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Isoperimetric inequalities for k-Hessian equations

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Abstract. We consider the homogeneous Dirichlet problem for a special k-Hessian equation of sub-linear type in a $(k-1)$-convex domain $\Omega \subset \mathbb{R}^n$, $1 \le k \le n$. We study the comparison between the solution of this problem and the (radial) solution of the corresponding problem in a ball having the same $(k-1)$-quermassintegral as Ω. Next, we consider the eigenvalue problem for the k-Hessian equation and study a comparison between its principal eigenfunction and the principal eigenfunction of the corresponding problem in a ball having the same $(k-1)$-quermassintegral as Ω. Symmetrization techniques and comparison principles are the main tools used to get these inequalities.

Keywords. Monge–Ampère type equations, rearrangements, eigenvalues, isoperimetric inequalities.

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1 Introduction

In the seminal paper [23], G. Talenti pioneered an important method for establishing sharp a priori estimates of quantities involving solutions to boundary value problems of second order elliptic linear PDE's. In the subsequent papers [24, 25], he obtained optimal estimates of a priori solutions to Dirichlet problems with homogeneous boundary conditions by comparing them to corresponding quantities of solutions of problems that have better symmetry. These papers have inspired the use of similar methods in numerous investigations involving both linear and nonlinear elliptic problems.

In the paper [28], K. Tso applies the ideas in the paper [24] to develop symmetrization schemes for obtaining isoperimetric inequalities of quantities involving solutions to k-Hessian equations. More specifically, let Ω be a bounded, open convex set in $\mathbb{R}^n$, and ψ be a positive, smooth real-valued function defined on Ω. Consider the Dirichlet problem

$$\mathcal{F}_k(D^2u) = \psi(x) \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega. \tag{1.1}$$

Here $1 \le k \le n$, $\mathcal{F}_k(\mathcal{A})$ is the sum of the principal minors of order k of the $n \times n$ real matrix $\mathcal{A}$, and D^2u is the Hessian of $u \in C^2(\Omega)$. In [28], K. Tso derives isoperimetric inequalities of quantities involving strictly convex solutions u to (1.1) in Ω. An appropriate rearrangement u^*_{k-1} of a solution u of (1.1) is compared with the solution v to the so-called symmetrized Dirichlet problem

$$\mathcal{F}_k(D^2 v) = \psi^\sharp(x) \text{ in } \Omega^*_{k-1}, \quad v = 0 \text{ on } \partial\Omega^*_{k-1}. \tag{1.2}$$

In equation (1.2), $\psi^\sharp(x)$ is the usual Schwarz decreasing rearrangement of ψ, while Ω^*_{k-1} is a ball centered at the origin and with radius equal to the $(k-1)$-mean radius of Ω (see definition in Section 2). The $(k-1)$-mean radius is tailored to the k-Hessian operator and for $k = 1$, the ball Ω^*_0 has the same Lebesgue measure as Ω, while for $k = 2$, the ball Ω^*_1 has the same perimeter as Ω. The work in [28] provides a direct generalization of the results of G. Talenti in the paper [24] for $n = k = 2$ case. We refer the reader to the papers [2,4,25,27] for works that use symmetrization methods to study sharp a priori estimates of solutions to elliptic Dirichlet problems. The monograph [13] contains a wide description of classical rearrangements and its application to semilinear equations. Results of existence, uniqueness and regularity of solutions to Hessian equations can be found in [6], [15] and [29]. For works related to Hessian equations we also mention the papers [8,14,18,19]. The monograph [3] contains geometrical methods for the investigation of Monge–Ampère equations.

The main focus of the present work is to employ similar methods as in [23,28] to get isoperimetric inequalities of various quantities that involve sub-solutions and super-solutions related to the following Dirichlet problems:

$$\mathcal{F}_k(D^2 u) = f(-u), \quad u < 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

The inhomogeneous term f is a non-decreasing, non-negative smooth function defined on $[0, \infty)$. A key result we find is the inequality

$$\left(\frac{du^*(r)}{dr}\right)^k \le n \binom{n}{k}^{-1} r^{k-n} \int_0^r t^{n-1} f(-u^*(t))dt, \tag{1.4}$$

where $u^*(r)$ is a suitable rearrangement of a super-solution to (1.3). When $f(t)$ satisfies a special growth condition (a typical example is $f(t) = t^q$, $0 < q < k$), using inequality (1.4) we find a comparison result between $u^*(r)$ and the solution v to problem (1.3) in the ball Ω^*_{k-1}. In case of $k = n = 2$, this result is proved in [5].

We also study the case $f(t) = \lambda t^k$ and, again using (1.4), we provide estimates of symmetrands of appropriately normalized principal eigenfunctions for the k-Hessian operators in terms of eigenfunctions of the symmetrized eigenvalue problem. In case of $k = n = 2$, these estimates are proved in [5] by using different arguments. For properties of nonlinear eigenvalues, see also [1].

The paper is organized as follows. In Section 2, we provide notations and basic definitions that will be used in this work. We also recall some useful results from the literature. In Section 3, we find various estimates related to problem (1.3) in case $f^{\frac{1}{k}}$ is sublinear. In Section 4, we investigate isoperimetric inequalities involving the eigenvalues and appropriately normalized eigenfunctions associated with the k-Hessian operator.

2 Notations and preliminaries

Throughout this work, suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary. Let $\kappa_1, \dots, \kappa_{n-1}$ be the principal curvatures of $\partial\Omega$. For $m = 1, \dots, n-1$ we define the m-th mean curvature of $\partial\Omega$ by

$$\mathcal{H}_m(\partial\Omega) = S_m(\kappa_1, \dots, \kappa_{n-1}),$$

where S_m denotes the elementary symmetric function of order m of $\kappa_1, \dots, \kappa_{n-1}$. We also define $\mathcal{H}_0(\partial\Omega) = 1$.

For $h = 1, \dots, n-1$ we consider domains Ω which are h-convex, that is,

$$\mathcal{H}_j(\partial\Omega) \geq 0, \quad j = 1, \dots, h.$$

When $\partial\Omega$ is connected, the above condition is equivalent to $\mathcal{H}_h(\partial\Omega) \geq 0$. We also note that Ω is $(n-1)$-convex when the components of Ω are convex in the usual sense.

