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Multiple solutions to a Dirichlet problem with p-Laplacian and nonlinearity depending on a parameter

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Abstract. The homogeneous Dirichlet problem for an elliptic equation with p-Laplacian and concave-convex reaction term depending on a parameter $\lambda > 0$ is investigated. At least five nontrivial solutions for all λ sufficiently small are obtained via variational methods, truncation techniques and sub- and super-solution arguments. The special case $p = 2$ is also examined.

Keywords. Concave-convex nonlinearities, p-Laplacian, constant-sign solutions, multiple solutions.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, let $p \in]1, +\infty[$, and let p^* be the critical Sobolev exponent. Consider the homogeneous Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x, u, \lambda) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where Δ_p denotes the p-Laplace differential operator, namely

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

while the reaction term $f : \Omega \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ satisfies Carathéodory's conditions.

The literature concerning (P_λ) is by now very wide and many existence, multiplicity, or bifurcation-type results are already available. In particular, a meaningful case occurs when

$$f(x, t, \lambda) := \lambda|t|^{q-2}t + |t|^{r-2}t, \quad (x, t, \lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^+, \tag{1.1}$$

with $1 < q < p < r < p^*$. If $p = 2$, then (1.1) reduces to a so-called concave-convex nonlinearity and, after the seminal paper [2], the corresponding problem has been thoroughly investigated; see [26] for an exhaustive account. A similar comment holds true also when $p \neq 2$, in which case we cite [3, 11, 16, 21]. The very recent work [21] contains a bifurcation theorem, describing the dependence of positive solutions to (P_λ) on the parameter $\lambda > 0$, where the reaction term f takes the form

$$f(x,t,\lambda) := \lambda g(x,t) + h(x,t), \quad (x,t,\lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^+,$$

for suitable $g, h : \Omega \times \mathbb{R} \to \mathbb{R}$ of Carathéodory's type.

Roughly speaking, in this paper, one requires that $f(x,\cdot,\lambda)$ has a $(p-1)$-sub-linear growth at zero and is $(p-1)$-super-linear at infinity, i.e.,

$$\lim_{t\to 0} \frac{f(x,t,\lambda)}{|t|^{p-2}t} = +\infty,$$

$$\lim_{|t|\to +\infty} \frac{f(x,t,\lambda)}{|t|^{p-2}t} = +\infty \quad \text{uniformly with respect to } x \in \Omega.$$

Specifically, the behavior of $f(x,\cdot,\lambda)$ at infinity stems from an hypothesis, patterned after that of [19], which does not imply the usual Ambrosetti–Rabinowitz condition; cf. Remarks 2.5–2.7. Under these assumptions, the existence of $\lambda^* > 0$ such that, for all $\lambda \in]0, \lambda^*[$, (P_λ) possesses at least five nontrivial weak solutions belonging to $C_0^1(\overline{\Omega})$, four of which have constant sign, is established in Theorem 4.1. Moreover, when $p = 2$, (P_λ) has a further nontrivial solution; see Theorem 4.3.

As an example, if $1 < q < p < r < p^*$ and $\lambda > 0$ is sufficiently small, then Theorem 4.1 ensures that the equation

$$-\Delta_p u = \lambda |u|^{q-2}u + |u|^{p-2}u \log(1 + |u|^p) \quad \text{in } \Omega$$

possesses at least five nontrivial solutions $v_0, v_1, u_0, u_1, w \in C_0^1(\overline{\Omega})$ such that $v_i < 0 < u_i$, $i = 1, 2$, in Ω.

The technical approach we follow employs truncation methods, sub- and super-solution arguments, besides a careful computation of critical groups of the involved energy functionals and Morse's identity.

Let us finally point out that, in latest years, also numerous multiplicity theorems for Neumann problems with nonlinearities having various growth rates have been published; see for example [8, 20, 22] and their bibliographies.

2 Preliminaries and basic assumptions

Let $(X, \|\cdot\|)$ be a real Banach space. If V is a subset of X, we write $\overline{V}$ for the closure of V, ∂V for the boundary of V, and $\mathrm{int}(V)$ for the interior of V. Given $x \in X$, $\rho > 0$, the symbol $B_X(x, \rho)$ indicates the open ball of radius ρ centered at x, $(X^*, \|\cdot\|_{X^*})$ denotes the dual space of X, $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X and X^*, while $x_n \to x$ (respectively, $x_n \rightharpoonup x$) in X means 'the sequence $\{x_n\}$ converges strongly (respectively, weakly) in X'.

The next elementary but useful result [21, Proposition 2.1] will be used in Section 3.

Proposition 2.1. *Suppose $(X, \|\cdot\|)$ is an ordered Banach space with order cone K. If $x_0 \in \mathrm{int}(K)$, then to every $z \in K$ there corresponds a $t_z > 0$ such that*

$$t_z x_0 - z \in K.$$

A function $\Phi : X \to \mathbb{R}$ fulfilling

$$\lim_{\|x\|\to+\infty} \Phi(x) = +\infty$$

is called coercive. Let $\Phi \in C^1(X)$. The Cerami compactness condition for Φ reads as follows:

(C) *Every sequence $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ is bounded and*

$$\lim_{n\to+\infty} (1 + \|x_n\|)\|\Phi'(x_n)\|_{X^*} = 0$$

has a convergent subsequence.

It is known that (C) includes the classical Palais–Smale condition, namely

(PS) *Every sequence $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ is bounded and*

$$\|\Phi'(x_n)\|_{X^*} \to 0$$

possesses a convergent subsequence,

as a special case. Moreover, the version below of the Mountain Pass Theorem holds true; see for instance [13, Corollary 5.2.7] or [24]. Define, for every $c \in \mathbb{R}$,

$$\Phi^c := \{x \in X : \Phi(x) \le c\}, \quad K_c(\Phi) := K(\Phi) \cap \Phi^{-1}(c),$$

where, as usual, $K(\Phi)$ denotes the critical set of Φ, i.e.,

$$K(\Phi) := \{x \in X : \Phi'(x) = 0\}.$$

Theorem 2.2. *Assume $\Phi \in C^1(X)$ satisfies* (C) *and there exist $x_0, x_1 \in X$ as well as $\rho \in]0, \|x_1 - x_0\|[$ such that*

$$\max\{\Phi(x_0), \Phi(x_1)\} < \inf_{x\in\partial B_X(x_0,\rho)} \Phi(x).$$

If $c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi(\gamma(t))$, where

$$\Gamma := \{\gamma \in C^0([0,1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $K_c(\Phi) \neq \emptyset$.

