$\lim_{k\to\infty} d^2(\Phi_\lambda^{N-2}y_{n_k}, \Phi_\lambda^{N-3}y_{n_k}) = \lim_{k\to\infty} d^2(\Phi_\lambda^{N-3}y_{n_k}, \Phi_\lambda^{N-4}y_{n_k}) = \cdots = \lim_{k\to\infty} d^2(\Phi_\lambda^1 y_{n_k}, y_{n_k}) = 0. \tag{35}$$

Thus,

$$d(v_{n_k}, y_{n_k}) \leq d(\Phi_\lambda^N y_{n_k}, \Phi_\lambda^{N-1}y_{n_k}) + d(\Phi_\lambda^{N-1}y_{n_k}, \Phi_\lambda^{N-2}y_{n_k}) + \cdots + d(\Phi_\lambda^1 y_{n_k}, y_{n_k}).$$

This implies from (33), (34) and (35) that

$$\lim_{k\to\infty} d(v_{n_k}, y_{n_k}) = \lim_{k\to\infty} d\left(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_{n_k}, y_{n_k}\right) = 0. \tag{36}$$

Furthermore, from (32) and (36), we obtain

$$\lim_{k\to\infty} d(v_{n_k}, S_\mu v_{n_k}) = 0. \tag{37}$$

Finally, we show that $\{x_n\}$ converges strongly to $z \in \Gamma$.

From (19) and Condition $C1$, we obtain

$$d(y_{n_k}, x_{n_k}) = d((1-\alpha_{n_k})x_{n_k} \oplus \alpha_{n_k} u, x_{n_k}) = \alpha_{n_k} d(u, x_{n_k}) \to 0 \quad \text{as } k \to \infty. \tag{38}$$

Since $\Delta - \lim_{k\to\infty} x_{n_k} = z$, we obtain from (38) that $\Delta - \lim_{k\to\infty} y_{n_k} = z$, and from (36) that $\Delta - \lim_{k\to\infty} v_{n_k} = zz$. By the demicloseness of S_μ, (37) and Lemma 3.6, we obtain that $z \in F(S_\mu) = F(T)$. Since J_λ^i, $i = 1, 2, \ldots, N$ are nonexpansive mappings and the composition of nonexpansive mappings is nonexpansive, we obtain from (36), Lemma 2.7 and Lemma 2.12 that $z \in F(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1) = F(J_\lambda^N) \cap F(J_\lambda^N) \cap F(J_\lambda^{N-1}) \cap \cdots \cap F(J_\lambda^2) \cap F(J_\lambda^1)$. Hence $z \in \Gamma$.

Furthermore, by Lemma 2.6, we have

$$\limsup_{n\to\infty} \langle \overrightarrow{zu}, \overrightarrow{x_{n_k}z} \rangle \geq 0.$$

Thus, we obtain from (29) and (31) that

$$\liminf_{n\to\infty} d_n = 2\lim_{k\to\infty} \langle \overrightarrow{zu}, \overrightarrow{x_{n_k}z} \rangle \geq 0.$$

Hence from (28), we have

$$\limsup_{n\to\infty} d^2(x_n, z) \leq -\liminf_{n\to\infty} d_n \leq 0.$$

Therefore, $\lim_{n\to\infty} d(x_n, z) = 0$ and this implies that $\{x_n\}$ converges strongly to $z \in \Gamma$. □

Setting $T \equiv I$ in Theorem 3.7, we have the following result.

Corollary 3.8. *Let X be a Hadamard space and X^* be its dual space. Let $A_i: X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be a finite family of multivalued monotone mappings satisfying the range condition. Suppose that $\Gamma := \bigcap_{i=1}^N A_i^{-1}(\mathbf{0}) \neq \emptyset$ and for arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n, \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \quad n \geq 1, \end{cases} \tag{39}$$

where $\lambda \in (0, \infty)$ and $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ are in $(0, 1)$ satisfying the following:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,

(C2) $0 < a \leq \beta_n, \gamma_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

Setting $N = 1$ in Theorem 3.7, we have the following result.

Corollary 3.9. *Let X be a Hadamard space and X^* be its dual space. Let $A: X \to 2^X$ be a multivalued monotone mapping that satisfies the range condition and $T: X \to X$ be a θ-generalized demimetric mapping with $\theta \neq 0$. Suppose that $\Gamma := F(T) \cap A^{-1}(\mathbf{0}) \neq \emptyset$ and for arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n S_\mu (J_\lambda^A y_n), \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \quad n \geq 1, \end{cases} \tag{40}$$

where $S_\mu x := \mu x \oplus (1-\mu)Tx$ *such that* S_μ is *Δ-demiclosed, with* $\theta \le \frac{2}{1-\mu}$, $\mu \in (0,1)$, $\lambda \in (0,\infty)$ *and* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ *are in* $(0,1)$ *satisfying the following*:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < a \le \beta_n,\ \gamma_n \le b < 1$.