If $\mathcal{A}$ is an $n \times n$ matrix, we denote by $\mathcal{F}_h(\mathcal{A})$ the sum of the principal minors of order h of $\mathcal{A}$. For $h = 1, \dots, n-1$, assume Ω is h-convex, and let $A_h(\Omega)$ denote the class of functions

$$\begin{aligned} A_h(\Omega) := \{u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega}) : u < 0 \text{ and } (h+1)\text{-convex in } \Omega, \\ u = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Recall that u is $(h+1)$-convex if and only if

$$\mathcal{F}_j(D^2u) > 0, \quad j = 1, \dots, h+1.$$

It is well known (see [30]) that if u is $(h+1)$-convex, then the operator $\mathcal{F}_{h+1}(D^2u)$ is elliptic. If $u \in A_h(\Omega)$, then from Sard's theorem it follows that for almost all $t \in (m_0, 0)$, $m_0 = \min u$, the sub-level set

$$\Omega_t = \{x \in \Omega : u(x) < t\} \tag{2.1}$$

will have smooth boundary Σ_t given by the level surface

$$\Sigma_t := \{x \in \Omega : u(x) = t\}.$$

We have $\Omega_0 = \Omega$. Furthermore, by (2.28) of [27], whenever $|Du| > 0$ on Σ_t, we have

$$\mathcal{H}_j(\Sigma_t) = \mathcal{F}_j\left(D\frac{Du}{|Du|}\right) \geq 0, \quad j = 1, \dots, h$$

on Σ_t, so that Ω_t is h-convex. When Ω is convex, all functions belonging to the class $A_{n-1}(\Omega)$ are convex in the usual sense. We also define $A_0(\Omega)$ as the class of functions $u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega})$ such that $u < 0$ in Ω, $u = 0$ on $\partial\Omega$. In this case, no convexity conditions on Ω are needed.

In what follows, k is a fixed integer such that $1 \leq k \leq n$. Whenever $2 \leq k \leq n$, assume Ω to be $(k-1)$-convex. For $0 < q < k$, we consider the Dirichlet problem

$$\mathcal{F}_k(D^2u) = (-u)^q, \quad u < 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{2.2}$$

For existence and regularity questions related with problem (2.2) we refer the reader to [9, 30, 31]. Let u_{ij} denote the second order derivative of u with respect to x_i and x_j. Since $\mathcal{F}_k(D^2u)$ is k-homogeneous with respect to u_{ij}, we have

$$\frac{\partial \mathcal{F}_k}{\partial u_{ij}} u_{ij} = k\mathcal{F}_k(D^2u).$$

Denote by $T^{(k-1)}(D^2u)$ the $(k-1)$-order Newton tensor of D^2u. Note that

$$T_{ij}^{(k-1)}(D^2u) = \frac{\partial \mathcal{F}_k}{\partial u_{ij}}.$$

Since $T^{(k-1)}(D^2u)$ is divergence free (see [20, 21]), we have

$$\mathcal{F}_k(D^2u) = \frac{1}{k}\frac{\partial \mathcal{F}_k}{\partial u_{ij}} u_{ij} = \frac{1}{k}\left(T_{ij}^{(k-1)}(D^2u)u_i\right)_j,$$

and

$$\int_\Omega (-u)\mathcal{F}_k(D^2u)dx = \frac{1}{k}\int_\Omega T_{ij}^{(k-1)}(D^2u)u_iu_j\,dx.$$

Define

$$S_{k,q}(\Omega) = \inf\left\{\int_\Omega (-v)\mathcal{F}_k(D^2v)dx : v \in A_{k-1}(\Omega),\ \int_\Omega (-v)^{q+1}dx = 1\right\}. \tag{2.3}$$

We can rewrite (2.3) as

$$S_{k,q}(\Omega) = \inf\left\{\frac{\int_\Omega (-v)\mathcal{F}_k(D^2v)dx}{\left(\int_\Omega (-v)^{q+1}dx\right)^{\frac{k+1}{q+1}}} : v \in A_{k-1}(\Omega)\right\}.$$

One can show (see [30, Section 6]) that a minimizer v of (2.3) satisfies the problem

$$\mathcal{F}_k(D^2 v) = S_{k,q}(\Omega)(-v)^q, \quad v \in A_{k-1}(\Omega). \tag{2.4}$$

Note that if v is a solution to problem (2.4), then

$$u = \big(S_{k,q}(\Omega)\big)^{\frac{1}{q-k}} v$$

is a solution to problem (2.2).

For $m = 1, \dots, n-1$, the *quermassintegral* $V_m(\Omega)$ is defined by

$$V_m(\Omega) = \frac{1}{n}\binom{n-1}{m}^{-1} \int_{\partial\Omega} \mathcal{H}_{n-m-1}(\partial\Omega) d\sigma, \tag{2.5}$$

where $d\sigma$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. We find

$$V_{n-1}(\Omega) = \frac{\sigma(\partial\Omega)}{n},$$

where $\sigma(\partial\Omega)$ denotes the $(n-1)$-measure of $\partial\Omega$. We also define $V_n(\Omega) = |\Omega|$, the usual Lebesgue measure of Ω. Furthermore, whenever Ω is connected, we put $V_0(\Omega) = \omega_n$, the measure of the unit ball in $\mathbb{R}^n$.

With Ω being $(n-m+1)$-convex whenever $m \le n-2$, following [27] we define the m-mean radius of Ω, denoted by $\eta_m(\Omega)$, as

$$\eta_m(\Omega) = \left(\frac{V_m(\Omega)}{\omega_n}\right)^{\frac{1}{m}}, \quad m = 1, \dots, n.$$

Note that $\eta_n(\Omega)$ is the radius of the ball having the same n-measure as Ω, and $\eta_{n-1}(\Omega)$ is the radius of the ball having the same perimeter as Ω. In case Ω is a ball we have $\eta_1(\Omega) = \cdots = \eta_n(\Omega)$. The following isoperimetric inequalities are well known for convex domains Ω:

$$\eta_\ell(\Omega) \le \eta_m(\Omega), \quad 1 \le m \le \ell \le n.$$

These inequalities are equivalent to the Alexandrov–Fenchel inequalities for the quermassintegrals of convex domains. Recently P. Guan and J. Li have shown in [11] that these isoperimetric inequalities hold for starshaped h-convex domains Ω. The paper [26] deals with the Alexandrov–Fenchel inequalities for h-convex (without additional information) but, as remarked in [11] and in [12], the proof in [26] is incomplete.

For $h = 1, \dots, n-2$, we consider domains Ω such that

$$\left(\frac{V_n(\Omega)}{\omega_n}\right)^{\frac{1}{n}} \le \left(\frac{V_{n-h}(\Omega)}{\omega_n}\right)^{\frac{1}{n-h}} \le \left(\frac{V_{n-h-1}(\Omega)}{\omega_n}\right)^{\frac{1}{n-h-1}}. \tag{2.6}$$

It is unclear if (2.6) holds for any h-convex domain (see [11]).

For each $h = 1, \dots, n-2$, we introduce the family of functions

$$\Phi_h(\Omega) := \{u \in A_h(\Omega) : \text{(2.6) holds for each } \Omega_t, t \leq 0\},$$

where Ω_t is as defined in (2.1). We note that, for $h = 0$, the left-hand side of (2.6) holds trivially, and the right-hand side reduces to the familiar isoperimetric inequality; therefore, we put $\Phi_0(\Omega) = A_0(\Omega)$. We also note that all functions belonging to $A_{n-1}(\Omega)$ are convex, and that the left-hand side of (2.6) holds for convex domains. We put $\Phi_{n-1}(\Omega) = A_{n-1}(\Omega)$.