An operator $A : X \to X^*$ is called of type $(\mathrm{S})_+$ if

$$x_n \rightharpoonup x \quad \text{in } X, \quad \limsup_{n\to+\infty} \langle A(x_n), x_n - x\rangle \le 0$$

imply $x_n \to x$. The next simple result is more or less known and will be employed in Section 4.

Proposition 2.3. *Let X be reflexive and let $\Phi \in C^1(X)$ be coercive. Assume Φ satisfies $\Phi' = A + B$, where $A : X \to X^*$ is of type* $(\mathrm{S})_+$ *while $B : X \to X^*$ is compact. Then Φ fulfils* (PS).

Proof. Pick a sequence $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ turns out to be bounded and

$$\lim_{n\to+\infty} \|\Phi'(x_n)\|_{X^*} = 0. \tag{2.1}$$

By the reflexivity of X, besides the coercivity of Φ, we may suppose, up to subsequences, $x_n \rightharpoonup x$ in X. Since B is compact, using (2.1) and taking a subsequence when necessary, one has

$$\lim_{n\to+\infty} \langle A(x_n), x_n - x\rangle = \lim_{n\to+\infty} \big(\langle \Phi'(x_n), x_n - x\rangle - \langle B(x_n), x_n - x\rangle\big) = 0\,.$$

This forces $x_n \to x$ in X, because A is of type $(\mathrm{S})_+$, as desired. □

Given a topological pair (A, B) satisfying $B \subset A \subseteq X$, the symbol $H_k(A, B)$, $k \in \mathbb{N}_0$, indicates the k^{th}-relative singular homology group of (A, B) with integer coefficients. If $x_0 \in K_c(\Phi)$ is an isolated point of $K(\Phi)$, then

$$C_k(\Phi, x_0) := H_k(\Phi^c \cap U, \Phi^c \cap U \setminus \{x_0\}), \quad k \in \mathbb{N}_0,$$

are the critical groups of Φ at x_0. Here, U stands for any neighborhood of x_0 such that $K(\Phi) \cap \Phi^c \cap U = \{x_0\}$. By excision, this definition does not depend on the choice of U. Suppose the function Φ fulfils (PS). When $\Phi|_{K(\Phi)}$ is bounded below while $c < \inf_{x\in K(\Phi)} \Phi(x)$, we write

$$C_k(\Phi, \infty) := H_k(X, \Phi^c), \quad k \in \mathbb{N}_0,$$

which is independent from c because of the second deformation theorem ([7, Lemma I.3.2]). If $K(\Phi)$ is finite, then setting

$$M(t, x) := \sum_{k=0}^{+\infty} \operatorname{rank} C_k(\Phi, x) t^k,$$

$$P(t, \infty) := \sum_{k=0}^{+\infty} \operatorname{rank} C_k(\Phi, \infty) t^k \quad \forall\, (t, x) \in \mathbb{R} \times K(\Phi),$$

the following Morse relation holds:

$$\sum_{x \in K(\Phi)} M(t,x) = P(t,\infty) + (1+t)Q(t),$$

where $Q(t)$ denotes a formal series with nonnegative integer coefficients; see for instance [6, Proposition 3.7]. Now, let X be a Hilbert space and let $x \in K(\Phi)$. Assume that $\Phi \in C^2(U)$, where U denotes any neighborhood of x. If $\Phi''(x)$ turns out to be invertible, then x is called non-degenerate. The Morse index d of x is the supremum of the dimensions of the vector subspaces of X on which $\Phi''(x)$ turns out to be negative definite. When x is non-degenerate and with Morse index d, one has

$$C_k(\Phi, x) = \delta_{k,d}\mathbb{Z}, \quad k \in \mathbb{N}_0.$$

The monograph [7] represents a general reference on the subject.

Throughout the paper, we denote by Ω a bounded domain of the real euclidean N-space $(\mathbb{R}^N, |\cdot|)$ with a smooth boundary $\partial\Omega$, $p \in]1, +\infty[$, $p' := p/(p-1)$, $\|\cdot\|_p$ is the usual norm of $L^p(\Omega)$, and $W_0^{1,p}(\Omega)$ indicates the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. On $W_0^{1,p}(\Omega)$ we introduce the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}, \quad u \in W_0^{1,p}(\Omega).$$

Write p^* for the critical exponent of the Sobolev embedding $W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$. Recall that $p^* = Np/(N-p)$ if $p < N$, $p^* = +\infty$ otherwise, and the embedding is compact whenever $1 \le q < p^*$.

Let $W^{-1,p'}(\Omega)$ be the dual space of $W_0^{1,p}(\Omega)$ and let

$$A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$$

be the nonlinear operator stemming from the negative p-Laplacian, i.e.,

$$\langle A(u), v \rangle := \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) \, dx \quad \forall\, u, v \in W_0^{1,p}(\Omega).$$

A standard argument yields the following auxiliary result; see, e.g., [23].

Proposition 2.4. *The operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is of type* $(\mathrm{S})_+$.

Define $C_0^1(\overline{\Omega}) := \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$. Obviously, $C_0^1(\overline{\Omega})$ is an ordered Banach space with order cone

$$C_0^1(\overline{\Omega})_+ := \{u \in C_0^1(\overline{\Omega}) : u(x) \ge 0 \ \forall\, x \in \overline{\Omega}\}.$$

Moreover, one has

$$\mathrm{int}(C_0^1(\overline{\Omega})_+) = \left\{ u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega,\ \frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega \right\},$$

where $n(x)$ is the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$; see, for example, [13, Remark 6.2.10].

