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# A class of degenerate elliptic eigenvalue problems

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**Abstract.** We consider a general class of eigenvalue problems where the leading elliptic term corresponds to a convex homogeneous energy function that is not necessarily differentiable. We derive a strong maximum principle and show uniqueness of the first eigenfunction. Moreover we prove the existence of a sequence of eigensolutions by using a critical point theory in metric spaces. Our results extend the eigenvalue problem of the $p$-Laplace operator to a much more general setting.

**Keywords.** Nonlinear eigenvalue problems, quasilinear elliptic equations, critical point theory, convex analysis, nonsmooth analysis.

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## 1 Introduction

The classical eigenvalue problem $-\Delta u = \lambda u$ and some of its nonlinear extensions involving the $p$-Laplace operator $\operatorname{div} |Du|^{p-2} Du$ have in common that they are related to a variational problem where a convex coercive function $E$ as, e.g.,

$$\int_\Omega |Du|^p \, dx, \quad 1 < p < \infty,$$

corresponds to the leading elliptic term in the partial differential equation. The usual techniques for deriving typical properties of elliptic problems like maximum principle, uniqueness of the solution, or the existence of a sequence of eigensolutions are known to work for problems where the corresponding function $E$ has certain differentiability properties. However the inherent property of ellipticity is rather convexity than smoothness of $E$. In this paper we want to demonstrate for a general class of eigenvalue problems that, by extending and supplementing the standard techniques, we can derive the typical properties also for elliptic problems where the leading term corresponds to a function $E$ that is convex but lacks the usual smoothness assumptions.

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More precisely, we consider the constrained variational problem

$$E(u) := \int_\Omega A(x, Du)\, dx \to \text{Min!}$$

subjected to

$$G(u) := \int_\Omega \omega |u|^p\, dx = 1$$

where $\Omega \subset \mathbb{R}^n$ might be unbounded, $\omega$ is allowed to change sign, and $A(x, \cdot)$ is assumed to be convex but not necessarily differentiable. This kind of problem has been studied by Szulkin & Willem [22] for the special case of $A(x, Du) = |Du|^p$ with $1 < p < \infty$ which corresponds to the nonlinear eigenvalue problem

$$-\operatorname{div} |Du|^{p-2} Du = \lambda \omega |u|^{p-2} u.$$

In our more general case we are led to the eigenvalue problem

$$-\operatorname{div} a = p \lambda \omega |u|^{p-2} u$$

where $a$ is coupled to $u$ by $a(x) \in \partial A(x, Du(x))$ a.e. on $\Omega$ with $\partial A$ denoting the convex subdifferential of $A(x, \cdot)$. Assuming some $p$-growth on $A(x, \cdot)$ we show that the minimizer $u$ of the variational problem, that exists in a suitable space, is a weak solution of this eigenvalue problem with homogeneous boundary conditions and that it is unique (up to sign). For the verification of uniqueness, we extend to our framework a strong maximum principle due to Ancona [2] and Brezis & Ponce [8]. Finally we show the existence of a sequence $\{u_k\}$ of eigenfunctions with corresponding eigenvalues $\lambda_k \to \infty$. If $\omega$ changes sign, our arguments also apply to $-\omega$ and provide a sequence of eigenfunctions with eigenvalues $\lambda_k \to -\infty$.

The leading term in our general eigenvalue problems is not only of analytical interest but also of practical relevance, since it describes, e.g., anisotropic diffusion and heat flux. Recall that the simplest model for diffusion is based on Fick's law where the flux $j$ has the form $-B \cdot Du$. Here $B$ is the diffusion tensor and we are led to the term $-\operatorname{div}(B \cdot Du)$ in a corresponding differential equation. Note that, for general $B$, the flux $j$ has not to be parallel to $Du$, i.e. some kind of anisotropic diffusion is covered. Basic nonlinear models (e.g. Perona-Malik) assume that $j = \beta(|Du|)Du$ for a scalar function $\beta$. While this describes an isotropic situation, the use of an anisotropic norm $|Du|_*$ as argument of $\beta$ leads to an anisotropic model. Combining the idea that the flux has not to be parallel to $Du$ and that the intensity of diffusion might depend on some magnitude of $Du$ through a function $\beta(\cdot)$, we are readily led to models where the flux has the form $j = A'(Du)$ with a convex function $A : \mathbb{R}^n \to \mathbb{R}$. Simple examples would be given, e.g., by $A(Du) = |Du|_q^p$, that is anisotropic for $q \neq 2$. In the limit cases $q = 1$ or $q = \infty$, the function $A(\cdot)$ is not differentiable anymore and we can replace the diffusion law by $j \in \partial A(Du)$ (where $\partial$ denotes the convex subdifferential). This way one can describe diffusion in structured materials where diffusion takes place only in special preferred directions and where the flux direction might jump at certain directions of $Du$. Notice that the theory we are presenting covers such general diffusion laws.

The precise formulation of the problem and explicit nonsmooth examples for function $A$ are provided in Section 2. Some preliminary material about the (sub-) differentiability of underlying functions is collected in Section 3. In Section 4 we state the existence of a minimizer of the constrained variational problem and we show that it is a weak solution of the corresponding eigenvalue problem. Here we use a general result from convex analysis that is derived in Section A.1. Section 5 is devoted to the strong maximum principle for non-negative weak solutions of the eigenvalue problem. Here we apply some results about capacity and a Poincaré inequality that are summarized in Section A.3. The uniqueness of the first eigenfunction is shown in Section 6 by using that maximum principle. Section 7 verifies the existence of a sequence of eigensolutions on the basis of a critical point theorem, derived in Section A.2, that is a modified version of a general result due to Degiovanni & Marzocchi [10].

**Notation.** The closure of a set $A$ is denoted by $\bar{A}$. We write $B_r(u)$ for the open ball of radius $r$ centered at $u$. For a Banach space $X$, its dual is $X^*$ and $\langle u^*, u\rangle$ stands for the duality pairing. In particular, $\langle \cdot\,, \cdot\rangle$ is the scalar product in $\mathbb{R}^n$. By $\operatorname{sgn}\alpha$ we denote the sign of $\alpha \in \mathbb{R}$ and

$$\operatorname{Sgn}\alpha := \begin{cases} \{\operatorname{sgn}\alpha\} & \text{if } \alpha \neq 0, \\ [-1, 1] & \text{if } \alpha = 0, \end{cases}$$

is the set-valued sign function. For $u : \Omega \to \mathbb{R}$ we define

$$\{u > 0\} := \{x \in \Omega \mid u(x) > 0\}$$

and, analogously, $\{u = 0\}$ etc. Moreover, $u^+$ and $u^-$ are the positive and negative part of $u$, respectively, such that $u = u^+ - u^-$. The subdifferential of a convex function $F$ at $u$ is denoted by $\partial F(u)$ and $F'(u; v)$ stands for the one-sided directional derivative of $F$ at $u$ in direction $v$. We write $|\cdot|_p$ for the $p$-norm in $\mathbb{R}^n$ and we set $|\cdot| := |\cdot|_2$. In several estimates $c > 0$ is a generic positive constant that can differ from one equation or inequality to the next one. Denote by $L^p(\Omega)$ the usual Lebesgue space of $p$-integrable functions, $L^{p'}(\Omega)$ its dual, and $W^{1,p}(\Omega)$ the Sobolev space of $p$-integrable functions having $p$-integrable weak derivatives. Then $L^p_{\text{loc}}(\Omega)$ and $W^{1,p}_{\text{loc}}(\Omega)$ are the spaces of functions where any restriction to a compact subset belongs to the corresponding spaces. We write a.e. for "almost everywhere" and q.e. for "quasi everywhere" (cf. Section A.3).

## 2 Problem

For $\Omega \subset \mathbb{R}^n$ open, $1 < p < \infty$, let $D_0^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|Du\|_{L^p} := \left( \int_\Omega |Du|^p \, dx \right)^{1/p}$$

where we exclude $p \geq n$ if $\Omega$ is unbounded (since the elements of $D_0^{1,p}(\Omega)$ cannot be identified with a function in $L^1_{\mathrm{loc}}(\Omega)$ in that case). Notice that

- $D_0^{1,p}(\Omega) \subset W^{1,p}_{\mathrm{loc}}(\Omega)$,
- $D_0^{1,p}(\Omega) = W_0^{1,p}(\Omega)$ if $\Omega$ is bounded,
- $W_0^{1,p}(\Omega) \subsetneqq D_0^{1,p}(\Omega)$ if $\Omega$ is unbounded.

Since $D_0^{1,p}(\Omega)$ can be considered as a closed subspace of a cross product $L^p(\Omega)^n$, it is uniformly convex and, thus, also reflexive (cf. Adams [1, p. 7f.] and Section A.4). Moreover,

- $D_0^{1,p}(\Omega) \hookrightarrow L^{p*}(\Omega)$ continuously for $p^* = \frac{np}{n-p}$ if $p < n$ (follows from the Gagliardo–Nirenberg–Sobolev inequality),
- $D_0^{1,n}(\Omega) \hookrightarrow L^{\tilde{p}}(\Omega)$ is compact if $1 \leq \tilde{p} < \infty$,
- $D_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact if $p > n$.

If $\Omega$ is bounded, then $D_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $1 \leq q < p^*$. Notice that any weakly convergent sequence $u_n \rightharpoonup u$ in $D_0^{1,p}(\Omega)$ has a subsequence converging pointwise a.e. on $\Omega$ (for $\Omega$ bounded this follows from the compact embedding $D_0^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$ and for $\Omega$ unbounded we can consider a sequence $B_n$ of balls covering $\Omega$ and then we use the compact embedding $W^{1,p}(B_n) \hookrightarrow L^1(B_n)$ and stepwise select subsequences).