For $h = 0, \dots, n-1$, we define the rearrangement of $u \in \Phi_h(\Omega)$ with respect to the quermassintegral V_{n-h} as

$$u_h^{\bigstar}(s) = \sup\bigl\{t \leq 0 : V_{n-h}(\Omega_t) \leq s,\ 0 \leq s \leq V_{n-h}(\Omega)\bigr\}.$$

The function $u_h^{\bigstar}(s)$ is non decreasing and satisfies

$$u_h^{\bigstar}(0) = \min_{\Omega} u(x), \quad u_h^{\bigstar}\bigl(V_{n-h}(\Omega)\bigr) = 0.$$

We also define

$$u_h^*(x) = u_h^{\bigstar}\bigl(\omega_n |x|^{n-h}\bigr), \quad 0 \leq |x| \leq \eta_{n-h}(\Omega).$$

The function $u_h^*(x)$ is called *the h-symmetrand of u* (see [27]) and can also be defined by

$$u_h^*(x) = \sup\bigl\{t \leq 0 : \eta_{n-h}(\Omega_t) \leq |x|,\ 0 \leq |x| \leq \eta_{n-h}(\Omega)\bigr\}.$$

Since $u_h^*(x)$ is radially symmetric, we often write $u_h^*(x) = u_h^*(r)$ for $|x| = r$. We have

$$u_h^*(0) = \min_{\Omega} u(x) \quad \text{and} \quad u_h^*\bigl(\eta_{n-h}(\Omega)\bigr) = 0.$$

Note that $u_0^*(x)$ is the usual Schwarz increasing symmetrization of $u(x)$.

By definition we have

$$\eta_{n-h}(\Omega_t) = \eta_{n-h}(\{u_h^*(x) < t\}).$$

By using this equation and the left-hand side of (2.6), we find

$$\eta_n(\Omega_t) \leq \eta_{n-h}(\{u_h^*(x) < t\}) = \eta_n(\{u_h^*(x) < t\}).$$

It follows that

$$|\{x \in \Omega : u(x) < t\}| \leq |\{x \in \Omega_h^* : u_h^*(x) < t\}|, \tag{2.7}$$

where $|E|$ denotes the usual Lebesgue measure of E, and Ω_h^* is the ball with radius $\eta_{n-h}(\Omega)$. Note that Ω_0^* is the ball such that $|\Omega_0^*| = |\Omega|$, and Ω_1^* is the ball with the same perimeter as Ω.

By (2.7) it follows that, for $p \geq 1$,

$$\|u\|_{L_p(\Omega)} \leq \|u_h^*\|_{L_p(\Omega_h^*)}. \tag{2.8}$$

Recall that k is an integer such that $1 \leq k \leq n$, and that Ω_{k-1}^* is the ball centered at the origin and radius $\eta_{n-k+1}(\Omega)$. To simplify our statements, we say that every smooth domain Ω is 0-convex.

Lemma 2.1. *Let Ω be $(k-1)$-convex and $u \in \Phi_{k-1}(\Omega)$. If $m_0 = \inf_\Omega u$, for almost every $t \in (m_0, 0)$ we have*

$$\frac{d}{dt}\int_{\Sigma_t} \mathcal{H}_{k-2}\, d\sigma = (k-1)\int_{\Sigma_t} \frac{\mathcal{H}_{k-1}}{|Du|}\, d\sigma, \quad k \geq 2, \tag{2.9}$$

$$T_{ij}^{(k-1)}(D^2u)u_i u_j = |Du|^{k+1}\mathcal{H}_{k-1} \quad \text{on } \Sigma_t. \tag{2.10}$$

Furthermore, if $z := u_{k-1}^*$, for $p \geq k+1$ we have

$$\begin{aligned} &\int_\Omega T_{ij}^{(k-1)}(D^2u)u_i u_j |Du|^{p-k-1}\, dx \\ &\quad \geq \int_{\Omega_{k-1}^*} T_{ij}^{(k-1)}(D^2z) z_i z_j |Dz|^{p-k-1}\, dx. \end{aligned} \tag{2.11}$$

Proof. For the proof of (2.9), (2.10) and (2.11) see (2.20), (2.28) and (4.1) of [27], respectively. □

3 Sub-linear equations

Proposition 3.1. *Let $S_{k,q}(\Omega)$ be defined as in (2.3) for a $(k-1)$-convex domain Ω, and let $S_{k,q}(\Omega_{k-1}^*)$ be defined as in (2.3) for Ω_{k-1}^*. If $S_{k,q}(\Omega)$ has a minimizer $u \in \Phi_{k-1}(\Omega)$, then we have*

$$S_{k,q}(\Omega) \geq S_{k,q}(\Omega_{k-1}^*). \tag{3.1}$$

Proof. If $u \in \Phi_{k-1}(\Omega)$ is a minimizer for $S_{k,q}(\Omega)$, we have

$$S_{k,q}(\Omega) = \frac{1}{k}\int_\Omega T_{ij}^{(k-1)}(D^2u)u_i u_j\, dx, \quad \int_\Omega (-u)^{q+1}\, dx = 1.$$

With notation as in Lemma 2.1, by using (2.11) with $p = k+1$ we find

$$S_{k,q}(\Omega) \geq \frac{1}{k}\int_{\Omega_{k-1}^*} T_{ij}^{(k-1)}(D^2z) z_i z_j\, dx.$$

By (2.8) with $h = k-1$ and $p = q+1$ we have

$$1 = \int_\Omega (-u)^{q+1}\, dx \leq \int_{\Omega_{k-1}^*} (-z)^{q+1}\, dx.$$

Therefore, we find

$$S_{k,q}(\Omega) \geq \frac{\frac{1}{k}\int_{\Omega^*_{k-1}} T^{(k-1)}_{ij}(D^2 z) z_i z_j\, dx}{\left(\int_{\Omega^*_{k-1}} (-z)^{q+1} dx\right)^{\frac{k+1}{q+1}}} \geq S_{k,q}(\Omega^*_{k-1}).$$

The proposition is proved. □

Remark 3.2. We may ask if a minimizer of $S_{k,q}(\Omega)$ always belongs to $\Phi_{k-1}(\Omega)$. Well, certainly this is true for $k = 1$ since we have $\Phi_0(\Omega) = A_0(\Omega)$, and for $k = n$ (Monge–Ampère case) since Ω must be convex and $\Phi_{n-1}(\Omega) = A_{n-1}(\Omega)$. For $1 < k < n$ and Ω convex we believe that a minimizer of $S_{k,q}(\Omega)$ has convex level sets, but we do not have a proof at this time. We have learned from Paolo Salani that the result holds in case $n = 3$ and $k = 2$, see [17] and [22].

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a $(k-1)$-convex domain. Suppose*

$$f : (0,\infty) \to (0,\infty)$$

is a non-decreasing smooth function, and $f(s) > 0$ for $s > 0$. Let $v \in \Phi_{k-1}(\Omega)$ be a super-solution of

$$\mathcal{F}_k(D^2 v) = f(-v) \quad in\ \Omega.$$

If $v^*_{k-1}(x)$ is the $(k-1)$-symmetrand of v, and if $v^*(r) := v^*_{k-1}(x)$ for $r = |x|$, then

$$\left(\frac{dv^*(r)}{dr}\right)^k \leq n \binom{n}{k}^{-1} r^{k-n} \int_0^r t^{n-1} f(-v^*(t))\,dt, \tag{3.2}$$

with equality if and only if v is a solution and Ω is a ball.