Theorem A.0.6 in [23] ensures that $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bijective. Thus, we can find a function $e \in W_0^{1,p}(\Omega)$ such that

$$-\Delta_p e(x) = 1 \quad \text{in } \Omega. \tag{2.2}$$

Theorems 1.5.6 and 1.5.7 of [12] then give $e \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$.

Finally, 'measurable' always signifies Lebesgue measurable while $m(E)$ indicates the Lebesgue measure of E. Put, provided $t \in \mathbb{R}$, $u, v : \Omega \to \mathbb{R}$,

$$t^- := \max\{-t, 0\}, \quad t^+ := \max\{t, 0\}, \quad \Omega(u < v) := \{x \in \Omega : u(x) < v(x)\}.$$

The meaning of $\Omega(u > v)$, etc., is analogous. If u, v belong to a given function space X and $u(x) \le v(x)$ in Ω, then we set

$$[u, v] := \{w \in X : u(x) \le w(x) \le v(x) \text{ in } \Omega\}.$$

To avoid unnecessary technicalities, 'for every $x \in \Omega$' will take the place of 'for almost every $x \in \Omega$' and the variable x will be omitted when no confusion can arise.

Let $(x, t, \lambda) \mapsto f(x, t, \lambda)$, $(x, t, \lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^+$, be measurable in x for all $(t, \lambda) \in \mathbb{R} \times \mathbb{R}^+$, continuous with respect to t for every $(x, \lambda) \in \Omega \times \mathbb{R}^+$, and such that $f(x, 0, \lambda) = 0$, $(x, \lambda) \in \Omega \times \mathbb{R}^+$. Write, as usual,

$$F(x, z, \lambda) := \int_0^z f(x, t, \lambda)dt \quad \forall\, (x, z, \lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^+.$$

The hypotheses below will be posited in the sequel.

(f_1) *There exist* $a_1 : \mathbb{R}^+ \to \mathbb{R}^+$, $a_2 > 0$, $r \in\,]p, p^*[$ *satisfying*

$$\lim_{\lambda \to 0^+} a_1(\lambda) = 0, \quad |f(x, t, \lambda)| \le a_1(\lambda) + a_2 |t|^{r-1} \quad \text{in } \Omega \times \mathbb{R} \times \mathbb{R}^+.$$

(f_2) $\lim_{|z| \to +\infty} \frac{F(x,z,\lambda)}{|z|^p} = +\infty$ *uniformly with respect to* $(x, \lambda) \in \Omega \times \mathbb{R}^+$.

(f_3) *To every* $\lambda, \rho > 0$ *there corresponds* $\mu_{\lambda,\rho} > 0$ *such that the function*

$$t \mapsto f(x, t, \lambda) + \mu_{\lambda,\rho} |t|^{p-2} t$$

is non-decreasing in $[-\rho, \rho]$ *for all* $x \in \Omega$.

(f_4) *Let* $\lambda > 0$ *and let*

$$\xi_\lambda(x, z) := zf(x, z, \lambda) - pF(x, z, \lambda), \quad (x, z) \in \Omega \times \mathbb{R}.$$

Then there exists $\alpha_\lambda \in L^1(\Omega)$ *fulfilling*

$$\alpha_\lambda(x) \geq 0\,, \quad \xi_\lambda(x, z') \leq \xi_\lambda(x, z'') + \alpha_\lambda(x) \quad \text{in } \Omega$$

provided $z', z'' \in \mathbb{R}$, $|z'| \leq |z''|$, *and* $z'z'' \geq 0$.

(f_5) *To every* $\lambda > 0$ *there correspond* $\delta_\lambda > 0$, $\theta_\lambda \in]1, p[$ *such that*

$$\operatorname{ess\,inf}_{x\in\Omega} F(x, \delta_\lambda, \lambda) > 0, \quad \theta_\lambda F(x, z, \lambda) \geq zf(x, z, \lambda) > 0 \tag{2.3}$$

for all $(x, z) \in \Omega \times \mathbb{R}$ *with* $0 < |z| \leq \delta_\lambda$.

If we fix $\lambda > 0$, then a useful comparison between (f_1)–(f_5) and some classical assumptions can be done.

Remark 2.5. Hypothesis (f_2) is weaker than the Ambrosetti–Rabinowitz condition below, as a simple computation shows.

(AR) There exist $\theta > p$, $M > 0$ such that

$$0 < \theta F(x, z, \lambda) \leq zf(x, z, \lambda)$$

for every $x \in \Omega$, $|z| \geq M$.

Moreover, from [19, Lemma 2.4] we know that (f_2) and (f_4) force

$$\lim_{|t|\to+\infty} \frac{f(x, t, \lambda)}{|t|^{p-2}t} = +\infty \quad \text{uniformly in } x \in \Omega,$$

i.e., $f(x, \cdot, \lambda)$ turns out to be $(p-1)$-superlinear at infinity.

Remark 2.6. Assume $f(\cdot, \cdot, \lambda) \in C^0(\overline{\Omega} \times \mathbb{R})$ while the function $z \mapsto \xi_\lambda(x, z)$ is non-decreasing in $[z_0, +\infty[$ and non-increasing in $]-\infty, -z_0]$, where $z_0 > 0$. Then (f_4) holds for constant α_λ; see [19, Lemma 2.4].

Remark 2.7. After integration, (2.3) produces a constant $a_3(\lambda) > 0$ such that

$$a_3(\lambda)|z|^{\theta_\lambda} \leq F(x, z, \lambda)$$

for all $(x, z) \in \Omega \times \mathbb{R}$ with $0 < |z| \leq \delta_\lambda$. Since $\theta_\lambda < p$, we get

$$\lim_{z\to 0} \frac{F(x, z, \lambda)}{|z|^p} = +\infty \quad \text{uniformly in } x \in \Omega.$$

So, if the limit $\lim_{t\to 0} \frac{f(x,t,\lambda)}{|t|^{p-2}t}$ exists, then it must be $+\infty$, and $f(x, \cdot, \lambda)$ exhibits a $(p-1)$-sub-linear behavior at zero.