Henceforth we set $X := D_0^{1,p}(\Omega)$, and for a weight function $\omega \in L^1_{\mathrm{loc}}(\Omega)$ with $\omega^+ \not\equiv 0$ we define

$$Y := \left\{ u \in X \;\middle|\; \int_\Omega |\omega| \, |u|^p \, dx < \infty \right\}$$

and

$$Y^+ := \left\{ u \in Y \;\middle|\; \int_\Omega \omega \, |u|^p \, dx > 0 \right\}.$$

We want to consider the minimization problem

$$\lambda := \inf_{u \in Y^+} \frac{\int_\Omega A(x, Du) \, dx}{\int_\Omega \omega \, |u|^p \, dx} \tag{2.1}$$

and, first, let us formulate basic assumptions we want to use.

(A) $A\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ with $A(x,\cdot)$ convex for all $x \in \Omega$, and $A(\cdot,q)$ measurable for all $q \in \mathbb{R}^n$. (Notice that $A(x,\cdot)$ is continuous for all $x \in \Omega$, and that the subdifferential $\partial A(x,q)$ of $A(x,\cdot)$ at $q$ is nonempty for all $x \in \Omega$, $q \in \mathbb{R}^n$.)

(A1) There are $c_1, c_2, c_3 > 0$, $\beta \in L^1(\Omega)$ such that for a.e. $x \in \Omega$

- (a) $0 \le A(x,q) \le c_1|q|^p + \beta(x)$ for all $q \in \mathbb{R}^n$,
- (b) $\langle a, q\rangle \ge c_2|q|^p$ for all $a \in \partial A(x,q)$, $q \in \mathbb{R}^n$,
- (c) $|a| \le c_3|q|^{p-1}$ for all $a \in \partial A(x,q)$, $q \in \mathbb{R}^n$.

(A2) $A(x,tq) = |t|^p A(x,q)$ for all $t \in \mathbb{R}$, $x \in \Omega$, $q \in \mathbb{R}^n$.

($\mathrm{W}^+$) $p \neq n$, $\omega^+ \not\equiv 0$ and furthermore

- if $p < n$, then $\omega^+ \in L^{n/p}(\Omega)$,
- if $p > n$, then $\omega^+ \in L^1(\Omega)$ and $\Omega$ bounded.

Let us start with some simple consequences of theses assumptions. From (A), (A2) we get

$$\langle a, q\rangle = pA(x,q) \quad \text{for all } q \in \mathbb{R}^n,\ a \in \partial A(x,q) \tag{2.2}$$

by evaluating $A(x,tq) - A(x,q) \ge \langle a, tq - q\rangle$ for $t \to 1$. Consequently, if we assume (A), (A2), then (A1b) implies

$$c_2|q|^p \le pA(x,q) \quad \text{for all } x \in \Omega,\ q \in \mathbb{R}^n, \tag{2.3}$$

and (A1c) implies

$$pA(x,q) \le c_3|q|^p \quad \text{for all } x \in \Omega,\ q \in \mathbb{R}^n.$$

Hence (A), (A1b), (A1c), (A2) imply (A1a).

Let us mention that we work with condition ($\mathrm{W}^+$) for the sake of simplicity. It can be relaxed by working in some Lorentz spaces as in [3]. Our main intention is to focus on techniques necessary in the absence of differentiability in the elliptic term. While it is standard to treat that kind of problem for integrands $A$ where $A(x,\cdot)$ is smooth, here we want to demonstrate that smoothness is not needed.

**Examples.** Let us first provide some simple examples of $A$ satisfying all of our assumptions stated above but without being smooth. First we consider

$$A_1(q) := \left(\sum_{j=1}^{n} |q_j|\right)^p = |q|_1^p \quad \text{for all } q = (q_1,\dots,q_n) \in \mathbb{R}^n.$$

Obviously, the functions $f_j(q) := |q_j|$, $j = 1, \ldots, n$, are convex on $\mathbb{R}^n$ with the subdifferential

$$\partial f_j(q) = \{(0, \ldots, 0, \varphi_j, 0, \ldots, 0) \in \mathbb{R}^n \mid \varphi_j \in \mathrm{Sgn}(q_j)\}.$$

For $f(q) := \sum_{j=1}^n |q_j|$ the sum rule implies that

$$\partial f(q) = \mathrm{Sgn}(q_1) \times \cdots \times \mathrm{Sgn}(q_n).$$

Since $g(t) := |t|^p$ is continuously differentiable on $\mathbb{R}$ for $p > 1$, we can apply the chain rule of Clarke [9, Theorem 2.3.9] to get

$$\partial A_1(q) = p\left(\sum_{j=1}^n |q_j|\right)^{p-1} \big(\mathrm{Sgn}(q_1) \times \cdots \times \mathrm{Sgn}(q_n)\big) \quad \text{for all } q \in \mathbb{R}^n.$$

Using the equivalence of norms on $\mathbb{R}^n$ and (2.2) we readily verify (A1a)–(A1c) for $A_1$. Analogously we can argue for

$$A_2(q) := \Big(\max_{j=1,\ldots,n} |q_j|\Big)^p = |q|_\infty^p \quad \text{for all } q = (q_1, \ldots, q_n) \in \mathbb{R}^n$$

where

$$\partial A_2(q) = p\Big(\max_{j=1,\ldots,n} |q_j|\Big)^{p-1} \overline{\mathrm{conv}}\{\mathrm{Sgn}(q_j)e_j \mid |q_j| = A_2(q),\ j = 1, \ldots, n\}$$

with $e_1, \ldots, e_n$ being the standard unit vectors in $\mathbb{R}^n$. While $A_1$ and $A_2$ are not strictly convex, we can easily construct strictly convex nonsmooth functions $A$ by, e.g.,

$$A_3(q) := (|q|_1 + |q|_2)^p \quad \text{for all } q = (q_1, \ldots, q_n) \in \mathbb{R}^n.$$

## 3 Preliminary considerations

Before we formulate some general existence result and derive further properties of a minimizer, let us start with some preliminary considerations.

With $\lambda$ as in (2.1), problem (2.1) is equivalent to

$$F(u) := \int_\Omega A(x, Du)\, dx - \lambda \int_\Omega \omega\, |u|^p\, dx \to \text{Min!}, \quad u \in Y^+, \tag{3.1}$$

where $\inf_{Y^+} F = 0$. By (A2) we readily see that any $t\bar{u}$ with $t \in \mathbb{R}$ solves (2.1) as long as $\bar{u} \in Y^+$ is a solution. Therefore attention can be restricted to minimizer normalized by $\int_\Omega \omega\, |u|^p\, dx = 1$ and, in this sense, (2.1) is also equivalent to the constraint problem

$$E(u) := \int_\Omega A(x, Du)\, dx \to \text{Min!}, \quad u \in Y^+, \tag{3.2}$$

subject to

$$G(u) := \int_\Omega \omega\, |u|^p\, dx = 1. \tag{3.3}$$

Later we also use the notation

$$G_\pm(u) := \int_\Omega \omega^\pm\, |u|^p\, dx.$$

**Lemma 3.1.** *Let* (A), (A1a) *be satisfied. Then:*

(1) *$E$ is convex and continuous on $X$ and, thus, also locally Lipschitz continuous on $X$.*

(2) *The (one-sided) directional derivative $E'(u; v)$ exists for all $u, v \in X$.*

(3) *We have $\partial E(u) \neq \emptyset$ for all $u \in X$ and for any $E^* \in \partial E(u)$ there is some $a \in L^{p'}(\Omega, \mathbb{R}^n)$ with*

$$a(x) \in \partial A(x, Du(x)) \quad \text{for a.e. } x \in \Omega$$

*such that*

$$\langle E^*, v\rangle = \int_\Omega \langle a(x), Dv(x)\rangle\, dx \quad \text{for all } v \in X.$$

*Proof.* Let $v_1, v_2 \in X$. Then, using convexity of $A(x, \cdot)$,

$$\begin{aligned} E\left(\frac{v_1 + v_2}{2}\right) &= \int_\Omega A\left(x, \frac{Dv_1(x) + Dv_2(x)}{2}\right) dx \\ &\leq \int_\Omega \frac{A(x, Dv_1(x)) + A(x, Dv_2(x))}{2}\, dx = \frac{E(v_1) + E(v_2)}{2} \end{aligned}$$

and, thus, $E$ is convex.

Let now $v_n \to v$ in $X$. Thus $Dv_n \to Dv$ in $L^p(\Omega)$ and, possibly for a subsequence, $Dv_n(x) \to Dv(x)$ a.e. on $\Omega$ and

$$A(x, Dv_n(x)) \to A(x, Dv(x)) \quad \text{for a.e. } x \in \Omega.$$

By (A1a),

$$|A(x, Dv_n(x))| \leq c_1 |Dv_n(x)|^p + \beta(x) \quad \text{for a.e. } x \in \Omega.$$

With generalized dominated convergence we get $E(v_n) \to E(v)$. The subsequence principle then implies continuity of $E$. By convexity, $E$ is even locally Lipschitz continuous on $X$ (cf. [9, p. 34]).