Proof. Note that v^* is defined in $B_R = \Omega^*_{k-1}$, the ball centered in the origin and radius $R := \eta_{n-k+1}(\Omega)$. We also recall that

$$\Omega_t = \{x \in \Omega : v(x) < t\}, \quad \Sigma_t = \{x \in \Omega : v(x) = t\}.$$

Since v is a super-solution we have

$$\mathcal{F}_k(D^2 v) \leq f(-v).$$

Integration over Ω_t, for almost all values of t, leads to

$$\frac{1}{k}\int_{\Sigma_t} T^{(k-1)}_{ij}(D^2 v) v_i v_j |Dv|^{-1} d\sigma \leq \int_{\Omega_t} f(-v)\,dx.$$

On using (2.10), this inequality becomes

$$\int_{\Sigma_t} |Dv|^k \mathcal{H}_{k-1}\, d\sigma \leq k \int_{\Omega_t} f(-v)\,dx.$$

An application of Hölder's inequality then gives

$$\begin{aligned}\int_{\Sigma_t} \mathcal{H}_{k-1} d\sigma &\leq \left(\int_{\Sigma_t} |Dv|^k \mathcal{H}_{k-1} d\sigma\right)^{\frac{1}{k+1}} \left(\int_{\Sigma_t} |Dv|^{-1} \mathcal{H}_{k-1} d\sigma\right)^{\frac{k}{k+1}} \\ &\leq \left(k \int_{\Omega_t} f(-v) dx\right)^{\frac{1}{k+1}} \left(\int_{\Sigma_t} |Dv|^{-1} \mathcal{H}_{k-1} d\sigma\right)^{\frac{k}{k+1}}.\end{aligned} \tag{3.3}$$

For the rest of the argument we will write $V_h(t)$ for $V_h(\Omega_t)$. Consider first the case $2 \leq k \leq n-1$. By (2.5) with $m = n-k$ we find

$$\int_{\Sigma_t} \mathcal{H}_{k-1} d\sigma = n \binom{n-1}{n-k} V_{n-k}(t) = k \binom{n}{k} V_{n-k}(t). \tag{3.4}$$

On the other hand, by (2.9) we get

$$\int_{\Sigma_t} |Dv|^{-1} \mathcal{H}_{k-1} d\sigma = \frac{1}{k-1} \frac{d}{dt} \int_{\Sigma_t} \mathcal{H}_{k-2} d\sigma. \tag{3.5}$$

But, by (2.5) with $m = n-k+1$ we have

$$\int_{\Sigma_t} \mathcal{H}_{k-2} d\sigma = n \binom{n-1}{n-k+1} V_{n-k+1}(t).$$

Therefore, we can rewrite (3.5) as

$$\int_{\Sigma_t} |Dv|^{-1} \mathcal{H}_{k-1} d\sigma = \binom{n}{k-1} \frac{d}{dt} V_{n-k+1}(t).$$

Inserting the above equation and (3.4) into (3.3) yields

$$k \binom{n}{k} V_{n-k}(t) \leq \left[k \int_{\Omega_t} f(-v) dx\right]^{\frac{1}{k+1}} \left[\binom{n}{k-1} \frac{d}{dt} V_{n-k+1}(t)\right]^{\frac{k}{k+1}}.$$

After some simplification we obtain

$$\left(V_{n-k}(t)\right)^{\frac{k+1}{k}} \leq \left[\binom{n}{k}^{-1} \int_{\Omega_t} f(-v) dx\right]^{\frac{1}{k}} \frac{1}{n-k+1} \frac{d}{dt} V_{n-k+1}(t).$$

Using the right-hand side of (2.6) with $h = k-1$, we find

$$V_{n-k}(t) \geq \omega_n \left(\frac{V_{n-k+1}(t)}{\omega_n}\right)^{\frac{n-k}{n-k+1}} = \omega_n^{\frac{1}{n-k+1}} \left(V_{n-k+1}(t)\right)^{\frac{n-k}{n-k+1}}.$$

Therefore,

$$\omega_n^{\frac{k+1}{k(n-k+1)}} \big(V_{n-k+1}(t)\big)^{\frac{n-k}{n-k+1}\,\frac{k+1}{k}} \leq \left[\binom{n}{k}^{-1} \int_{\Omega_t} f(-v)dx\right]^{\frac{1}{k}} \frac{1}{n-k+1} \frac{d}{dt} V_{n-k+1}(t). \tag{3.6}$$

If $m_0 = \min_\Omega v(x)$ and $\mu(t) = |\Omega_t|$, we have (see [10, Theorem 3.36 and Proposition 6.23])

$$\int_{\Omega_t} f(-v)dx = \int_{m_0}^t f(-\tau)\mu'(\tau)d\tau = f(-t)\mu(t) + \int_{m_0}^t f'(-\tau)\mu(\tau)d\tau.$$

Using the left-hand side of (2.6) with $h = k-1$, we find

$$\mu(t) \leq \omega_n^{\frac{1-k}{n-k+1}} (V_{n-k+1}(t))^{\frac{n}{n-k+1}}.$$

Therefore,

$$\begin{aligned}\int_{\Omega_t} f(-v)dx &\leq \omega_n^{\frac{1-k}{n-k+1}} \Bigg[f(-t)(V_{n-k+1}(t))^{\frac{n}{n-k+1}} \\ &\qquad + \int_{m_0}^t f'(-\tau)(V_{n-k+1}(\tau))^{\frac{n}{n-k+1}} d\tau \Bigg] \\ &= \frac{n\omega_n^{\frac{1-k}{n-k+1}}}{n-k+1} \int_{m_0}^t f(-\tau)(V_{n-k+1}(\tau))^{\frac{k-1}{n-k+1}} (V_{n-k+1}(\tau))' d\tau.\end{aligned}$$

Putting $\rho = V_{n-k+1}(\tau)$, since $v^\star(\rho)$ is essentially the inverse of $V_{n-k+1}(\tau)$, we get

$$\int_{\Omega_t} f(-v)dx \leq \frac{n\omega_n^{\frac{1-k}{n-k+1}}}{n-k+1} \int_0^{V_{n-k+1}(t)} f(-v^\star(\rho))\rho^{\frac{k-1}{n-k+1}} d\rho. \tag{3.7}$$

Inserting the latter estimate into (3.6), we find

$$\begin{aligned}&\big(V_{n-k+1}(t)\big)^{\frac{n-k}{n-k+1}\,\frac{k+1}{k}} \\ &\quad\leq \left[\binom{n}{k}^{-1} \frac{n\omega_n^{\frac{-2k}{n-k+1}}}{(n-k+1)^{k+1}} \int_0^{V_{n-k+1}(t)} f(-v^\star(\rho))\rho^{\frac{k-1}{n-k+1}} d\rho\right]^{\frac{1}{k}} \\ &\qquad \times \frac{d}{dt} V_{n-k+1}(t).\end{aligned}$$