Example 2.8. A simple but meaningful situation when all the assumptions stated above are satisfied is the following:

$$f(x,t,\lambda) := \lambda |t|^{q-2} t + |t|^{r-2} t, \quad (x,t,\lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^+,$$

where $1 < q < p < r < p^*$. The same conclusion remains true if

$$f(x,t) := \lambda |t|^{q-2} t + |t|^{p-2} t \log(1 + |t|^p).$$

However, in such a case, condition (AR) does not hold.

Finally, in the semi-linear case, i.e., $p = 2$, the following stronger version of (f_1) will be exploited:

(f_1') *$f(x,\cdot,\lambda) \in C^1(\mathbb{R} \setminus \{0\})$ for all $(x,\lambda) \in \Omega \times \mathbb{R}^+$. Moreover, there exist a function $a_1 : \mathbb{R}^+ \to \mathbb{R}^+$, $a_2 > 0$, $r \in]2, 2^*[$ such that*

$$\lim_{\lambda \to 0^+} a_1(\lambda) = 0, \quad |f_t'(x,t,\lambda)| \le a_1(\lambda) + a_2 |t|^{r-2} \quad \text{in } \Omega \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^+.$$

3 Constant-sign solutions

To shorten notation, write $X := W_0^{1,p}(\Omega)$. Define

$$\varphi_\lambda(u) := \frac{1}{p} \|u\|^p - \int_\Omega F(x, u(x), \lambda)\, dx, \quad u \in X. \tag{3.1}$$

Obviously, $\varphi_\lambda \in C^1(X)$. Moreover, one has

Theorem 3.1. *Let* (f_1), (f_3), *and* (f_5) *be fulfilled. Then there exists $\lambda^* > 0$ such that, for all $\lambda \in]0, \lambda^*[$, problem* (P_λ) *possesses two solutions $u_0 \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, $v_0 \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$, which are local minima of φ_λ.*

Proof. We claim that there is $\lambda^* > 0$ such that to every $\lambda \in]0, \lambda^*[$ there corresponds a constant $t_\lambda > 0$ satisfying

$$a_1(\lambda) + (t_\lambda \|e\|_\infty)^{r-1} a_2 < t_\lambda^{p-1}, \tag{3.2}$$

where $e \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ comes from (2.2). Indeed, if the assertion were false, then we could construct a sequence $\{\lambda_k\} \subseteq \mathbb{R}^+$ with the properties

$$\lim_{k \to +\infty} \lambda_k = 0, \quad t^{p-1} \le a_1(\lambda_k) + (t \|e\|_\infty)^{r-1} a_2, \quad t > 0,\ k \in \mathbb{N}.$$

Letting $k \to +\infty$, from (f_1) it would follow

$$t^{p-r} \le a_2 \|e\|_\infty^{r-1} \quad \forall\, t > 0.$$

However, this is impossible because $p < r$. Pick any $\lambda \in]0, \lambda^*[$ and put

$$\bar{u} := t_\lambda e. \tag{3.3}$$

Moreover, set

$$\bar{f}_+(x,t,\lambda) := \begin{cases} 0 & \text{if } t < 0, \\ f(x,t,\lambda) & \text{if } 0 \le t \le \bar{u}(x), \\ f(x,\bar{u}(x),\lambda) & \text{if } \bar{u}(x) < t, \end{cases} \quad (x,t) \in \Omega \times \mathbb{R}, \tag{3.4}$$

as well as

$$\bar{\varphi}_{\lambda,+}(u) := \frac{1}{p}\|u\|^p - \int_\Omega \bar{F}_+(x,u(x),\lambda)\,dx \quad \forall\, u \in X, \tag{3.5}$$

where

$$\bar{F}_+(x,z,\lambda) := \int_0^z \bar{f}_+(x,t,\lambda)\,dt.$$

By (3.4) the functional $\bar{\varphi}_{\lambda,+}$ turns out to be coercive. A simple argument, based on the compact embedding $X \subseteq L^p(\Omega)$, shows that it is also weakly sequentially lower semi-continuous. So, there exists $u_0 \in X$ fulfilling

$$\bar{\varphi}_{\lambda,+}(u_0) = \inf_{u\in X} \bar{\varphi}_{\lambda,+}(u). \tag{3.6}$$

Let us verify that $u_0 \neq 0$. Indeed, if $u \in C_0^1(\overline{\Omega})_+ \setminus \{0\}$, then, due to Proposition 2.1, for any $t > 0$ sufficiently small we have

$$tu(x) \le \bar{u}(x), \quad tu(x) \le \delta_\lambda \quad \text{in } \overline{\Omega}.$$

On account of (3.4), (f_5), and (f_1), this implies

$$\bar{\varphi}_{\lambda,+}(tu) = \frac{t^p}{p}\|u\|^p - \int_\Omega \bar{F}_+(x,tu(x),\lambda)\,dx \le \frac{t^p}{p}\|u\|^p - a_4 t^{\theta_\lambda}\|u\|^{\theta_\lambda},$$

where $a_4 > 0$. Since $\theta_\lambda < p$, fixing $t > 0$ small enough furnishes $\bar{\varphi}_{\lambda,+}(tu) < 0$. Hence,

$$\bar{\varphi}_{\lambda,+}(u_0) = \inf_{u\in X} \bar{\varphi}_{\lambda,+}(u) < 0 = \bar{\varphi}_{\lambda,+}(0),$$

which clearly means $u_0 \neq 0$, as desired. Now, from (3.6) it follows $\bar{\varphi}'_{\lambda,+}(u_0) = 0$, namely

$$\langle A(u_0), v\rangle = \int_\Omega \bar{f}_+(x,u_0(x),\lambda)v(x)\,dx, \quad v \in X. \tag{3.7}$$