Since $E$ is convex and (locally Lipschitz) continuous on $X$, the existence of a finite one-sided directional derivative $E_1'(u;v)$ for all $u, v \in X$ is a standard result of convex analysis and, moreover, $\partial E(u) \neq \emptyset$ for all $u \in X$ (cf. Barbu & Precupanu [5, Chapter 2.1]). The structure assertion about $E^*$ is a direct consequence of Proposition A.3 in the Appendix. □

**Lemma 3.2.** *We have:*

(1) *$Y$ is a linear subspace in $X$.*

(2) *Let $u \in Y^+$ and $v \in Y$. Then there is $t_0 > 0$ such that $u + tv \in Y^+$ for all $|t| < t_0$.*

*Proof.* Let $u_1, u_2 \in Y$. Obviously $tu_1 \in Y$ for all $t \in \mathbb{R}$. Moreover, by convexity there is some $c > 0$ with

$$\int_\Omega |\omega|\, |u_1 + u_2|^p \, dx \le c \int_\Omega |\omega|\, (|u_1|^p + |u_2|^p)\, dx$$

which implies the first assertion. For the second assertion we use the analogous estimate that

$$|\omega(x)|\,|u(x)+tv(x)|^p \le c|\omega(x)|\big(|u(x)|^p + |t|^p|v(x)|^p\big) \quad \text{for all } t \in \mathbb{R},\ x \in \Omega,$$

and that

$$\lim_{t\to 0} \omega(x)\,|u(x) + tv(x)|^p = \omega(x)\,|u(x)|^p \quad \text{for all } x \in \Omega.$$

Thus, by dominated convergence,

$$\lim_{t\to 0} \int_\Omega \omega\,|u + tv|^p \, dx = \int_\Omega \omega\,|u|^p \, dx > 0$$

which readily implies (2). □

**Lemma 3.3.** *We have that $G : Y \to \mathbb{R}$ is well-defined and the directional derivative $G'(u;v)$ is given by*

$$G'(u;v) = p \int_\Omega \omega |u|^{p-2} uv \, dx \quad \textit{for all } u \in Y^+,\ v \in Y. \tag{3.4}$$

*If in addition* (W$^+$) *is satisfied, then*

$$G_+(u) = \int_\Omega \omega^+ |u|^p \, dx < \infty \quad \textit{for all } u \in X$$

*and $G_+$ is weakly continuous on $X$ (i.e. $v_n \rightharpoonup v$ implies $G_+(v_n) \to G_+(v)$).*

*Proof.* The definition of $Y$ directly implies that $G$ is well-defined on $Y$. Now, let $u \in Y^+$, $v \in Y$ be fixed. Then, for $|t|$ small, $u + tv \in Y$ and

$$G(u + tv) = \int_\Omega \omega\, |u + tv|^p\, dx$$

is finite. For $\psi(x,t) := \omega(x)\, |u(x) + tv(x)|^p$ we have

$$\psi_t(x,t) = p\omega\, |u + tv|^{p-2}(u + tv)v \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Hence, using Young's inequality and convexity,

$$\begin{aligned} \tfrac{1}{p}\, |\psi_t(x,t)| &= |\omega|^{\frac{p-1}{p}} |u + tv|^{p-1} |\omega|^{\frac{1}{p}} |v| \\ &\le \frac{p-1}{p}\, |\omega|\, |u + tv|^p + \tfrac{1}{p}\, |\omega|\, |v|^p \\ &\le c\, \frac{p-1}{p}\, |\omega| \left(|u|^p + |v|^p\right) + \frac{1}{p}\, |\omega|\, |v|^p \end{aligned} \tag{3.5}$$

for a.e. $x \in \Omega$ and all $|t| < 1$. Since the right hand side is integrable, we obtain (cf. Zeidler [24, p. 1018])

$$G'(u; v) = \lim_{t \to 0} \frac{G(u + tv) - G(u)}{t} = \int_\Omega \psi_t(x,0)\, dx = p \int_\Omega \omega |u|^{p-2} uv\, dx.$$

Let us now verify that $G_+$ is weakly continuous. The arguments are well known, and we give it here for the sake of completeness. We first consider $p < n$. Let $u_n \rightharpoonup u$ in $X$, and let $\varepsilon > 0$. On the one hand, the fact that $u_n$ is bounded in $L^{p^*}(\Omega)$ together with Hölder inequality imply the existence of a ball $B$ such that

$$\int_{\Omega \setminus B} \omega^+ \big| |u_n|^p - |u|^p \big|\, dx < \varepsilon. \tag{3.6}$$

On the other hand, since $|u_n|^p$ is bounded in $L^{p^*/p}(\Omega \cap B)$ and $|u_n|^p \to |u|^p$ strongly in $L^1_{\mathrm{loc}}(\Omega)$, we easily deduce that $|u_n|^p \rightharpoonup |u|^p$ in $L^{p^*/p}(\Omega \cap B)$. Therefore, since $\omega^+$ is in the dual of $L^{p^*/p}(\Omega)$, we deduce

$$\lim_{n \to \infty} \int_{\Omega \cap B} \omega^+ \left(|u_n|^p - |u|^p\right) dx = 0. \tag{3.7}$$

From (3.6) and (3.7) we deduce $G_+(u_n) \to G_+(u)$.

For $p > n$ we recall that $\Omega$ is bounded and that the embedding

$$D_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$$

is compact. This readily implies the assertion. □

## 4 Existence and Euler–Lagrange equation

Let us first verify the existence of a solution to the minimizing problem (2.1).

**Theorem 4.1.** *Let* (A), (A1b), (A2), ($\mathrm{W}^+$) *be satisfied. Then* (2.1) *has a minimizer in* $Y^+$.

*Proof.* We verify the existence of a solution of (3.2), (3.3). For that we consider a minimizing sequence $\{u_n\} \subset Y^+$. By (2.3) it is bounded in $X$ and, thus, we may assume that $u_n \rightharpoonup: u \in X$. Since $E$ is convex and continuous on $X$ by Lemma 3.1, it is weakly lower semicontinuous and $E(u) \le \liminf_{n\to\infty} E(u_n)$. Since $G_+$ is weakly continuous on $X$ by Lemma 3.3, Fatou's lemma implies that

$$G_+(u) = \lim_{n\to\infty} G_+(u_n) = \lim_{n\to\infty} G_-(u_n) + 1 \ge G_-(u) + 1.$$

Hence $u \in Y^+$ with $G(u) \ge 1$, i.e. $u \in Y^+$. By (A2) we conclude that $u$ has to be a minimizer. □

We now assume that $\bar{u} \in Y^+$ is a solution of (2.1) or, equivalently, of (3.1) and we claim to derive a necessary condition for it.

**Lemma 4.2.** *Let* (A), (A1a) *be satisfied. Then the solution* $\bar{u}$ *of* (3.1) *satisfies*

$$E'(\bar{u}; v) - \lambda G'(\bar{u}; v) \ge 0 \quad \textit{for all } v \in Y. \tag{4.1}$$

*Moreover, there is* $E^* \in \partial E(\bar{u})$ *such that*

$$\langle E^*, v\rangle - \lambda G'(\bar{u}; v) = 0 \quad \textit{for all } v \in Y.$$

*Proof.* Fix any $v \in Y$. Since $\bar{u} \in Y^+$, we know from Lemma 3.2 that there is $t_0 > 0$ such that $\bar{u} + tv \in Y^+$ for all $|t| < t_0$. Hence all $\bar{u} + tv$ with $|t|$ small are admissible for the variational problem (3.1) and

$$F(\bar{u} + tv) - F(\bar{u}) \ge 0 \quad \text{for all } |t| < t_0.$$

By the existence of the directional derivatives according to Lemmas 3.1 and 3.3 we readily obtain (4.1).

Since $E$ is locally Lipschitz continuous on $X$ according to Lemma 3.1, there is $\tilde{c} > 0$ such that

$$E'(\bar{u}; v) \le \tilde{c}\, \|v\| \quad \text{for all } v \in X.$$

Thus, by (4.1),

$$\lambda G'(\bar{u}; v) \le E'(\bar{u}; v) \le \tilde{c}\, \|v\| \quad \text{for all } v \in Y.$$

Consequently, $G'(\bar{u};\cdot)$ is linear and continuous on $Y$. Since $E'(\bar{u};\cdot)$ is sublinear on $X$, the Hahn–Banach Theorem provides the existence of some $E^* \in X^*$ with

$$\langle E^*, v\rangle = \lambda G'(\bar{u}; v) \quad \text{for all } v \in Y, \tag{4.2}$$

$$\langle E^*, v\rangle \leq E'(\bar{u}; v) \quad \text{for all } v \in X. \tag{4.3}$$

Since $E$ is continuous at $\bar{u}$, the last inequality implies that $E^* \in \partial E(\bar{u})$ (cf. Barbu [4, Proposition I.1.6]) and the assertion is a consequence of (4.2). □

As a direct consequence of Lemma 3.1, Lemma 3.3 and Lemma 4.2 we obtain the following theorem where we use the decomposition $\omega = \omega^+ - \omega^-$ for the last inequality.

**Theorem 4.3.** *Let* (A), (A1a) *be satisfied and let $\bar{u}$ be a solution of* (3.1). *Then there is $a \in L^{p'}(\Omega, \mathbb{R}^n)$ with*

$$a(x) \in \partial A(x, \bar{u}(x)) \quad \textit{for a.e. } x \in \Omega \tag{4.4}$$

*such that*

$$\int_\Omega \langle a(x), Dv(x)\rangle\, dx - p\lambda \int_\Omega \omega |\bar{u}|^{p-2}\bar{u} v\, dx = 0 \quad \textit{for all } v \in Y. \tag{4.5}$$

*Moreover, if $\bar{u} \geq 0$ a.e. on $\Omega$, then*

$$\int_\Omega \langle a(x), Dv(x)\rangle\, dx + p\lambda \int_\Omega \omega^- |\bar{u}|^{p-2}\bar{u} v\, dx \geq 0$$

*for all $v \in C_0^\infty(\Omega)$ with $v \geq 0$.*

Notice that (4.5) can be written, in the distributional sense, as

$$-\operatorname{div} a = p\lambda\omega|\bar{u}|^{p-2}\bar{u} \quad \text{on } \Omega$$

where $a$ is coupled to $\bar{u}$ by (4.4). In the special case of $A(x,q) = |q|^p$, that meets all of our assumptions, we recover the (nonlinear) eigenvalue problem for the $p$-Laplace operator.