Now we put $V_{n-k+1}(t) = s$. Since

$$\frac{d}{dt} V_{n-k+1}(t) = \left(\frac{dv^\star(s)}{ds}\right)^{-1},$$

we get

$$\left(\frac{dv^\star(s)}{ds}\right)^k \le \binom{n}{k}^{-1} \frac{n\omega_n^{\frac{-2k}{n-k+1}}}{(n-k+1)^{k+1}} s^{\frac{(k-n)(k+1)}{n-k+1}} \int_0^s f(-v^\star(\rho))\rho^{\frac{k-1}{n-k+1}} d\rho.$$

With the change of variable $s = \omega_n r^{n-k+1}$ we have $v^\star(s) = v^*(r)$, and

$$\frac{dv^*(r)}{dr} = \frac{dv^\star(s)}{ds}\omega_n(n-k+1)r^{n-k}.$$

With this new variable we find

$$\left(\frac{dv^*(r)}{dr}\right)^k \le \binom{n}{k}^{-1} \frac{n\omega_n^{\frac{-n}{n-k+1}}}{n-k+1} r^{k-n} \int_0^s \rho^{\frac{k-1}{n-k+1}} f(-v^\star(\rho)) d\rho.$$

Putting $\rho = \omega_n t^{n-k+1}$ and recalling that $s = \omega_n r^{n-k+1}$, after simplification we get the inequality stated in the lemma for $2 \le k \le n-1$.

If $k = 1$, equation (3.4) continues to hold, and inequality (3.5) can be replaced by the classical inequality

$$\int_{\Sigma_t} |Dv|^{-1} d\sigma = \frac{d}{dt} V_n(t).$$

The proof continues as in the previous case.

Finally, let $k = n$. Instead of (3.4) now we use the well-known inequality

$$\int_{\Sigma_t} \mathcal{H}_{n-1} d\sigma = n\,\omega_n.$$

Inequality (3.5) continues to hold when $k = n$, therefore, by (3.3) we find

$$(\omega_n)^{\frac{n+1}{n}} \le \left[\int_{\Omega_t} f(-v)dx\right]^{\frac{1}{n}} \frac{d}{dt} V_1(t).$$

This inequality may be viewed as the analogous of (3.6) in case of $k = n$. From now on the proof continues as before and yields the inequality stated in the lemma for $k = n$.

If Ω is a ball and z is a solution of the given equation, all the inequalities used in the proof of the lemma are equalities, therefore, the inequality of the lemma holds with equality sign. More easily, in this case the equality follows directly from the equation which, for radial functions $z = z(r)$, reads as

$$\frac{1}{n}\binom{n}{k} r^{-n+1}\left(r^{n-k}\left(\frac{dz}{dr}\right)^k\right)' = f(-z). \tag{3.8}$$

Finally, if equality holds for all $r \in (0, \eta_{n-k+1}(\Omega))$, then all the inequalities involved in the proof must be equalities. Furthermore, by equation (3.8) we see that $z'(r) > 0$ for $r > 0$. Hence, Ω must be a ball. The lemma is proved. $\square$

To proceed further, we will need a comparison principle. By [31, Lemma 5.1], a result of uniqueness holds for problem (2.2) (and, equivalently, for problem (2.4)). By using a method similar to the one used in [31], we prove a comparison lemma which we shall use later on. Consider a smooth function $f : [0, \infty) \to [0, \infty)$ such that $f(s) > 0$ for $s > 0$ and

$$\begin{aligned} &f'(s) \geq 0 \text{ and there is } 0 < q < k \text{ such that} \\ &f(\alpha s) \geq \alpha^q f(s) \ \ \forall (\alpha, s) \in (0,1) \times (0, \infty). \end{aligned} \tag{3.9}$$

A typical example which satisfies (3.9) is $f(s) = s^q$ with $0 < q < k$.

Lemma 3.4. *Suppose f satisfies condition* (3.9)*, and let $\Omega \subset \mathbb{R}^n$ be a $(k-1)$-convex domain. Suppose $w, u \in A_{k-1}(\Omega)$ satisfy*

$$\mathcal{F}_k(D^2 w) \geq f(-w), \quad \mathcal{F}_k(D^2 u) \leq f(-u) \quad \text{in } \Omega. \tag{3.10}$$

Then, $w \leq u$ in Ω.

Proof. Since $w \in A_{k-1}(\Omega)$, there is a positive constant $C(n,k)$ such that

$$\Delta w \geq C(n,k)(\mathcal{F}_k(D^2 w))^{\frac{1}{k}} \geq C(n,k)(f(-w))^{\frac{1}{k}} > 0.$$

It follows from Hopf's lemma for subharmonic functions (rather using the barrier constructed in its proof) that $w(x) \leq -c_1 d(x)$ in $\overline{\Omega}$, where $d(x)$ denotes the distance from x to $\partial\Omega$ and c_1 is a suitable positive constant. For $x \in \Omega$, let $x_b \in \partial\Omega$ such that $d(x) = |x - x_b|$. Then, since $u \in C^{0,1}(\overline{\Omega})$, we find that

$$-u(x) = |u(x)| \leq \mathrm{Lip}(u, \overline{\Omega})|x - x_b| = \mathrm{Lip}(u; \overline{\Omega}) d(x).$$

It follows that $u(x) \geq -c_2 d(x)$ for $x \in \overline{\Omega}$ and some positive constant c_2. Consequently we get

$$w(x) \leq \frac{c_1}{c_2} u(x). \tag{3.11}$$

Let us now suppose, by way of contradiction, that $w \leq u$ in $\overline{\Omega}$ does not hold. Consider the set

$$S := \{\lambda \in [0,1] : w(x) \leq \lambda u(x) \ \ \forall x \in \overline{\Omega}\}.$$

Let $\Lambda := \sup S$. By using (3.11) we see that $0 < \Lambda < 1$, and we have $w \leq \Lambda u$ in Ω. By condition (3.9), there is $0 < q < k$ such that

$$f(-\Lambda u(x)) \geq \Lambda^q f(-u(x)) \quad \forall x \in \Omega. \tag{3.12}$$

Since $0 < q < k$, we choose $\varepsilon > 0$, sufficiently small, that $\Lambda^q > (\Lambda + \varepsilon)^k$. Using the monotonicity of f, (3.10) and (3.12), the following chain of inequalities hold in Ω:

$$\begin{aligned}\mathcal{F}_k(D^2 w) &\geq f(-w)\\ &\geq f\big(-\Lambda u\big)\\ &\geq \Lambda^q f(-u)\\ &> (\Lambda + \varepsilon)^k f(-u)\\ &\geq (\Lambda + \varepsilon)^k \mathcal{F}_k(D^2 u)\\ &= \mathcal{F}_k\big(D^2((\Lambda + \varepsilon)u)\big).\end{aligned}$$

Thus, by the comparison principle, we have

$$w(x) \leq (\Lambda + \varepsilon)u(x) \quad \forall x \in \Omega.$$

But this contradicts the choice of Λ. The lemma is proved. □

Remark 3.5. By Lemma 3.4, a result of uniqueness holds for problem (2.4). If B is a ball then the minimizer v for $S_{k,q}(B)$ is radially symmetric. Indeed, v satisfies (2.4) with $\Omega = B$. Since the operator $\mathcal{F}_k(D^2 v)$ is invariant for rotations, if v were not symmetric, then, by a suitable rotation, we would find a different solution $\tilde{v}$, contradicting the uniqueness for problem (2.4).