Through (3.7), written for $v := -u_0^-$, we obtain that $\|u_0^-\|^p = 0$. Thus, $u_0 \geq 0$. Thanks to (3.4), (f_1), and (3.2), letting $v := (u_0 - \bar{u})^+$ in (3.7) yields

$$\begin{aligned}\langle A(u_0), (u_0 - \bar{u})^+\rangle &= \int_\Omega \bar{f}_+(x, u_0, \lambda)(u_0 - \bar{u})^+ \, dx \\ &= \int_\Omega f(x, \bar{u}, \lambda)(u_0 - \bar{u})^+ \, dx \\ &\leq t_\lambda^{p-1} \int_\Omega (u_0 - \bar{u})^+ \, dx = \langle A(\bar{u}), (u_0 - \bar{u})^+\rangle\end{aligned}$$

because $\bar{u} := t_\lambda e$. This forces

$$\int_{\Omega(u_0 > \bar{u})} \left(|\nabla u_0|^{p-2}\nabla u_0 - |\nabla \bar{u}|^{p-2}\nabla \bar{u}\right) \cdot \nabla(u_0 - \bar{u}) \, dx \leq 0,$$

which, on account of [23, Lemma A.0.5], leads to $m(\Omega(u_0 > \bar{u})) = 0$. Consequently,

$$u_0(x) \leq \bar{u}(x) \quad \text{a.e. in } \Omega, \tag{3.8}$$

and, by (3.4) and (3.7) again, the function u_0 solves problem (P_λ). Standard regularity results ([12, Theorems 1.5.5–1.5.6]) give $u_0 \in C_0^1(\overline{\Omega})_+$. If $\rho := \|\bar{u}\|_\infty$, then, due to ($f_3$), we have

$$-\Delta_p u_0(x) + \mu_{\lambda,\rho} u_0(x)^{p-1} = f(x, u_0(x), \lambda) + \mu_{\lambda,\rho} u_0(x)^{p-1} \geq 0.$$

Thus, Theorem 1.5.7 in [12] ensures that

$$u_0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+). \tag{3.9}$$

Now, combining (f_1) with (3.2) provides

$$\begin{aligned}-\Delta_p u_0(x) + \mu_{\lambda,\rho} u_0(x)^{p-1} &= f(x, u_0(x), \lambda) + \mu_{\lambda,\rho} u_0(x)^{p-1} \\ &< t_\lambda^{p-1} + \mu_{\lambda,\rho} \bar{u}(x)^{p-1} \\ &= -\Delta_p \bar{u}(x) + \mu_{\lambda,\rho} \bar{u}(x)^{p-1}.\end{aligned}$$

On account of [4, Proposition 2.6] this implies $\bar{u} - u_0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, namely

$$B_{C_0^1(\overline{\Omega})}(u_0, \delta) \subseteq \bar{u} - C_0^1(\overline{\Omega})_+ \tag{3.10}$$

for some $\delta > 0$. By decreasing δ if necessary, (3.9)–(3.10) lead to

$$B_{C_0^1(\overline{\Omega})}(u_0, \delta) \subseteq [0, \bar{u}]\,.$$

Since $\varphi_\lambda|_{[0,\bar{u}]} = \bar{\varphi}_{\lambda,+}|_{[0,\bar{u}]}$ and (3.6) holds, the function u_0 turns out to be a $C_0^1(\overline{\Omega})$-local minimum of φ_λ. Thanks to [11, Theorem 1.1], the same is true with X in place of $C_0^1(\overline{\Omega})$.

Finally, setting, for every $(x,t) \in \Omega \times \mathbb{R}$,

$$\bar{f}_-(x,t,\lambda) := \begin{cases} f(x,-\bar{u}(x),\lambda) & \text{if } t < -\bar{u}(x), \\ f(x,t,\lambda) & \text{if } -\bar{u}(x) \le t \le 0, \\ 0 & \text{if } 0 < t, \end{cases} \tag{3.11}$$

and arguing as before yields a function $v_0 \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$ that satisfies (P_λ) and is a local minimum of φ_λ. □

Lemma 3.2. *Under the assumptions of Theorem* 3.1*, if* $\lambda \in]0,\lambda^*[$*, then problem* (P_λ) *has a smallest solution* $\hat{v}$ *in the order interval* $[-\bar{u},0]$ *and a biggest solution* $\hat{u}$ *in the order interval* $[0,\bar{u}]$*, with* $\bar{u}$ *given by* (3.3).

Proof. Define $S_{\lambda,+} := \{u \in [0,\bar{u}] : u \text{ is nontrivial and fulfils } (\mathrm{P}_\lambda)\}$. The proof of Theorem 3.1 shows that $S_{\lambda,+} \neq \emptyset$, because $u_0 \in S_{\lambda,+}$, and $S_{\lambda,+} \subseteq \mathrm{int}(C^1(\overline{\Omega})_+)$. Moreover, $S_{\lambda,+}$ turns out to be upward directed; cf. [16, Lemma 2.4]. Hence, by Zorn's lemma, a biggest solution of (P_λ) in $[0,\bar{u}]$ exists once we know that each chain $C \subseteq S_{\lambda,+}$ is bounded above. Due to [10, p. 336] one has

$$\sup C = \sup\{u_k : k \in \mathbb{N}\} \tag{3.12}$$

for some $\{u_k\} \subseteq C$, while Lemma 1.1.5 of [15] allows this sequence to be increasing. Since

$$u_k \in [0,\bar{u}] \quad \text{and} \quad A(u_k) = f(\cdot,u_k,\lambda) \quad \text{in } W^{-1,p'}(\Omega) \quad \forall\, k \in \mathbb{N}, \tag{3.13}$$

$\{u_k\}$ turns out to be bounded in $W_0^{1,p}(\Omega)$. Passing to a subsequence when necessary, we may thus suppose $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as well as $u_k \to u$ in $L^q(\Omega)$, with

$$u = \sup\{u_k : k \in \mathbb{N}\}. \tag{3.14}$$

This forces

$$\lim_{k\to+\infty} \int_\Omega f(x,u_k(x),\lambda)(u_k(x)-u(x))\,dx = 0.$$

Therefore, on account of (3.13),

$$\lim_{k\to+\infty} \langle A(u_k), u_k - u\rangle = 0.$$

Proposition 2.4 ensures that $u_k \to u$ in $W_0^{1,p}(\Omega)$. From (3.13) it follows, letting $k \to +\infty$,

$$u \in [0,\bar{u}], \quad A(u) = f(\cdot,u,\lambda) \quad \text{in } W_0^{-1,p'}(\Omega),$$

namely $u \in S_{\lambda,+}$. Now, (3.12) and (3.14) lead to $\sup C \in S_{\lambda,+}$, as desired. A similar argument produces a smallest solution of (P_λ) in the order interval $[-\bar{u},0]$. □

Through Lemma 3.2 we can find two further constant-sign solutions. This exactly is the goal of the next result.