## 5 Strong maximum principle

Let us state the strong maximum principle for the type of problems we have considered before where we follow ideas of Brezis & Ponce, cf. [8]. The notion of capacity as used here is precisely formulated in Section A.3 below.

**Proposition 5.1.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and assume that $A : \Omega \times \mathbb{R}^n \to \mathbb{R}$ and $\omega \in L^1_{\text{loc}}(\Omega)$ satisfy* (A), (A1b) *and* (A1c)*. Moreover let $u \in W^{1,p}_{\text{loc}}(\Omega)$ with $u \geq 0$ a.e. on $\Omega$ be quasi-continuous and let it satisfy*

$$\int_\Omega \langle a(x), Dv(x)\rangle\, dx + p\lambda \int_\Omega \omega^- |u|^{p-2} uv\, dx \geq 0 \tag{5.1}$$

*for all $v \in C_0^\infty(\Omega)$ with $v \geq 0$ for some measurable selection*

$$a(x) \in \partial A(x, Du(x)) \quad \textit{for a.e. } x \in \Omega.$$

*Then, denoting the set of zeros by*

$$\mathcal{Z} := \{x \in \Omega \mid u(x) = 0\}, \tag{5.2}$$

*we have that either* $\text{cap}_p(\mathcal{Z}) = 0$ *or* $u \equiv 0$.

**Remark 5.2.** (1) Let us mention that the result is not true in the case $p = 1$ where typical solutions of the eigenvalue problem of the 1-Laplace operator vanish on a set with positive Lebesgue measure (cf. Kawohl & Schuricht [17]).

(2) Notice that in the case $p > n$, where we only consider bounded $\Omega$, the capacity $\text{cap}_p(\{x\}) > 0$ for any $x \in \Omega$. Therefore, either $\mathcal{Z} = \emptyset$ or $u \equiv 0$.

*Proof.* We need to show that $u \equiv 0$ if $\text{cap}_p(\mathcal{Z}) > 0$. We borrow here the arguments found in [8], and the main idea is to prove that for any $\xi \in C_0^\infty(\Omega)$ with $0 \leq \xi \leq 1$ there is a constant $c_0 := c_0(\xi) > 0$ such that

$$\int_\Omega \left| D \log\left(1 + \frac{u}{\delta}\right)\right|^p \xi^p\, dx \leq c_0 \quad \text{for all } \delta > 0. \tag{5.3}$$

To derive (5.3) we proceed as follows:

$$\begin{aligned}
&\int_\Omega \left| D \log\left(1 + \frac{u}{\delta}\right)\right|^p \xi^p\, dx \\
&\quad = \int_\Omega |Du|^p (u+\delta)^{-p}\, \xi^p\, dx \\
&\quad \leq \frac{1}{c_2} \int_\Omega \langle a(x), Du\rangle (u+\delta)^{-p}\, \xi^p\, dx \qquad \text{(by (A1b))} \\
&\quad = -\frac{1}{c_2(p-1)} \int_\Omega \langle a(x), D(u+\delta)^{1-p}\rangle\, \xi^p\, dx \\
&\quad = -\frac{1}{c_2(p-1)} \int_\Omega \left\langle a(x), D\left(\frac{\xi^p}{(u+\delta)^{p-1}}\right) - \frac{D\xi^p}{(u+\delta)^{p-1}}\right\rangle dx.
\end{aligned}$$

Note that the differential inequality (5.1) with $v = \xi^p/(u+\delta)^{p-1}$ and a density argument (since $v \notin C_0^\infty(\Omega)$ in general) shows that

$$-\int_\Omega \left\langle a(x), D\left(\frac{\xi^p}{(u+\delta)^{p-1}}\right)\right\rangle dx \le p\lambda \int_\Omega \omega^-(x)|u|^{p-1}\frac{\xi^p}{(u+\delta)^{p-1}}\,dx. \quad (5.4)$$

From (5.4) and the assumption (A1c), we then obtain

$$\begin{aligned} c_2(p-1)\int_\Omega \left|D\log\left(1+\frac{u}{\delta}\right)\right|^p \xi^p\,dx \\ &\le p\lambda\int_\Omega \omega^-(x)|u|^{p-1}\frac{\xi^p}{(u+\delta)^{p-1}}\,dx + c_3\int_\Omega\left(\frac{|Du|}{u+\delta}\right)^{p-1}|D\xi^p|\,dx \\ &= \underbrace{p\lambda\int_\Omega \omega^-(x)|u|^{p-1}\frac{\xi^p}{(u+\delta)^{p-1}}\,dx}_{I_1} \\ &\quad + \underbrace{c_3 p\int_\Omega\left|D\log\left(1+\frac{u}{\delta}\right)\right|^{p-1}\xi^{p-1}\,|D\xi|\,dx}_{I_2}. \end{aligned} \quad (5.5)$$

To estimate $I_1$, we note that $0 \le \frac{|u|}{u+\delta} < 1$ and get

$$I_1 \le p\lambda\int_\Omega \omega^-\xi^p\,dx. \quad (5.6)$$

Let us now estimate $I_2$. By using the inequality $ab \le \varepsilon^r\frac{a^r}{r} + \frac{b^s}{s\varepsilon^s}$ ($\frac{1}{r}+\frac{1}{s}=1$, $\varepsilon > 0$) with $r = \frac{p}{p-1}$, $s = p$, we can find a constant $\tilde{c} = \tilde{c}(\xi)$ such that

$$I_2 \le c_2\frac{p-1}{2}\int_\Omega\left|D\log\left(1+\frac{u}{\delta}\right)\right|^p\xi^p\,dx + \tilde{c}. \quad (5.7)$$

Hence, by plugging estimates (5.6) and (5.7) in (5.5), we get

$$c_2\frac{p-1}{2}\int_\Omega\left|D\log\left(1+\frac{u}{\delta}\right)\right|^p\xi^p\,dx \le p\lambda\int_\Omega\omega^-\xi^p\,dx + \tilde{c}.$$

Therefore, given $\xi \in C_0^\infty(\Omega)$, we can find a constant $c_0 = c_0(\xi)$ such that (5.3) holds.

To proceed with the proof of Proposition 5.1, let us first notice that the set of zeros of the function $x \mapsto \log(1+\frac{u(x)}{\delta})$ coincides with $\mathcal{Z}$ for any $\delta > 0$. Moreover, by modifying $u$ on a set of capacity zero, we may assume that $\mathcal{Z}$ is a Borel set (cf. Proposition A.7).

Now let the set $\mathcal{Z}$ have positive capacity, i.e. $\mathrm{cap}_p \mathcal{Z} > 0$. We can find open balls $B_k \subset\subset \Omega$, $k \in \mathbb{N}$, such that

$$\bigcup_{k \in \mathbb{N}} B_k = \Omega.$$

Then

$$0 < \mathrm{Cap}_p \mathcal{Z} \leq \sum_{k \in \mathbb{N}} \mathrm{Cap}_p(\mathcal{Z} \cap B_k).$$

Clearly there is some $j \in \mathbb{N}$ such that $\mathrm{Cap}_p(\mathcal{Z} \cap B_j) > 0$. Let us now choose any domain $U \subset\subset \Omega$ with $B_j \subset U$ and with Lipschitz boundary $\partial U$. We consider $\xi \in C_0^\infty(\Omega)$ with $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $U$. Then we can apply the Poincaré inequality stated in Proposition A.6 to the functions $\log(1 + \frac{u}{\delta})$ (note that $u$ equals its precise representative $u^\star$ q.e., since it is quasi-continuous, cf. Section A.3). Still using (5.3), we find a constant $c := c(U)$ such that

$$\int_U \left| \log\left(1 + \frac{u}{\delta}\right) \right|^p dx \leq c \quad \text{for all } \delta > 0.$$

Since $\delta$ is arbitrary, the above uniform bound implies that $u = 0$ a.e. in $U$. Since $U$ can be chosen arbitrarily, we deduce that $u = 0$ a.e. in $\Omega$ as claimed. □

## 6 Uniqueness

**Proposition 6.1.** *Let* (A), (A1a) *be satisfied, let* $A(x, 0) = 0$*, and let*

$$u \in Y^+ \subset D_0^{1,p}(\Omega)$$

*be a minimizer of* (2.1). *Then either* $u^+$ *or* $u^-$ *is also a minimizer.*

*Proof.* Recall that $Du^\pm = Du$ a.e. on $\{u^\pm > 0\}$ and $Du = 0$ a.e. on $\{u = 0\}$ (cf. [13, p. 130]). Since a minimizer satisfies (4.5) with $v = u^\pm$,

$$\int_\Omega \langle a, Du^\pm \rangle \, dx = \lambda \int_\Omega \omega |u^\pm|^p \, dx.$$

By convexity of $A(x, \cdot)$,

$$-A(x, q) = A(x, 0) - A(x, q) \geq a \cdot (-q) \quad \text{for all } x \in \Omega, q \in \mathbb{R}^n, a \in \partial A(x, q).$$