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n$ be a $(k-1)$-convex domain. Suppose f satisfies condition* (3.9)*. Let $v \in \Phi_{k-1}(\Omega)$ be a super-solution of*

$$\mathcal{F}_k(D^2 v) = f(-v) \quad \text{in } \Omega,$$

and let $v^*_{k-1}(x)$ be its $(k-1)$-symmetrand. Let $z \in \Phi_{k-1}(B_R)$ be a sub-solution of the above equation in the ball B_R, where $R := \eta_{n-k+1}(\Omega)$. If $v^*(r) = v^*_{k-1}(x)$ and $z(x) = z(r)$ for $|x| = r$, we have

$$v^*(r) \geq z(r), \quad 0 \leq r \leq R.$$

Proof. Let $w < 0$ be a (radial) solution of

$$\mathcal{F}_k(D^2 w) = f(-v^*) \text{ in } B_R \quad \text{and} \quad w = 0 \text{ on } \partial B_R. \tag{3.13}$$

In fact w is given explicitly by

$$w(r) = -n^{\frac{1}{k}} \binom{n}{k}^{-\frac{1}{k}} \int_r^R \left(s^{k-n} \int_0^s t^{n-1} f(-v^*(t))dt \right)^{\frac{1}{k}} ds.$$

Therefore w satisfies

$$\left(\frac{dw(r)}{dr}\right)^k = n\binom{n}{k}^{-1} r^{k-n}\int_0^r t^{n-1} f(-v^*(t))dt.$$

Comparing this equation with the inequality (3.2) in Lemma 3.3, we find

$$\frac{dv^*(r)}{dr} \le \frac{dw(r)}{dr}, \quad 0 < r < R.$$

Integrating this on (r, R) for any $0 < r < R$, we get

$$v^*(r) \ge w(r) \quad \text{for } 0 \le r \le R.$$

With $v^*(x) = v^*(r)$ and $w(x) = w(r)$ for $|x| = r$, we have

$$v^*(x) \ge w(x) \quad \text{for } x \in B_R. \tag{3.14}$$

Using (3.13) and (3.14), we find that

$$\mathcal{F}_k(D^2 w) = f(-v^*) \le f(-w) \quad \text{in } B_R.$$

In summary we see that w and z satisfy

$$\begin{cases} \mathcal{F}_k(D^2 w) \le f(-w) & \text{in } B_R \quad \text{and} \quad w = 0 \quad \text{on } \partial B_R, \\ \mathcal{F}_k(D^2 z) \ge f(-z) & \text{in } B_R \quad \text{and} \quad z = 0 \quad \text{on } \partial B_R. \end{cases}$$

By Lemma 3.4, we have

$$w(x) \ge z(x) \quad \text{in } B_R. \tag{3.15}$$

Thus, from (3.14) and (3.15) we conclude

$$v^*(x) \ge z(x) \quad \text{in } B_R.$$

The theorem follows. □

Corollary 3.7. *Let $g : [0, \infty) \to [0, \infty)$ be a non-decreasing smooth function such that $g(0) = 0$ and $g(t) > 0$ for $t > 0$. Under the assumptions of Theorem 3.6, we have*

$$\int_\Omega g(-v)dx \le \int_{\Omega^*_{k-1}} g(-z)dx.$$

Moreover we have

$$\inf_\Omega v(x) \ge \inf_{\Omega^*_{k-1}} z(x).$$

Furthermore, equality holds in each of these inequalities if and only if v and z are solutions of the corresponding equation and Ω is a ball.

Proof. Let $\mu(t) = |\{x \in \Omega : v(x) < t\}|$, and let

$$\mu^*(t) = |\{x \in \Omega^*_{n-1} : v^*_{n-1}(x) < t\}|.$$

By (2.7) we have

$$\mu(t) \le \mu^*(t) \quad \forall t \in (m_0, 0), \quad m_0 = \min_\Omega v.$$

We note that $m_0 = v^*_{n-1}(0)$, and

$$\begin{aligned}\int_\Omega g(-v)dx &= \int_{m_0}^0 g(-t)d\mu(t) = \int_{m_0}^0 g'(-t)\mu(t)dt \\ &\le \int_{m_0}^0 g'(-t)\mu^*(t)dt = \int_{m_0}^0 g(-t)d\mu^*(t) \\ &= \int_{\Omega^*_{n-1}} g(-v^*_{n-1})dx.\end{aligned}$$

Since by Theorem 3.6 we have $-v^*_{k-1}(x) \le -z(x)$, the first statement follows. The second statement is true because

$$\inf_\Omega v(x) = v^*_{k-1}(0) \ge z(0) = \inf_{\Omega^*_{k-1}} z(x).$$

Finally, when equality holds in each of these inequalities we must have

$$v^*_{k-1}(x) = z(x) \quad \forall x \in \Omega^*_{k-1}.$$

Hence, we must have equality in the inequality of Lemma 3.3, but this implies Ω is a ball. The corollary is proved. □

For $u \in \Phi_{k-1}(\Omega)$, we define the Hessian integral

$$I(\Omega, u) := \int_\Omega (-u)\mathcal{F}_k(D^2u)dx.$$

Proposition 3.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $(k-1)$-convex domain, and suppose f satisfies condition* (3.9). *Let $v \in \Phi_{k-1}(\Omega)$ be a super-solution of*

$$\mathcal{F}_k(D^2v) = f(-v) \quad \text{in } \Omega,$$

and let $v^*_{k-1}(x)$ be its $(k-1)$-symmetrand. If $z \in \Phi_{k-1}(\Omega^*_{k-1})$ is a sub-solution of the above equation in the ball Ω^*_{k-1}, we have

$$I(\Omega, v) \le I(\Omega^*_{k-1}, z).$$

Proof. We have

$$
\begin{aligned}
I(\Omega, v) &= \int_\Omega (-v)\mathcal{F}_k(D^2 v)dx \\
&\le \int_\Omega (-v) f(-v)dx && \text{since } v \text{ is a super-solution} \\
&\le \int_{\Omega^*_{k-1}} (-z) f(-z)dx && \text{by Corollary 3.7} \\
&\le \int_{\Omega^*_{k-1}} (-z)\mathcal{F}_k(D^2 z)dx && \text{since } z \text{ is a sub-solution} \\
&= I(\Omega^*_{k-1}, z).
\end{aligned}
$$

The proposition is proved. □

4 Eigenvalues

As already observed, a typical example of f which satisfies condition (3.9) is as follows: $f(t) = t^q, 0 < q < k$. In this section we consider the case $q = k$, that is, the eigenvalue problem

$$\mathcal{F}_k(D^2 u) = \lambda(\Omega)(-u)^k, \quad \lambda(\Omega) > 0, \quad u < 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{4.1}$$