Then, $\{x_n\}$ *converges strongly to an element of* Γ.

If T is nonexpansive in Corollary 3.9, we obtain the following result.

Corollary 3.10. *Let* X *be a Hadamard space and* X^* *be its dual space. Let* $A\colon X \to 2^{X^*}$ *be a multivalued monotone mapping satisfying the range condition and* $T\colon X \to X$ *be a nonexpansive mapping. Suppose that* $\Gamma := F(T) \cap A^{-1}(\mathbf{0}) \ne \emptyset$ *and for arbitrary* $u, x_1 \in X$, *the sequence* $\{x_n\}$ *is defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n T(J_\lambda^A y_n), \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \quad n \ge 1, \end{cases} \tag{41}$$

with $\lambda \in (0,\infty)$ *and* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ *are in* $(0,1)$ *satisfying the following*:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < a \le \beta_n,\ \gamma_n \le b < 1$.

Then, $\{x_n\}$ *converges strongly to an element of* Γ.

4 Application to some optimization problems

In this section, we apply our results to solve some optimization problems.

Definition 4.1. Let X be a Hadamard space and $f\colon X \to (-\infty,\infty]$ be a proper, convex and lower semicontinuous function with domain $D(f) := \{u \in X : f(u) < +\infty\}$. The function $f\colon X \to (-\infty,\infty]$ is called

(i) proper, if $D(f) \ne \emptyset$,

(ii) convex, if

$$f(\lambda u \oplus (1-\lambda)v) \le \lambda f(u) + (1-\lambda)f(v) \quad \text{for all} \quad u, v \in X \quad \text{and} \quad \lambda \in (0,1),$$

(iii) lower semicontinuous at a point $u \in D(f)$, if

$$f(u) \le \liminf_{n\to\infty} f(x_n),$$

for each sequence $\{x_n\}$ in $D(f)$ such that $\lim\limits_{n\to\infty} x_n = u$,

(iv) f is lower semicontinuous on $D(f)$, if it is lower semicontinuous at any point in $D(f)$.

Definition 4.2. [44] Let X be a Hadamard space and X^* be its dual space. The subdifferential of f is the multivalued function $\partial f\colon X \to 2^{X^*}$ defined by

$$\partial f(u) = \begin{cases} \{u^* \in X^* : f(w) - f(u) \ge \langle u^*, \overrightarrow{uw}\rangle \quad \text{for all} \quad w \in X\}, & \text{if } u \in D(f), \\ \emptyset, \quad \text{otherwise}. \end{cases} \tag{42}$$

Theorem 4.3. [44] *Let* $f\colon X \to (-\infty,+\infty]$ *be a proper, convex and lower semicontinuous function on a Hadamard space* X *with dual* X^*, *then*

(i) *f attains its minimum at* $u \in X$ *if and only if* $0 \in \partial f(u)$,

(ii) $\partial f\colon X \to 2^{X^*}$ *is a monotone operator*,

(iii) *for any* $v \in X$ *and* $\alpha > 0$, *there exists a unique point* $u \in X$ *such that* $[\alpha\overrightarrow{uv}] \in \partial f(u)$, *that is* $D(J_\lambda^{\partial f}) = X$, *for all* $\lambda > 0$.

Definition 4.4. Let C be a nonempty, closed and convex subset of X. Then, the indicator function $\delta_C: X \to \mathbb{R}$ is defined by

$$\delta_C u = \begin{cases} 0, & \text{if } u \in C, \\ +\infty, & \text{otherwise.} \end{cases} \tag{43}$$

It is generally known that δ_C is a proper convex. Thus, by Theorem 4.3(ii) and (iii), we have that the subdifferential of δ_C, given by

$$\partial\delta_C(u) = \begin{cases} \{u^* \in X^*: \langle u^*, \overrightarrow{uw}\rangle \le 0 \ \text{ for all } \ w \in C\}, & \text{if } u \in C, \\ \emptyset, \quad \text{otherwise} \end{cases} \tag{44}$$

is a monotone operator that satisfies the range condition.

4.1 VIP

Recently, Khatibzadeh and Ranjbar [51] formulated a VIP associated with a nonexpansive mapping in a Hadamard space as follows: Find $x \in C$ such that

$$\langle \overrightarrow{Txx}, \overrightarrow{xy}\rangle \ge 0 \quad \text{for all } \ y \in C. \tag{45}$$

Recall that the metric projection $P_C: X \to C$ is defined for $x \in X$ by $d(x, P_C x) = \inf_{y \in C} d(x, y)$ and is characterized by $z = P_C x$ if and only if $\langle \overrightarrow{zx}, \overrightarrow{zy}\rangle \le 0$, for all $y \in C$ (see [51]). Using the characterization of P_C, we obtain that

$$x = P_C Tx \quad \text{if and only if } \ \langle \overrightarrow{Txx}, \overrightarrow{xy}\rangle \ge 0 \ \text{ for all } \ y \in C.$$