Using $a(x) \in \partial A(x, Du(x)) = \partial A(x, Du^\pm(x))$ a.e. on $\{u^\pm > 0\}$, we get

$$\int_\Omega \omega |u^\pm|^p \, dx \geq \frac{1}{\lambda} \int_\Omega A(x, Du^\pm) \, dx \geq 0.$$

Consequently,

$$\lambda = \frac{\int_\Omega A(x,Du)\,dx}{\int_\Omega \omega|u|^p\,dx} = \frac{\int_\Omega A(x,Du^+)\,dx + \int_\Omega A(x,Du^-)\,dx}{\int_\Omega \omega|u|^p\,dx}$$
$$= \frac{\int_\Omega A(x,Du^+)\,dx}{\int_\Omega \omega|u^+|^p\,dx}\frac{\int_\Omega \omega|u^+|^p\,dx}{\int_\Omega \omega|u|^p\,dx} + \frac{\int_\Omega A(x,Du^-)\,dx}{\int_\Omega \omega|u^-|^p\,dx}\frac{\int_\Omega \omega|u^-|^p\,dx}{\int_\Omega \omega|u|^p\,dx}$$
$$\geq \min\left\{\frac{\int_\Omega A(x,Du^+)\,dx}{\int_\Omega \omega|u^+|^p\,dx}, \frac{\int_\Omega A(x,Du^-)\,dx}{\int_\Omega \omega|u^-|^p\,dx}\right\}. \tag{6.1}$$

Thus either $u^+$ or $u^-$ has to be a minimizer. □

**Proposition 6.2.** *Let* $\Omega \subset \mathbb{R}^n$ *be a domain, let* (A), (A1a)–(A1c) *be satisfied, let* $A(x,0) = 0$ *for all* $x \in \Omega$*, and let* $u \in Y^+ \subset D_0^{1,p}(\Omega)$ *be a minimizer of* (2.1). *Then either* $u > 0$ *or* $u < 0$ *q.e. on* $\Omega$.

*Proof.* According to the previous proposition let, without loss of generality, $u^+$ be also minimizer. Since $u^+ \not\equiv 0$, the strong maximum principle implies that

$$\mathrm{Cap}_p\{u^+ = 0\} = 0$$

and, thus, $u^+ > 0$ q.e. on $\Omega$. But this implies the assertion. □

**Proposition 6.3.** *Let* (A), (A2) *be satisfied and let* $u, v \in Y^+ \subset D_0^{1,p}(\Omega)$ *be minimizers of* (2.1) *satisfying* $u > 0$, $v > 0$ *q.e. on* $\Omega$ *and being normalized by*

$$\int_\Omega \omega u^p\,dx = \int_\Omega \omega v^p\,dx = 1.$$

*Then*

$$\psi := \left(\frac{u^p + v^p}{2}\right)^{\frac{1}{p}}$$

*is also a minimizer of* (2.1) *with*

$$\int_\Omega \omega \psi^p\,dx = 1.$$

*In addition, let* $A(x,\cdot)$ *be strictly convex for all* $x \in \Omega$. *Then*

$$vDu = uDv \quad \textit{q.e. on } \Omega. \tag{6.2}$$

*Proof.* Here we follow some arguments from Belloni & Kawohl [7]. We readily see that $\psi$ is normalized and we show that it is a minimizer. Obviously

$$D\psi = \frac{1}{2^{\frac{1}{p}}}\frac{u^{p-1}Du + v^{p-1}Dv}{(u^p+v^p)^{1-\frac{1}{p}}}.$$

Thus, using convexity and homogeneity of $A(x,\cdot)$,

$$\begin{aligned} A(x,D\psi) &= \frac{u^p+v^p}{2} A\left(x, \frac{u^p}{u^p+v^p}\frac{Du}{u} + \frac{v^p}{u^p+v^p}\frac{Dv}{v}\right) \\ &\le \frac{u^p+v^p}{2}\left(\frac{u^p}{u^p+v^p}A\left(x,\frac{Du}{u}\right) + \frac{v^p}{u^p+v^p}A\left(x,\frac{Dv}{v}\right)\right) \\ &= \frac{1}{2}(A(x,Du)+A(x,Dv)). \end{aligned} \tag{6.3}$$

By integrating over $\Omega$ we see that $\psi$ is a minimizer and we must have equality in (6.3) a.e. on $\Omega$. But, under the additional assumption of strict convexity, this implies the final assertion. □

As shown in [19], condition (6.2) implies the final uniqueness statement.

**Proposition 6.4.** *Let* $\Omega \subset \mathbb{R}^n$ *be a domain, let* (A), (A1a)–(A1c), (A2) *be satisfied, and let* $A(x,\cdot)$ *be strictly convex. If* (2.1) *admits a minimizer* $u \in Y^+ \subset D_0^{1,p}(\Omega)$*, then* $u$ *is unique in the sense that any minimizer has to be a multiple of* $u$.

## 7 Higher eigenvalues

A minimizer $u \in Y^+$ of (2.1) satisfies the eigenvalue equation (4.5), i.e. there is some function $a \in L^{p'}(\Omega,\mathbb{R}^n)$ with

$$a(x) \in \partial A(x,u(x)) \quad \text{for a.e. } x \in \Omega$$

such that

$$\int_\Omega \langle a(x), Dv(x)\rangle\, dx - p\lambda \int_\Omega \omega |u|^{p-2} uv\, dx = 0 \quad \text{for all } v \in Y. \tag{7.1}$$

Let us now look for higher eigensolutions, i.e. let us verify the existence of a sequence of solutions $u_n$ of (7.1) with corresponding eigenvalues $\lambda_n \to \infty$. For that we are looking for critical points of $E$ subject to the constraint

$$K := \{u \in Y \mid G(u) = 1\}.$$

Notice that the special case of the $p$-Laplace operator has been treated by Szulkin & Willem [22]. In our more general setting we are confronted with the difficulty that $E$ may not be differentiable. Thus we cannot define critical points $u$ in the classical way that $E'(u) - \lambda G'(u) = 0$. Instead we say that $u$ is a critical point of (3.2), (3.3) if the weak slope $|dE|_K(u)$ in the metric space $K$, that replaces $\|E'(u) - \lambda G'(u)\|$, vanishes (cf. Degiovanni & Marzocchi [10]).

For our analysis we need some Palais–Smale condition as compactness condition. Therefore we assume that $\omega$ satisfies ($\mathrm{W}^+$) and we work in the different Banach space

$$\tilde{X} := \left\{ v \in D_0^{1,p}(\Omega) \;\middle|\; \|v\| := \left( \int_\Omega |Dv|^p \, dx + \int_\Omega \omega^- |v|^p \, dx \right)^{\frac{1}{p}} < \infty \right\}.$$

Obviously $\tilde{X} \hookrightarrow X$ continuously and $Y \subset \tilde{X}$. From (7.4) below we get $\tilde{X} \subset Y$ and, hence, $Y = \tilde{X}$ as sets. We start by collecting properties of the space $\tilde{X}$, and of the maps

$$G_\pm(u) = \int_\Omega \omega^\pm |u|^p \, dx.$$

**Proposition 7.1.** *We have that:*

(1) *$\tilde{X}$ is a uniformly convex Banach space and therefore reflexive.*

(2) *$G_-$ is continuously differentiable on $\tilde{X}$ with*

$$\langle G_-'(u), v\rangle = \int_\Omega \omega^- |u|^{p-2} uv \, dx \quad \textit{for all } u, v \in \tilde{X}.$$

(3) *If* ($\mathrm{W}^+$) *is satisfied, then $G_+$ is continuously differentiable on $\tilde{X}$ with*

$$\langle G_+'(u), v\rangle = \int_\Omega \omega^+ |u|^{p-2} uv \, dx \quad \textit{for all } u, v \in \tilde{X}$$

*and $G_+' : \tilde{X} \to \tilde{X}^*$ is completely continuous.*

Notice that in a uniformly convex Banach space $u_n \to u$ as long as $u_n \rightharpoonup u$ and $\|u_n\| \to \|u\|$ (cf. Zeidler [23, p. 604]).

*Proof.* (1) To show completeness, let $\{u_n\}$ be Cauchy sequence in $\tilde{X}$. Then it is a Cauchy sequence in $X$ and $u_n \to u$ in $X$. Thus $u_n \to u$ in $L^{p^*}(\Omega)$ if $p < n$ or in $L^\infty(\Omega)$ if $p > n$ and, up to a subsequence, $u_n \to u$ a.e. on $\Omega$. Since $\{u_n\}$ is bounded in $\tilde{X}$, we have by Fatou's lemma

$$\int_\Omega \omega^- |u|^p \, dx \le \liminf \int_\Omega \omega^- |u_n|^p \, dx < \tilde{c}$$

and, thus, $u \in \tilde{X}$. Clearly, $\{(\omega^-)^{\frac{1}{p}} u_n\}$ is also Cauchy sequence in $L^p(\Omega)$. Hence there is some $\tilde{u}$ with

$$(\omega^-)^{\frac{1}{p}} u_n \to (\omega^-)^{\frac{1}{p}} \tilde{u} \quad \text{in } L^p(\Omega)$$

and we get $\tilde{u} = u$ a.e. on $\{\omega^- > 0\}$. Consequently, $G_-(u_n - u) \to 0$ and $u_n \to u$ in $\tilde{X}$. Therefore $\tilde{X}$ is complete. Uniform convexity follows as in Adams [1, p. 7f.], since $\tilde{X}$ can be considered as cross product of $L^p$-spaces (note that the weight $\omega^-$ can be considered as the density of a measure on $\Omega$ and see Section A.4). Consequently, $\tilde{X}$ is reflexive (cf. [23, p. 604]).