Recall that $1 \le k \le n$. In case of $k > 1$ we assume Ω being $(k-1)$-convex. In this situation, it is known (see [9, Theorem 1.3]) that there exists $\lambda(\Omega) > 0$ corresponding to which problem (4.1) admits a solution. Of course $\lambda(\Omega)$ depends also on k, but in this section k will be fixed. We have

$$\lambda(\Omega) = \inf\left\{\int_\Omega (-v)\mathcal{F}_k(D^2 v)dx : v \in A_{k-1}(\Omega),\ \int_\Omega (-v)^{k+1}dx = 1\right\}. \tag{4.2}$$

A minimizer u of (4.2) satisfies (4.1) and is unique (see [31, Theorem 4.1]). Of course, we also have

$$\lambda(\Omega) = \inf\left\{\frac{\int_\Omega (-v)\mathcal{F}_k(D^2 v)dx}{\int_\Omega (-v)^{k+1}dx} : v \in A_{k-1}(\Omega)\right\}.$$

Remark 4.1. As in Remark 3.2, we may ask if a minimizer u of $\lambda(\Omega)$ always belongs to $\Phi_{k-1}(\Omega)$. The answer is positive for $k = 1$ and for $k = n$, for the same reasons as in Remark 3.2. For $1 < k < n$ and Ω convex, we believe that a minimizer u of $\lambda(\Omega)$ has convex level sets, but we do not have a proof. The result holds in case of $n = 3$ and $k = 2$, see [16] and [22].

Proposition 4.2. *Given a $(k-1)$-convex domain Ω, let Ω^*_{k-1} be the ball with radius $R := \eta_{n-k+1}(\Omega)$. If $\lambda(\Omega)$ has a minimizer $u \in \Phi_{k-1}(\Omega)$, then*

$$\lambda(\Omega) \geq \lambda(\Omega^*_{k-1}). \tag{4.3}$$

Equality holds in (4.3) *if and only if Ω is a ball.*

Proof. If $u \in \Phi_{k-1}(\Omega)$ is a minimizer for $\lambda(\Omega)$, we have

$$\lambda(\Omega) = \frac{1}{k}\int_\Omega T^{(k-1)}_{ij}(D^2u)u_iu_j\,dx, \quad \int_\Omega (-u)^{k+1}dx = 1.$$

Putting $z = u^*_{k-1}$, by using (2.11) with $p = k+1$ we find

$$\lambda(\Omega) \geq \frac{1}{k}\int_{\Omega^*_{k-1}} T^{(k-1)}_{ij}(D^2z)z_iz_j\,dx.$$

By (2.8) with $h = k-1$ and $p = k+1$ we have

$$1 = \int_\Omega (-u)^{k+1}dx \leq \int_{\Omega^*_{k-1}} (-z)^{k+1}dx.$$

Therefore, we find

$$\lambda(\Omega) \geq \frac{\frac{1}{k}\int_{\Omega^*_{k-1}} T^{(k-1)}_{ij}(D^2z)z_iz_j\,dx}{\int_{\Omega^*_{k-1}}(-z)^{k+1}dx} \geq \lambda(\Omega^*_{k-1}).$$

Now suppose equality holds in (4.3), and let $\lambda := \lambda(\Omega) = \lambda(\Omega^*_{k-1})$. Let us consider an eigenfunction $u \in \Phi_{k-1}(\Omega)$ of problem (4.1) in Ω and let $u^* := u^*_{k-1}$ be its $(k-1)$-symmetrand. Then, by Lemma 3.3 with $f(s) = \lambda s^k$, we see that

$$\left(\frac{du^*(r)}{dr}\right)^k \leq n\binom{n}{k}^{-1} r^{k-n}\lambda\int_0^r t^{n-1}(-u^*(t))^k dt, \quad 0 < r < R. \tag{4.4}$$

We show that equality holds, and hence by Lemma 3.3, it will follow that Ω is a ball. To this end, let v be an eigenfunction of problem (4.1) in Ω^*_{k-1}, that is, in the ball B_R centered at the origin and radius $R = \eta_{n-k+1}(\Omega)$. We assume that v is normalized so that $u^*(0) = v(0) < 0$. Then,

$$\left(\frac{dv(r)}{dr}\right)^k = n\binom{n}{k}^{-1} r^{k-n}\lambda\int_0^r t^{n-1}(-v(t))^k dt, \quad 0 < r < R. \tag{4.5}$$

Let $w \in \Phi_{k-1}(B_R)$ be a (radial) solution of $\mathcal{F}_k(D^2w) = \lambda(-u^*)^k$ in B_R. Then,

$$\left(\frac{dw(r)}{dr}\right)^k = n\binom{n}{k}^{-1} r^{k-n}\lambda\int_0^r t^{n-1}(-u^*(t))^k dt, \quad 0 < r < R.$$

By comparing this with (4.4) we see that

$$\frac{du^*(r)}{dr} \le \frac{dw(r)}{dr}, \quad 0 < r < R.$$

Integrating this on (r, R) for any $0 < r < R$, we get $-u^*(r) \le -w(r)$ in $(0, R)$. Hence,

$$\mathcal{F}_k(D^2 w) = \lambda(-u^*)^k \le \lambda(-w)^k \text{ in } B_R, \quad w = 0 \text{ on } \partial B_R.$$

Therefore we have

$$\frac{\int_{B_R}(-w)\mathcal{F}_k(D^2 w)dx}{\int_{B_R}(-w)^{k+1}dx} \le \lambda.$$

This implies that w is a λ-eigenfunction of problem (4.1) in B_R, and therefore we have $w = cv$ for some constant $c > 0$. Consequently,

$$\lambda(-u^*)^k = \mathcal{F}_k(D^2 w) = c^k \mathcal{F}_k(D^2 v) = \lambda(-cv)^k$$

in B_R. That is $cv = u^*$ in $(0, R)$. In view of the normalization $u^*(0) = v(0)$, we find that $u^* = v$. Thus, using this and (4.5), we see that equality holds in (4.4), as claimed. The proposition is proved. □

Assume Ω is $(k-1)$-convex, and suppose problem (4.1) has an eigenfunction $u \in \Phi_{k-1}(\Omega)$. Let B_{R_0} be the ball centered at the origin and such that

$$\lambda(B_{R_0}) = \lambda(\Omega).$$

Let v be the (radial) eigenfunction corresponding to $\lambda(B_{R_0})$ normalized either such that

$$\inf_{B_{R_0}} v(x) = \inf_{\Omega} u(x), \tag{4.6}$$

or such that

$$\int_{B_{R_0}} (-v(x))^p dx = \int_{\Omega^*_{k-1}} (-u^*(x))^p dx, \quad 0 < p < \infty. \tag{4.7}$$

Theorem 4.3. *Let Ω be $(k-1)$-convex, and let $u \in \Phi_{k-1}(\Omega)$ be a fixed eigenfunction of problem* (4.1). *Let B_{R_0} be a ball with radius R_0 centered at the origin such that $\lambda(B_{R_0}) = \lambda(\Omega) =: \lambda$. Let v be an eigenfunction of problem* (4.1) *with $\Omega = B_{R_0}$. If v is normalized as in* (4.6)*, then*

$$u^*(r) \le v(r), \quad 0 < r < R_0;$$

if v is normalized as in (4.7)*, then*

$$\int_0^r t^{n-1}(-u^*(t))^p dt \le \int_0^r t^{n-1}(-v(t))^p dt, \quad 0 < r < R_0.$$

Proof. The idea of the proof is inspired by [7]. If R is the radius of Ω_{k-1}^*, by Proposition 4.1 we have $\lambda(B_{R_0}) \geq \lambda(B_R)$, which yields $R_0 \leq R$. If $R_0 = R$, then we have $\Omega = B_R$, and there is nothing to prove. Thus, assume $R_0 < R$. We write $u^*(x) = u^*(r)$ and $v(x) = v(r)$ for $|x| = r$.