Theorem 3.3. *Let* (f_1)–(f_5) *be satisfied, let* $\lambda \in]0, \lambda^*[$, *and let* u_0, v_0 *be as in Theorem* 3.1. *Then problem* (P_λ) *possesses two solutions* $u_1 \in \mathrm{int}(C_0^1(\overline{\Omega})_+) \setminus \{u_0\}$ *and* $v_1 \in -\mathrm{int}(C_0^1(\overline{\Omega})_+) \setminus \{v_0\}$ *such that* $v_1 \le v_0 < 0 < u_0 \le u_1$ *in* Ω.

Proof. Keep the same notation of Lemma 3.2 and define, provided $(x,t) \in \Omega \times \mathbb{R}$,

$$\hat{f}_+(x,t,\lambda) := \begin{cases} f(x,\hat{u}(x),\lambda) & \text{if } t \le \hat{u}(x), \\ f(x,t,\lambda) & \text{if } \hat{u}(x) < t. \end{cases} \tag{3.15}$$

Moreover, set

$$\hat{\varphi}_{\lambda,+}(u) := \frac{1}{p}\|u\|^p - \int_\Omega \hat{F}_+(x,u(x),\lambda)\,dx \quad \forall\, u \in X, \tag{3.16}$$

where $\hat{F}_+(x,z,\lambda) := \int_0^z \hat{f}_+(x,t,\lambda)\,dt$. Obviously, one has

$$K(\hat{\varphi}_{\lambda,+}) \subseteq \{u \in X : \hat{u} \le u \text{ in } \Omega\}. \tag{3.17}$$

Combining the arguments adopted in the proof of Theorem 3.1 with the maximality of $\hat{u}$ (cf. Lemma 3.2) shows that $\hat{u} \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ is a local minimum for $\hat{\varphi}_{\lambda,+}$. Hence, $\hat{u} \in K(\hat{\varphi}_{\lambda,+})$. We can certainly assume $\hat{u}$ turns out to be an isolated critical point of $\hat{\varphi}_{\lambda,+}$, otherwise a whole sequence of positive solutions to (P_λ) will exist. So, like in the proof of [1, Proposition 29],

$$\hat{\varphi}_{\lambda,+}(\hat{u}) < \inf\{\hat{\varphi}_{\lambda,+}(u) : \|u - \hat{u}\| = \hat{r}\} \tag{3.18}$$

for some $\hat{r} > 0$. Since (f_1)–(f_4) hold true, the functional $\hat{\varphi}_{\lambda,+}$ fulfils condition (C); the relevant verification goes on exactly as that of [21, Lemma 3.1]. Finally, from (f_2) it evidently follows

$$\lim_{t\to+\infty} \hat{\varphi}_{\lambda,+}(tu) = -\infty$$

whenever $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. At this point, Theorem 2.2 can be applied and, on account of (3.18), we obtain a point $u_1 \in W_0^{1,p}(\Omega) \setminus \{\hat{u}\}$ such that

$$\hat{\varphi}'_{\lambda,+}(u_1) = 0. \tag{3.19}$$

Inclusion (3.17) forces $\hat{u} \le u_1$. Thanks to (3.15) and (3.19) one thus has

$$A(u_1) = f(\cdot, u_1, \lambda) \quad \text{in } W^{-1,p'}(\Omega),$$

while standard regularity results ([12, Theorems 1.5.5–1.5.6]) produce

$$u_1 \in \mathrm{int}(C_0^1(\overline{\Omega})_+).$$

Since $u_0 \le \hat{u} \le u_1$ in Ω, the function u_1 complies with our conclusion. The construction of a fourth solution $v_1 \le v_0$ to (P_λ) is analogous. □

4 Multiplicity results

Problem (P_λ) possesses another nontrivial solution in addition to those given by Theorems 3.1–3.3. The notation is the same as in Section 3.

Theorem 4.1. *Under assumptions* (f_1)–(f_5)*, if* $\lambda \in]0, \lambda^*[$*, then there exists a solution* $w \in W_0^{1,p}(\Omega)$ *to* (P_λ) *such that* $w \in C_0^1(\overline{\Omega}) \setminus \{0, v_0, v_1, u_0, u_1\}$*.*

Proof. Define, for every $(x,t) \in \Omega \times \mathbb{R}$,

$$f_0(x,t,\lambda) := \begin{cases} f(x, v_0(x), \lambda) & \text{if } t < v_0(x), \\ f(x,t,\lambda) & \text{if } v_0(x) \le t \le u_0(x), \\ f(x, u_0(x), \lambda) & \text{if } u_0(x) < t, \end{cases} \tag{4.1}$$

$$f_{0,+}(x,t,\lambda) := f_0(x, t^+, \lambda), \quad f_{0,-}(x,t,\lambda) := f_0(x, -t^-, \lambda). \tag{4.2}$$

Moreover, set, provided $u \in X$,

$$\psi_\lambda(u) := \frac{1}{p}\|u\|^p - \int_\Omega F_0(x, u(x), \lambda)\, dx,$$
$$\psi_{\lambda,\pm}(u) := \frac{1}{p}\|u\|^p - \int_\Omega F_{0,\pm}(x, u(x), \lambda)\, dx,$$

where $F_0(x,z,\lambda) := \int_0^z f_0(x,t,\lambda)\, dt$ and $F_{0,\pm}(x,z,\lambda) := \int_0^z f_{0,\pm}(x,t,\lambda)\, dt$. By (4.1)–(4.2) one has

$$K(\psi_\lambda) \subseteq [v_0, u_0], \quad K(\psi_{\lambda,-}) \subseteq [v_0, 0], \quad K(\psi_{\lambda,+}) \subseteq [0, u_0]. \tag{4.3}$$

We may suppose that

$$K(\psi_{\lambda,-}) \subseteq \{v_0, 0\}, \quad K(\psi_{\lambda,+}) \subseteq \{0, u_0\}. \tag{4.4}$$

Indeed, if, for example, $u \in K(\psi_{\lambda,+}) \setminus \{0, u_0\}$, then $u \in [0, u_0] \setminus \{0, u_0\}$ due to (4.3). Thus, $u \neq \hat{u}$, because $u_0 \le \hat{u}$, and, a fortiori, $u \neq u_1$. On account of (4.1) the conclusion would follow with $w := u$.