(2) We argue as in the proof of Lemma 3.3 with $\omega^-$ instead of $\omega$ to get the directional derivative

$$G'_-(u;v) = \int_\Omega \omega^-|u|^{p-2}uv\,dx \quad \text{for all } u,v \in \tilde{X}.$$

By Hölder's inequality

$$\begin{aligned}
|G'_-(u;v)| &\le \int_\Omega |\omega^-|^{\frac{p-1}{p}}|u|^{p-1}|\omega^-|^{\frac{1}{p}}|v|\,dx \\
&\le \left(\int_\Omega \omega^-|u|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_\Omega \omega^-|v|^p\,dx\right)^{\frac{1}{p}} \\
&\le c\|v\|_{\tilde{X}} \quad \text{for all } v \in \tilde{X}. \qquad (7.2)
\end{aligned}$$

Thus $G'_-(u)$ exists as Gâteaux derivative. For a sequence $u_n \to u$ in $\tilde{X}$, which we can assume to converge pointwise a.e. on $\Omega$, we find $v_n \in \tilde{X}$ with $\|v_n\|_{\tilde{X}} = 1$ such that

$$\begin{aligned}
\|G'_-(u_n) - G'_-(u)\| &= \langle G'_-(u_n) - G'_-(u), v_n\rangle \\
&= \int_\Omega \omega^-\big||u_n|^{p-2}u_n - |u|^{p-2}u\big|\,v_n\,dx \\
&= \int_\Omega (\omega^-)^{\frac{p-1}{p}}\big||u_n|^{p-2}u_n - |u|^{p-2}u\big|\,(\omega^-)^{\frac{1}{p}}|v_n|\,dx \\
&\overset{\text{Hölder}}{\le} \left(\int_\Omega \omega^-\big||u_n|^{p-2}u_n - |u|^{p-2}u\big|^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} \\
&\qquad \cdot \left(\int_\Omega \omega^-|v_n|^p\,dx\right)^{\frac{1}{p}}. \qquad (7.3)
\end{aligned}$$

Since

$$\int_\Omega \omega^-|v_n|^p\,dx \le 1$$

and

$$\omega^-\big||u_n|^{p-2}u_n - |u|^{p-2}u\big|^{\frac{p}{p-1}} \le \omega^-(|u_n|^p + |u|^p),$$

generalized dominated convergence and the subsequence principle imply

$$G'_-(u_n) \to G'_-(u)$$

and, thus, the continuity of $G'_-(\cdot)$ on $\tilde{X}$.

(3) For the directional derivative $G'_+(u;v)$ we basically argue as in the proof of Lemma 3.3. But we now use ($\mathrm{W}^+$) for the integrability of the right hand side in (3.5). More precisely, we still use $\tilde{X} \subset L^\infty(\Omega)$ if $p > n$ and for $p < n$ we apply Young's inequality to get

$$|\omega^+|\,|u|^p \le c|\omega^+|^{n/p} + c|u|^{p^*} \quad \text{and} \quad |\omega^+|\,|v|^p \le c|\omega^+|^{n/p} + c|v|^{p^*}. \tag{7.4}$$

This way we obtain

$$G'_+(u;v) = \int_\Omega \omega^+|u|^{p-2}uv\,dx \quad \text{for all } u, v \in \tilde{X}.$$

As in (7.2) we get

$$|G'_+(u;v)| \le \left(\int_\Omega \omega^+|u|^p\,dx\right)^{\frac{p-1}{p}} \left(\int_\Omega \omega^+|v|^p\,dx\right)^{\frac{1}{p}}.$$

For $p > n$ we readily conclude

$$|G'_+(u;v)| \le \|\omega^+\|_{L^1}\|u\|_{L^\infty}^{p-1}\|v\|_{L^\infty} \le c\,\|v\|_X \le c\,\|v\|_{\tilde{X}} \quad \text{for all } v \in \tilde{X}.$$

If $p < n$, then

$$\begin{aligned}
|G'_+(u;v)| &\overset{\text{Hölder}}{\le} \left(\left(\int_\Omega |\omega^+|^{\frac{n}{p}}\,dx\right)^{\frac{p}{n}} \left(\int_\Omega |u|^{p^*}\,dx\right)^{\frac{n-p}{n}}\right)^{\frac{p-1}{p}} \\
&\qquad \cdot \left(\left(\int_\Omega |\omega^+|^{\frac{n}{p}}\,dx\right)^{\frac{p}{n}} \left(\int_\Omega |v|^{p^*}\,dx\right)^{\frac{n-p}{n}}\right)^{\frac{1}{p}} \\
&\le c\left(\int_\Omega |v|^{p^*}\,dx\right)^{\frac{1}{p^*}} \\
&\le c\|v\|_X \le c\|v\|_{\tilde{X}} \quad \text{for all } v \in \tilde{X}.
\end{aligned} \tag{7.5}$$

Thus $G'_+(u)$ exists as Gâteaux derivative for any $u \in \tilde{X}$.

Let now $u_n \rightharpoonup u$ in $\tilde{X}$. Then there are $v_n \in \tilde{X}$ with $\|v_n\|_{\tilde{X}} = 1$ such that, as in (7.3),

$$\begin{aligned}
\|G'_+(u_n) - G'_+(u)\| &\le \left(\int_\Omega \omega^+\left||u_n|^{p-2}u_n - |u|^{p-2}u\right|^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} \\
&\qquad \cdot \left(\int_\Omega \omega^+|v_n|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned} \tag{7.6}$$

As in the arguments above we obtain

$$\left(\int_\Omega \omega^+|v_n|^p\,dx\right)^{\frac{1}{p}} \le c\,\|v_n\|_{\tilde{X}} \le c.$$

For $p > n$ we have $u_n \to u$ in $L^\infty(\Omega)$ and we readily derive $G'_+(u_n) \to G'_+(u)$. Thus $G'_+(\,\cdot\,)$ is completely continuous on $\tilde{X}$ in that case. For $p < n$ we consider

$$\tilde{u}_n := \big||u_n|^{p-2}u_n - |u|^{p-2}u\big|^{\frac{p}{p-1}}$$

and, by convexity, we have

$$|\tilde{u}_n|^{n/(n-p)} \le \big(|u_n|^{p-1} + |u|^{p-1}\big)^{\frac{p^*}{p-1}} \le c\big(|u_n|^{p^*} + |u|^{p^*}\big).$$

Since $\tilde{X} \hookrightarrow L^{p^*}(\Omega)$, we conclude that $\{\tilde{u}_n\}$ is a bounded sequence in $L^{\frac{n}{n-p}}(\Omega)$. Thus, at least for a subsequence,

$$\big||u_n|^{p-2}u_n - |u|^{p-2}u\big|^{\frac{p}{p-1}} \rightharpoonup: \tilde{u} \quad \text{in } L^{\frac{n}{n-p}}(\Omega) = L^{\frac{n}{p}}(\Omega)'.$$

Since, up to a subsequence, $u_n \to u$ a.e. on $\Omega$ (by $u_n \rightharpoonup u$ in $\tilde{X}$), we obtain $\tilde{u}_n \to 0$ a.e. on $\Omega$. Therefore $\tilde{u}_n \rightharpoonup 0$ in $L^{\frac{n}{n-p}}(\Omega)$. By $\omega^+ \in L^{\frac{n}{p}}(\Omega)$ and the subsequence principle we derive from (7.6) that $G'_+(u_n) \to G'_+(u)$. Hence, $G'_+(\,\cdot\,)$ is completely continuous on $\tilde{X}$ also for $p < n$. □

In order to apply some abstract theorem about critical points let us first introduce some terminology. For a function $F : M \to \mathbb{R}$ on a metric space $M$ we denote the weak slope of $F$ at $u$ by $|dF|(u) = |dF|_M(u)$ and we call $u \in M$ a *critical point* of $F$ if $|dF|(u) = 0$. We say that $F$ satisfies the *Palais–Smale condition* at level $\gamma \in \mathbb{R}$ if any sequence $\{u_n\}$ in $M$ with $F(u_n) \to \gamma$ and $|dF|_M(u_n) \to 0$ has a convergent subsequence $u_{n'} \to u$ in $M$. By $\operatorname{gen} S$ we denote the Krasnoselkii genus of a closed symmetric set $S$ with $0 \notin S$ given by

$$\operatorname{gen} S := \inf\{k \in \mathbb{N} \mid \text{there is } f : S \to \mathbb{R}^k \setminus \{0\} \text{ odd, continuous}\}$$

where $\inf \emptyset = \infty$ and $\operatorname{gen} \emptyset := 0$ (cf. [23, p. 319]).

We will apply Proposition A.5 to function $E$ on the metric space $K$ where the metric is induced by $\tilde{X}$ (notice that $Y = \tilde{X}$ as sets). Let us still recall the estimate

$$|dE|_K(u) \ge \min\{\|E^* - \lambda G'(u)\| \mid E^* \in \partial E(u),\ \lambda \in \mathbb{R}\} \tag{7.7}$$

from Degiovanni & Schuricht [11, Theorem 3.5] which implies that critical points satisfy a corresponding eigenvalue problem (7.1). We also use the condition

(A3) $u_n \rightharpoonup u$ in $X$, $E_n^* \in \partial E(u_n)$, and $\langle E_n^*, u_n - u\rangle \to 0$ implies $u_n \to u$ in $X$

to verify a Palais–Smale condition for $E$. Notice that (A3) is a nonsmooth version of condition (S) that is typically used in critical point theory and that implies the stronger conditions $(\mathrm{S})_0$ and $(\mathrm{S})_1$. In our case of a convex function $E$ a nonsmooth version of the weaker condition $(\mathrm{S})_+$ would be equivalent to (A3) by monotonicity of $\partial E(\,\cdot\,)$ (cf. Zeidler [24, 27.1]).