Let v be normalized as in (4.6). Since $u^*(0) = v(0)$ and $u^*(R_0) < v(R_0) = 0$, if $u^*(r) \leq v(r)$ does not hold, there exists a point $r_0 \in (0, R_0)$ such that $u^*(r_0) = v(r_0)$ and either $u^*(r) \geq v(r)$ or $u^*(r) \leq v(r)$ for $0 < r < r_0$ with the inequalities being strict at some points. By Lemma 3.3 with $f(t) = \lambda t^k$, we have

$$\left(\frac{du^*(r)}{dr}\right)^k \leq n\binom{n}{k}^{-1} r^{k-n}\lambda \int_0^r t^{n-1}(-u^*(t))^k dt, \tag{4.8}$$

and

$$\left(\frac{dv(r)}{dr}\right)^k = n\binom{n}{k}^{-1} r^{k-n}\lambda \int_0^r t^{n-1}(-v(t))^k dt. \tag{4.9}$$

In case of $u^*(r) \geq v(r)$ on $(0, r_0)$ by (4.8) and (4.9), we get

$$\frac{du^*(r)}{dr} \leq \frac{dv(r)}{dr}, \quad 0 < r < r_0,$$

with the inequality being strict at some points. Integrating over $(0, r_0)$ and recalling that $u^*(0) = v(0)$ we find $u^*(r_0) < v(r_0)$, a contradiction.

In case of $u^*(r) \leq v(r)$ on $(0, r_0)$, we proceed as follows. Define

$$w(r) = \begin{cases} u^*(r), & r \in (0, r_0], \\ v(r), & r \in (r_0, R_0]. \end{cases}$$

By (4.8) and (4.9) we get

$$\big(w'(r)\big)^k \leq n\binom{n}{k}^{-1} r^{k-n}\lambda \int_0^r t^{n-1}(-w(t))^k dt, \quad 0 < r < R_0. \tag{4.10}$$

Furthermore, we have $w(r) < 0$ for $r \in (0, R_0)$, $w(R_0) = 0$, and clearly $w(r)$ is not equal to $cv(r)$ for any constant c. Therefore,

$$\lambda < \frac{\int_{B_{R_0}}(-w)\mathcal{F}_k(D^2 w)dx}{\int_{B_{R_0}}(-w)^{k+1}dx}. \tag{4.11}$$

As w is a radial function, with $w(r) = w(x)$ for $r = |x|$ we have (see [28, p. 99])

$$\mathcal{F}_k(D^2 w) = \binom{n-1}{k-1} r^{-n+1}\left(\frac{r^{n-k}}{k}(w')^k\right)'.$$

Therefore, using the latter equation and (4.10) we find

$$\begin{aligned}
\int_{B_{R_0}} (-w)\mathscr{F}_k(D^2w)dx &= n\omega_n \int_0^{R_0} (-w)\binom{n-1}{k-1}\left(\frac{r^{n-k}}{k}(w')^k\right)' dr \\
&= \omega_n \binom{n}{k} \int_0^{R_0} w' r^{n-k}(w')^k dr \\
&\leq \lambda n\omega_n \int_0^{R_0} w'(r)dr \int_0^r t^{n-1}(-w(t))^k dt \\
&= \lambda n\omega_n \int_0^{R_0} r^{n-1}(-w(r))^{k+1} dr \\
&= \lambda \int_{B_{R_0}} (-w)^{k+1} dx.
\end{aligned}$$

Insertion of this inequality into (4.11) yields $\lambda < \lambda$, a contradiction. Hence, we must have $u^*(r) \leq v(r)$ on $[0, R_0]$, as claimed.

Let v be normalized as in (4.7). It follows that there exists (at least) one point $r_0 \in (0, R_0)$ such that $u^*(r_0) = v(r_0)$. We claim that there is only one point r_0 such that $u^*(r_0) = v(r_0)$. Arguing by contradiction, we consider two possibilities. Assume $u^*(r_1) = v(r_1)$, $u^*(r_0) = v(r_0)$, $u^*(r) < v(r)$ on $(0, r_1)$ and $u^*(r) \geq v(r)$ on (r_1, r_0). Putting

$$w(r) = \begin{cases} u^*(r), & r \in (0, r_1], \\ v(r), & r \in (r_1, R_0]. \end{cases}$$

we see that w satisfies inequality (4.10). Arguing as in the previous case, we get a contradiction.

Now let $u^*(r_1) = v(r_1)$, $u^*(r_0) = v(r_0)$, $u^*(r) \geq v(r)$ on $(0, r_1)$ and $u^*(r) < v(r)$ on (r_1, r_0). Putting

$$w(r) = \begin{cases} v(r), & r \in (0, r_1], \\ u^*(r), & r \in (r_1, r_0], \\ v(r), & r \in (r_0, R_0], \end{cases}$$

w satisfies again inequality (4.10), and arguing as in the previous case we still get a contradiction.

Hence, we must have

$$u^*(r) > v(r), \quad 0 < r < r_0,$$

and

$$u^*(r) < v(r), \quad r_0 < r < R_0.$$

By (4.7) we have

$$\int_0^{R_0} t^{n-1}(-u^*(t))^p dt \le \int_0^{R_0} t^{n-1}(-v(t))^p dt.$$

Let us rewrite this inequality as

$$\int_{r_0}^{R_0} t^{n-1}\big[(-u^*(t))^p - (-v(t))^p\big]dt \le \int_0^{r_0} t^{n-1}\big[(-v(t))^p - (-u^*(t))^p\big]dt.$$

Since $-u^*(t) > -v(t)$ for $r_0 < t < R_0$, it follows that, for any $r \in [r_0, R_0]$,

$$\int_{r_0}^{r} t^{n-1}\big[(-u^*(t))^p - (-v(t))^p\big]dt \le \int_0^{r_0} t^{n-1}\big[(-v(t))^p - (-u^*(t))^p\big]dt,$$

that is,

$$\int_0^{r} t^{n-1}(-u^*(t))^p dt \le \int_0^{r} t^{n-1}(-v(t))^p dt, \quad r \in [0, R_0].$$

The proof of the theorem is complete. □

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