Thus, we have that $x \in F(P_C \circ T)$ if and only if x solves (45). From (10), we have that

$$z = J_\lambda^{\partial\delta_C} x \Leftrightarrow \left[\frac{1}{\lambda}\overrightarrow{zx}\right] \in \partial\delta_C z \Leftrightarrow \langle \overrightarrow{zx}, \overrightarrow{zy}\rangle \le 0, \quad \text{for all } \ y \in C \Leftrightarrow z = P_C x. \tag{46}$$

Letting $z = x$, we obtain that $x = P_C x$ if and only if $x \in (\partial\delta_C)^{-1}(\mathbf{0})$. Thus,

$$x \in (\partial\delta_C)^{-1}(0) \cap F(T) x \in F(P_C) \cap F(T) x \in F(P_C \circ T).$$

Suppose the solution set of problem (45) is Υ. Setting $A = \partial\delta_C$ in Corollary 3.10, we apply Corollary 3.10 to obtain the following result for approximating solutions of VIP in Hadamard spaces.

Theorem 4.5. *Let C be a nonempty closed and convex subset of a Hadamard space X and X^* be its dual space. Let $T: X \to X$ be a nonexpansive mapping. Suppose that $\Upsilon \ne \emptyset$ and for arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n T\left(J_\lambda^{\partial\delta_C} y_n\right), \quad n \ge 1, \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \quad n \ge 1, \end{cases} \tag{47}$$

with $\lambda \in (0, \infty)$ and $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty \subset (0, 1)$ satisfying the following:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,

(C2) $0 < a \le \beta_n, \ \gamma_n \le b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Υ.

4.2 Convex feasibility problem

The convex feasibility problem is defined as follows: find $x \in C$ such that

$$x \in \bigcap_{i=1}^{N} C_i, \tag{48}$$

where C is a nonempty closed and convex subset of X and C_i, $i = 1, 2, \ldots, N$ is a finite family of nonempty closed and convex subsets of C such that $\bigcap_{i=1}^{N} C_i \neq \emptyset$.

From (46), we have that $x = J_\lambda^{\partial \delta_{C_i}} x$ if and only if $x = P_{C_i}x$, $i = 1, 2, \ldots, N$. Setting $A_i = \partial \delta_{C_i}$ in Corollary 3.8 and $J_\lambda^i = P_{C_i}$, $i = 1, 2, \ldots, N$ in Algorithm 39, we can apply Corollary 3.8 to approximate solutions of (48).

4.3 Convex minimization problem

The minimization problem is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{49}$$

Observe from Theorem 4.3(i) that (49) can be written as: find $x \in X$ such that

$$\mathbf{0} \in \partial f(x). \tag{50}$$

Thus, by setting $A = \partial f$ in Theorem 3.7, we obtain the following result.

Theorem 4.6. *Let X be a Hadamard space and X^* be its dual space. Let $f_i\colon X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper, convex and lower semicontinuous functions and $T\colon X \to X$ be a θ-generalized demimetric mapping with $\theta \neq 0$. Suppose that $\Upsilon := F(T) \cap (\bigcap_{i=1}^{N} \partial f_i^{-1}(\mathbf{0})) \neq \emptyset$ and for arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} y_n = (1 - \alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1 - \gamma_n)y_n \oplus \gamma_n S_\mu (J_\lambda^{\partial f_N} \circ J_\lambda^{\partial f_{N-1}} \circ \cdots \circ J_\lambda^{\partial f_2} \circ J_\lambda^{\partial f_1} y_n), \\ x_{n+1} = (1 - \beta_n)y_n \oplus \beta_n z_n, \quad n \geq 1, \end{cases} \tag{51}$$

where $S_\mu x := \mu x \oplus (1 - \mu)Tx$ such that S_μ is Δ-demiclosed, with $\theta \leq \frac{2}{1-\mu}$, $\mu \in (0, 1)$, $\lambda \in (0, \infty)$ and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty} \subset (0, 1)$, satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < a \leq \beta_n, \ \gamma_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Υ.

5 Conclusion

The class of θ-generalized demimetric mappings is introduced and studied in Hadamard space settings. In the study, it was shown (see Example 3.2) that this class of mappings is more general than the class of demimetric mappings previously studied in [1–5]. Moreover, the class of θ-generalized demimetric mappings contains many important classes of mappings (see Remark 3.3) known to be very useful for solving optimization problems. Furthermore, the strong convergence of a Halpern-type proximal point method for solving MIP and fixed point problem for this mapping is established in the framework of Hadamard space. An application of this method for solving other optimization problems is also considered. The obtained results of this article are a natural generalization of the results previously obtained in [2–5] from the study of demimetric mappings to θ-generalized demimetric mappings. The results also extend and complement the results in [25,32,34,39,52] established in Hadamard spaces for nonexpansive, strictly pseudocontractive, generalized hybrid and demicontractive mappings. Thus, the results of this article possess many possible applications compared to many other results in this direction.

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