**Proposition 7.2.** *Let* (A), (A1a), (A1b), (A2), (A3), *and* ($W^+$) *be satisfied. Then:*

(1) *$E$ satisfies the Palais–Smale condition on $K$ at any level $\gamma \in \mathbb{R}$.*

(2) $\sup_K E = \infty$.

(3) $\sup\{\mathrm{gen}(S) \mid S \subset K$ *compact, symmetric*$\} = \infty$.

Let us discuss condition (A3) before we prove the proposition.

**Remark 7.3.** (1) Condition (A3) is satisfied if there is some $c_4 > 0$ with

$$\langle a_1 - a_2, q_1 - q_2 \rangle \geq c_4 |q_1 - q_2|^p \quad \text{for all } q_i \in \mathbb{R}^n,\ a_i \in \partial A(x, q_i),\ i = 1, 2. \tag{7.8}$$

This can readily be deduced from

$$\langle E_n^* - E^*, u_n - u \rangle = \int_\Omega \langle a_n - a, Du_n - Du \rangle \, dx \geq c_4 \int_\Omega |Du_n - Du|^p \, dx \geq 0$$

where $a_n$ and $a$ correspond to $E_n^*$ and $E^*$, respectively.

(2) If we replace (A3) in Proposition 7.2 with monotonicity condition (7.8), then (A1b) can be omitted. To see that we first notice that $q = 0$ minimizes the function $A(x, \cdot)$ by (A1a) and (A2) and, thus, $0 \in \partial A(x, 0)$. With $q_2 = 0$, $a_2 = 0$ in (7.8) condition (A1b) follows.

(3) In the case of the $p$-Laplace operator (i.e. $A(x, q) = |q|^p$) the monotonicity condition (7.8) is satisfied for $p \geq 2$ but not for $p < 2$ (cf. Lindqvist [18, Lemma 4.2]). Nevertheless (A3) can be shown directly for all $p > 1$ (cf. Szulkin & Willem [22]).

**Corollary 7.4.** *Let* (A), (A1a), (A1b), (A2) *be satisfied. Then*

$$\|u\|_E := E(u)^{\frac{1}{p}} = \left( \int_\Omega A(x, Du) \, dx \right)^{\frac{1}{p}}$$

*is an equivalent norm on $X$. If $X$ is uniformly convex with norm $\|\cdot\|_E$, then* (A3) *is satisfied.*

*Proof.* First we notice that we can take $\beta = 0$ in (A1a) by (A2). Then we readily derive the equivalence of the norms $\|\cdot\|_X$ and $\|\cdot\|_E$ from (A1a) and (2.3).

Now let $u_n \rightharpoonup u$ be a sequence as in (A3). By convexity we have

$$E(u) \leq \liminf_{n \to \infty} E(u_n)$$

and

$$\langle E_n^*, u_n - u \rangle \geq E(u_n) - E(u) \quad \text{for } E_n^* \in \partial E(u_n).$$

Thus

$$\limsup_{n \to \infty} E(u_n) \leq E(u).$$

Consequently, $E(u_n) \to E(u)$, i.e. $\|u_n\|_E \to \|u\|_E$. Hence $\|u_n - u\|_E \to 0$ by uniform convexity and, therefore, $u_n \to u$ in $X$ by the equivalence of norms. But this implies condition (A3). □

*Proof of Proposition* 7.2. (1) Let $\{u_n\} \in K$ be a sequence with $E(u_n) \to \gamma$ and $|dE|_K(u_n) \to 0$. Since $\partial E(u_n) \neq \emptyset$, there are $\tilde{E}_n^* \in \partial E(u_n)$ and corresponding measurable functions $\tilde{a}_n$ with $\tilde{a}_n(x) \in \partial A(x, Du_n(x))$ a.e. on $\Omega$ given by Lemma 3.1.

We have that

$$\begin{aligned} \int_\Omega |Du_n|^p\,dx &\overset{\text{(A1b)}}{\leq} \frac{1}{c_2}\int_\Omega \langle \tilde{a}_n(x), Du_n(x)\rangle\,dx \\ &\overset{(2.2)}{=} \frac{p}{c_2}\int_\Omega A(x, Du_n(x))\,dx = \frac{p}{c_2}E(u_n). \end{aligned} \tag{7.9}$$

For $p < n$ we get

$$\begin{aligned} \int_\Omega \omega^-|u_n|^p\,dx + 1 &\overset{u_n\in K}{=} \int_\Omega \omega^+|u_n|^p\,dx \overset{\text{Hölder}}{\leq} \|\omega^+\|_{n/p}\|u_n\|_{p^*}^p \\ &\overset{X\hookrightarrow L^{p^*}}{\leq} c\,\|\omega^+\|_{n/p}\|Du_n\|_p^p \overset{(7.9)}{\leq} c\,\|\omega^+\|_{n/p}E(u_n). \end{aligned} \tag{7.10}$$

Analogously, with $X \hookrightarrow L^\infty$, we obtain

$$\int_\Omega \omega^-|u_n|^p\,dx + 1 \leq c\,\|\omega^+\|_1 E(u_n) \quad \text{for } p > n.$$

Hence $u_n$ is bounded in $\tilde{X}$ and, at least for a subsequence, $u_n \rightharpoonup u$ in $\tilde{X}$ and in $X$. If $\gamma = 0$, we get a contradiction in (7.10), since the most left term is positive, and thus $\gamma > 0$.

By (7.7) we find $\lambda_n \in \mathbb{R}$ and $E_n^* \in \partial E(u_n)$ with corresponding functions $a_n$ (notice that $\partial E(u_n) \neq \emptyset$ is weakly*-compact) such that

$$\begin{aligned} |dE|(u_n) &\geq \min\{\|E^* - \lambda G'(u_n)\| \mid E^* \in \partial E(u_n),\ \lambda \in \mathbb{R}\} \\ &= \|E_n^* - \lambda_n G'(u_n)\| \\ &\geq \left|\left\langle E_n^* - \lambda_n G'(u_n), \frac{u_n}{\|u_n\|}\right\rangle\right| \\ &= \frac{1}{\|u_n\|}\left|\int_\Omega \langle a_n, Du_n\rangle\,dx - \lambda_n p\int_\Omega \omega|u_n|^p\,dx\right| \\ &\overset{(2.2),\,u_n\in K}{=} \frac{p}{\|u_n\|}\left|\int_\Omega A(x, Du_n)\,dx - \lambda_n\right| \geq c\,|E(u_n) - \lambda_n| \end{aligned} \tag{7.11}$$

where, for the last estimate, we have used that the sequence $\{u_n\}$ is bounded. From $|dE|(u_n) \to 0$ we derive that

$$\lambda_n \to \gamma > 0.$$

Analogously to (7.11) we have that

$$\begin{aligned}\|u_n - u\|\,|dE|(u_n) &\geq \big\langle E_n^* - \lambda_n G'(u_n), u_n - u\big\rangle \\ &= \big\langle E_n^* - \lambda_n G'_+(u_n) + \lambda_n G'_-(u_n), u_n - u\big\rangle.\end{aligned}$$

Since $G'_+$ is completely continuous,

$$\lim_{n\to\infty} \lambda_n \big\langle G'_+(u_n), u_n - u\big\rangle = 0.$$

Choosing any $E^* \in \partial E(u) \neq \emptyset$, we have

$$\big\langle E^*, u_n - u\big\rangle \to 0.$$

Since $\{u_n\}$ is bounded, we get in the limit

$$\begin{aligned}0 &= \lim_{n\to\infty} \|u_n - u\|\,|dE|(u_n) \\ &\geq \limsup_{n\to\infty} \big\langle E_n^* - E^*, u_n - u\big\rangle + \limsup_{n\to\infty} \lambda_n \big\langle G'_-(u_n), u_n - u\big\rangle.\end{aligned}$$

The convexity of $G_-$ implies that

$$G_-(u) \leq \liminf_{n\to\infty} G_-(u_n) \leq \limsup_{n\to\infty} G_-(u_n)$$

and

$$\limsup_{n\to\infty} \big\langle G'_-(u_n), u_n - u\big\rangle \geq \limsup_{n\to\infty} G_-(u_n) - G_-(u) \geq 0\,.$$

By the convexity of $E$ we have

$$\big\langle E_n^* - E^*, u_n - u\big\rangle \geq 0.$$

Consequently $\big\langle E_n^*, u_n - u\big\rangle \to 0$ and, by (A3), we get $u_n \to u$ in $X$. Moreover,

$$\lim_{n\to\infty} G_-(u_n) = G_-(u)$$

by $\gamma > 0$. We conclude that $\|u_n\|_{\tilde{X}} \to \|u\|_{\tilde{X}}$ and, since $\tilde{X}$ is uniformly convex, $u_n \to u$ in $\tilde{X}$. Thus the Palais–Smale condition is satisfied.