Let us next verify that u_0, v_0 are local minima of ψ_λ. Thanks to (4.2) the functional $\psi_{\lambda,+}$ is weakly sequentially lower semi-continuous and coercive. So, there exists $\tilde{u} \in X$ such that

$$\psi_{\lambda,+}(\tilde{u}) = \inf_{u \in X} \psi_{\lambda,+}(u). \tag{4.5}$$

Arguing as in the proof of Theorem 3.1 yields

$$\psi_{\lambda,+}(\tilde{u}) < 0 = \psi_{\lambda,+}(0), \quad \text{i.e.,} \quad \tilde{u} \neq 0. \tag{4.6}$$

By (4.4) this implies $\tilde{u} = u_0 \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. Since $\psi_\lambda|_{X_+} = \psi_{\lambda,+}|_{X_+}$, where

$$X_+ := \{u \in X : u \geq 0 \text{ in } \Omega\}, \tag{4.7}$$

u_0 turns out to be a $C_0^1(\overline{\Omega})$-local minimum of ψ_λ. Theorem 1.1 in [11] ensures that the same is true with X in place of $C_0^1(\overline{\Omega})$. A similar reasoning then holds for v_0.

Now, observe that, due to (4.1), ψ_λ is coercive and if

$$\langle B(u), v\rangle := -\int_\Omega f_0(x, u(x), \lambda) v(x)\, dx \quad \forall\, u, v \in X,$$

then

$$\langle \psi_\lambda'(u), v\rangle = \langle A(u), v\rangle + \langle B(u), v\rangle.$$

The operator A turns out to be of type $(\mathrm{S})_+$ (cf. Proposition 2.4) while $B : X \to X^*$ is compact, because X compactly embeds in $L^p(\Omega)$. So, Proposition 2.3 guarantees that ψ_λ satisfies (PS). Through [25, Corollary 1] (cf. also [24, Corollary 8]) we thus obtain

$$K(\psi_\lambda) \setminus \{u_0, v_0\} \neq \emptyset.$$

Let $w \in K(\psi_\lambda) \setminus \{u_0, v_0\}$ be a critical point of Mountain Pass type. From (4.3) and (4.1) it follows

$$A(w) = f(\cdot, w, \lambda) \quad \text{in } W^{-1,p'}(\Omega),$$

namely w solves (P_λ), while standard regularity results [12, Theorems 1.5.5–1.5.6] produce $w \in C_0^1(\overline{\Omega})$. We may assume that

$$C_1(\psi_\lambda, w) \neq 0; \tag{4.8}$$

see [7, pp. 89–90]. By Proposition 2.1 of [17] one has

$$C_k(\psi_\lambda, 0) = 0 \quad \forall\, k \in \mathbb{N}_0. \tag{4.9}$$

Comparing (4.8) with (4.9) leads to $w \neq 0$, which completes the proof. □

A further nontrivial solution of (P_λ) exists whenever $p = 2$, i.e., in the semilinear framework. Towards this, the first step is the following.

Lemma 4.2. *Let $\lambda > 0$ and let* (f_1), (f_2), (f_4) *be satisfied. Then $C_k(\varphi_\lambda, \infty) = 0$ for all $k \in \mathbb{N}_0$.*

Proof. Pick any $u \in X$. Thanks to (f_4) we get

$$0 = \xi_\lambda(x, 0) \leq \xi_\lambda(x, u^+(x)) + \alpha_\lambda(x),$$
$$0 = \xi_\lambda(x, 0) \leq \xi_\lambda(x, -u^-(x)) + \alpha_\lambda(x),$$

that is,

$$pF(x,u(x),\lambda)-u(x)f(x,u(x),\lambda) = -\xi_\lambda(x,u(x)) \le \alpha_\lambda(x) \quad \text{a.e. in } \Omega. \tag{4.10}$$

If $t > 0$, then, by the chain rule,

$$\begin{aligned}\frac{d}{dt}\varphi_\lambda(tu) = \langle \varphi'_\lambda(tu), u\rangle &= \frac{1}{t}\langle \varphi'_\lambda(tu), tu\rangle \\ &= \frac{1}{t}\left[\|\nabla(tu)\|_p^p - \int_\Omega tuf(x,tu,\lambda)dx\right].\end{aligned} \tag{4.11}$$

Inequality (4.10), written for tu in place of u, and (4.11) yield

$$\frac{d}{dt}\varphi_\lambda(tu) \le \frac{1}{t}[p\varphi_\lambda(tu) + \|\alpha_\lambda\|_1]. \tag{4.12}$$

Since, on account of (f_2),

$$\lim_{t\to+\infty} \varphi_\lambda(tu) = -\infty \tag{4.13}$$

provided $u \neq 0$, given $M < -\frac{1}{p}\|\alpha_\lambda\|_1$, there exists $t_M > 0$ such that $\varphi_\lambda(tu) \le M$ for all $t > t_M$. From (4.12) it thus follows

$$\frac{d}{dt}\varphi_\lambda(tu) < 0, \quad t > t_M. \tag{4.14}$$

Fix $u \in \partial B_X(0,1)$. Through (4.13)–(4.14) we can find a unique $t_u > 0$ fulfilling

$$\varphi_\lambda(t_u u) = M. \tag{4.15}$$

Moreover, if $\theta_0(u) := t_u$, $u \in \partial B_X(0,1)$, then, due to the Implicit Function Theorem, $\theta_0 : \partial B_X(0,1) \to \mathbb{R}$ is continuous. Write

$$h(t,u) := (1-t)u + t\theta(u)u \quad \forall\,(t,u) \in [0,1] \times (X \setminus \{0\}),$$

where

$$\theta(u) := \begin{cases} 1 & \text{if } \varphi_\lambda(u) < M, \\ \frac{1}{\|u\|}\theta_0\left(\frac{u}{\|u\|}\right) & \text{otherwise.}\end{cases}$$

Evidently, the function $h : [0,1] \times (X \setminus \{0\}) \to X \setminus \{0\}$ is continuous and satisfies $h(0,u) = u$ in $X \setminus \{0\}$. By (4.15) one has $h(1,u) = \theta(u)u \in \varphi_\lambda^M$ as well as

$$h(t,\cdot)|_{\varphi_\lambda^M} = \mathrm{id}|_{\varphi_\lambda^M}.$$

Hence, φ_λ^M is a deformation retract of $X \setminus \{0\}$. Owing to [9, Theorem XV.6.5], the same holds for $\partial B_X(0,1)$. Therefore, φ_λ^M and $\partial B_X(0,1)$ turn out to be homotopy equivalent, which forces

$$H_k(X, \varphi_\lambda^M) = H_k(X, \partial B_X(0,1)), \quad k \in \mathbb{N}_0. \tag{4.16}$$

Now, observe that $\partial B_X(0,1)$ is contractible in itself, because X has infinite dimension ([13, Remark 5.5.7]). Proposition 4.9 and 4.10 in [14, p. 389] thus yield

$$H_k(X, \partial B_X(0,1)) = 0. \tag{4.17}$$

Combining (4.16) with (4.17) immediately leads to

$$C_k(\varphi_\lambda, \infty) = H_k(X, \varphi_\lambda^M) = 0$$

provided $M < -\frac{1}{p}\|\alpha_\lambda\|_1$ and $|M|$ is big enough. □

We shall adopt the notation used in Theorem 4.1. Since $p = 2$, the underlying space X becomes a very good Hilbert space, i.e., $H_0^1(\Omega)$, while Δ_p reduces to the classical Laplace operator.

Theorem 4.3. *If* $p = 2$, $\lambda \in]0, \lambda^*[$, *and* ($\mathrm{f}_1'$), ($\mathrm{f}_2$)–($\mathrm{f}_5$) *hold true, then* ($\mathrm{P}_\lambda$) *possesses a solution* $\bar{w} \in H_0^1(\Omega)$ *such that* $\bar{w} \in C_0^1(\overline{\Omega}) \setminus \{0, v_0, v_1, u_0, u_1, w\}$.

Proof. On account of [7, Example 1, p. 33], besides Theorem 3.1, one clearly has

$$C_k(\varphi_\lambda, u_0) = C_k(\varphi_\lambda, v_0) = \delta_{k,0}\mathbb{Z} \quad \forall\, k \in \mathbb{N}_0. \tag{4.18}$$

The proof of Theorem 3.3 ensures that u_1 turns out to be a Mountain Pass critical point for $\hat{\varphi}_{\lambda,+}$. So, by Proposition 2.5 in [5],

$$C_k(\hat{\varphi}_{\lambda,+}, u_1) = \delta_{k,1}\mathbb{Z}, \quad k \in \mathbb{N}_0. \tag{4.19}$$

Using (3.15) we obtain $\hat{\varphi}_{\lambda,+}|_{X_+} = \varphi_\lambda|_{X_+} + \hat{c}$, where X_+ is defined in (4.7) and $\hat{c} \in \mathbb{R}$. From (4.19) it thus follows

$$C_k(\varphi_\lambda, u_1) = \delta_{k,1}\mathbb{Z} \quad \forall\, k \in \mathbb{N}_0. \tag{4.20}$$

A similar argument shows that

$$C_k(\varphi_\lambda, v_1) = \delta_{k,1}\mathbb{Z}, \quad k \in \mathbb{N}_0. \tag{4.21}$$

The proof of Theorem 4.1 guarantees that w turns out to be a Mountain Pass critical point for ψ_λ. Hence, due to [5, Proposition 2.5] and [18, Theorem 2.7],

$$C_k(\psi_\lambda, w) = \delta_{k,1}\mathbb{Z} \quad \forall\, k \in \mathbb{N}_0. \tag{4.22}$$

Reasoning as in the proof of Theorem 3.1 we achieve $B_{C_0^1(\overline{\Omega})}(w,\delta) \subseteq [v_0, u_0]$ for some $\delta > 0$ while (4.1) yields

$$\psi_\lambda|_{[v_0,u_0]} = \varphi_\lambda|_{[v_0,u_0]}.$$

Thanks to [5, Proposition 2.6], (4.22) provides

$$C_k(\varphi_\lambda, w) = \delta_{k,1}\mathbb{Z}\,, \quad k \in \mathbb{N}_0\,. \tag{4.23}$$

Finally, Proposition 2.1 in [17] and Lemma 4.2, respectively, force

$$C_k(\varphi_\lambda, 0) = 0\,, \quad C_k(\varphi_\lambda, \infty) = 0 \quad \forall\, k \in \mathbb{N}_0\,. \tag{4.24}$$

Now, if

$$K(\varphi_\lambda) = \{0, u_0, v_0, u_1, v_1, w\},$$

then the Morse relation written for $t := -1$, combined with (4.18), (4.20), (4.21), (4.23), and (4.24), would give $2(-1)^0 + 3(-1)^1 = 0$, which is impossible. Therefore, there exists a point

$$\bar{w} \in K(\varphi_\lambda) \setminus \{0, v_0, v_1, u_0, u_1, w\}.$$

The function $\bar{w}$ evidently solves problem (P_λ). Moreover, through standard regularity results ([12, Theorems 1.5.5–1.5.6]) we obtain $\bar{w} \in C_0^1(\overline{\Omega})$. □

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