(2) Let $y \in \Omega$ be a point of density 1 of $\{\omega > 0\}$ and a Lebesgue point of $\omega$. Then we can find $0 < r_1 < r_2$ and a Lipschitz continuous function $v_0$ supported on $B_{r_2}(y)$ with $0 \leq v_0 \leq 1$ and $v_0 = 1$ on $B_{r_1}(y)$ such that $\alpha := G(v_0) > 0$. Moreover there is $0 < r_0 < r_1$ such that $\alpha/2 \leq G(v_0 + v) \leq 2\alpha$ for all Lipschitz con-

tinuous $v$ supported on $B_{r_0}(y)$ with $0 \le v \le 1$. Obviously there are Lipschitz continuous functions $v_k$ supported on $B_{r_0}(y)$ that are radially symmetric with respect to $y$ and that satisfy $0 \le v_k \le 1$ and $|Dv_k(x)| \ge k$ a.e. on $B_{r_0}(y)$ (take functions that oscillate "radially"). For

$$\tilde{v}_k := \frac{v_0 + v_k}{G(v_0 + v_k)^{\frac{1}{p}}}$$

we then get $G(\tilde{v}_k) = 1$ and there is some $c > 0$ with

$$\begin{aligned} E(\tilde{v}_k) &= \frac{E(v_0 + v_k)}{G(v_0 + v_k)} = \frac{E(v_0) + E(v_k)}{G(v_0 + v_k)} \\ &\ge \frac{1}{2\alpha}\left(E(v_0) + \frac{c_2}{p}\int_{B_{r_0}(y)} |Dv_k|^p\,dx\right) \ge c(E(v_0) + k^p). \qquad (7.12) \end{aligned}$$

But this gives the assertion.

(3) For $k \in \mathbb{N}$ let $B_1, \dots, B_k \subset \Omega$ be pairwise disjoint open balls centered at $y_1, \dots, y_k$, respectively, such that all $y_j$ are points of density 1 of $\{\omega > 0\}$ and Lebesgue points of $\omega$. Then we find $v_j \in \tilde{X}$ supported on $B_j$ such that $G(v_j) = 1$ for all $j = 1, \dots, k$. Set $X_k := \operatorname{lin}\{v_1, \dots, v_k\}$ for the linear hull of the $v_j$ and consider the convex hull $C_k := \operatorname{conv}\{\pm v_1, \dots, \pm v_k\}$. Then

$$\Gamma_k := \left\{ \sum_{j=1}^k \alpha_j v_j \;\middle|\; \sum_{j=1}^k |\alpha_j| = 1,\ \alpha_j \in \mathbb{R} \right\}$$

is the boundary of $C_k$ within $X_k$. By the equivalence of norms in $\mathbb{R}^k$ there is $c > 0$ such that for $v \in \Gamma_k$

$$\begin{aligned} G(v) &= \int_\Omega \omega \left| \sum_{j=1}^k \alpha_j v_j \right|^p dx = \sum_{j=1}^k |\alpha_j|^p \int_\Omega \omega |v_j|^p\,dx \\ &= \sum_{j=1}^k |\alpha_j|^p \ge c\left(\sum_{j=1}^k |\alpha_j|\right)^p = c > 0. \end{aligned}$$

Thus $v \to v/G(v)^{\frac{1}{p}}$ is an odd homeomorphism from $\Gamma_k$ to the normalized set

$$\tilde{\Gamma}_k := \{\tilde{v} |\ \tilde{v} = v/G(v)^{\frac{1}{p}},\ v \in \Gamma_k\} \subset K$$

and we readily verify that gen $\tilde{\Gamma}_k = k$ (cf. [23, p. 320]). □

In order to apply Proposition A.5 we still notice that Lemma 3.1 remains true in $\tilde{X}$. Combined with Proposition 7.1, Proposition 7.2, and (7.7) we obtain the next result.

**Theorem 7.5.** *Let* (A), (A1a), (A1b), (A2), (A3), ($W^+$) *be satisfied. Then for any* $k \in \mathbb{N}$ *there are critical points* $\pm u_k \in \tilde{X}$ *of* $E$ *subject to* $K$, *and the* $\pm u_k$ *are eigenfunctions of* (7.1) *with corresponding eigenvalues* $\lambda_k \to \infty$.

**Remark 7.6.** (1) If $\omega^-$ satisfies assumption ($W^+$) instead of $\omega^+$ and if we replace $\omega$ with $-\omega$ in the previous theorem, then we obtain a sequence of eigenvalues $\tilde{\lambda}_k \to -\infty$.

(2) Let us provide specific examples satisfying all the assumptions of Theorem 7.5. In each of the examples below, the assumptions (A), (A1a), (A1b), and (A2) can be checked easily (cf. also Section 2), and for (A3) we apply Corollary 7.4.

Of course the theorem applies to the prototype examples $A(x, q) = |q|^p$ for $1 < p < \infty$ and covers previous results. More generally we can take

$$A(x, q) = |q|_s^p \quad \text{for } 1 < s, p < \infty$$

(cf. Section A.4 for uniform convexity of the corresponding norm).

Nonsmooth examples satisfying all the assumptions of Theorem 7.5 are given, e.g., by

$$A_1(x, q) = |q|_1^p + |q|_s^p \quad \text{with } 1 < s, p < \infty$$

and

$$A_\infty(x, q) = |q|_\infty^p + |q|_s^p \quad \text{with } 1 < s, p < \infty.$$

To see that we first notice that the norms

$$\|u\|_{1,p} := \left(\int_\Omega |Du|_1^p \, dx\right)^{1/p} \quad \text{and} \quad \|u\|_{\infty,p} := \left(\int_\Omega |Du|_\infty^p \, dx\right)^{1/p}$$

are equivalent to the uniformly convex norm

$$\|u\|_{s,p} := \left(\int_\Omega |Du|_s^p \, dx\right)^{1/p}.$$

Then

$$\left(\|u\|_1^p + \|u\|_s^p\right)^{1/p} = \left(\int_\Omega A_1(x, Du) \, dx\right)^{1/p}$$

and

$$\left(\|u\|_\infty^p + \|u\|_s^p\right)^{1/p} = \left(\int_\Omega A_\infty(x, Du) \, dx\right)^{1/p}$$

are also uniformly convex norms on $X$ by Proposition A.8 in Section A.4 below (cf. also Beauzamy [6, Exercise 3.II.1]). Hence, by Corollary 7.4, condition (A3) is satisfied for $A_1$ and $A_\infty$.

# A Appendix

## A.1 Tools from convex analysis

Here we collect some results of convex analysis specialized to our setting. Let $\Omega \subset \mathbb{R}^n$ be open and, for $1 < p < \infty$, set $X := D_0^{1,p}(\Omega)$, $Z := L^p(\Omega, \mathbb{R}^n)$. Moreover, let $A : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be such that $A(x, \cdot)$ is convex for all $x \in \Omega$ and that $A(\cdot, q)$ is (Lebesgue) measurable for all $q \in \mathbb{R}^n$. We assume that there are $c \geq 0$, $\beta \in L^1(\Omega)$ with

$$|A(x,q)| \leq c|q|^p + \beta(x) \quad \text{for all } q \in \mathbb{R}^n, \text{ a.e. } x \in \Omega \tag{A.1}$$

and that there is $\tilde{\alpha} \in L^{p'}(\Omega, \mathbb{R}^n)$, $\tilde{\beta} \in L^1(\Omega)$ with

$$A(q,x) \geq \langle \tilde{\alpha}(x), q \rangle + \tilde{\beta}(x) \quad \text{for all } x \in \Omega,\ q \in \mathbb{R}^n. \tag{A.2}$$

Notice that the last condition is trivially satisfied if $A(q,x) \geq 0$ for all $x \in \Omega$, $q \in \mathbb{R}^n$. Now we define

$$F(\alpha) := \int_\Omega A(x, \alpha(x))\, dx \quad \text{for } \alpha \in Z.$$

**Lemma A.1.** *The function $F : Z \to \mathbb{R}$ is finite, convex, and continuous. Moreover*

$$\partial F(\alpha) = \{a \in L^{p'}(\Omega, \mathbb{R}^n) \mid a(x) \in \partial A(x, \alpha(x)) \text{ for a.e. } x \in \Omega\} \quad \text{for all } \alpha \in Z$$

*($\partial A$ denotes the subdifferential with respect to the second argument).*

*Proof.* Note that $F$ is finite by (A.1) and convex by the convexity of $A(x, \cdot)$. Let $\alpha_n \to \alpha$ in $Z$. Then, possibly for a subsequence, $\alpha_n(x) \to \alpha(x)$ a.e. on $\Omega$. By (A.1),

$$|A(x, \alpha_n(x))| \leq c|\alpha_n(x)|^p + \beta(x) \quad \text{for a.e. } x \in \Omega.$$

Thus, $F(\alpha_n) \to F(\alpha)$ by generalized dominated convergence and continuity of $F$ is verified. The statement about the subdifferential is a direct consequence of Barbu [4, Proposition 1.9] combined with (A.2). □

Consider the continuous linear operator $L : X \to Z$ with

$$Lu := Du$$

and let $L^* : Z^* \to X^*$ denote its adjoint operator.

**Lemma A.2.** *The function $F \circ L : X \to \mathbb{R}$ is finite, convex, and continuous on $X$. Furthermore*

$$\partial(F \circ L)(u) = L^* \partial F(Lu) \quad \text{for all } u \in X.$$

*Proof.* We readily verify that the function $F$ is finite, convex and continuous on $X$. The structure of the subdifferential follows from Ekeland & Temam [12, Proposition I.5.7]. $\square$

Define $E : X \to \mathbb{R}$ with

$$E(u) := \int_\Omega A(x, Du(x))\, dx.$$

Obviously, $E(u) = F(Lu)$. Thus $E$ is convex and continuous on $X$, and we obtain

**Proposition A.3.** *Let $E^* \in \partial E(u)$. Then there is some $a \in L^{p'}(\Omega, \mathbb{R}^n)$ such that*

$$a(x) \in \partial A(x, Du(x)) \quad \textit{for a.e. } x \in \Omega$$

*and*

$$\langle E^*, v\rangle = \int_\Omega \langle a(x), Dv(x)\rangle\, dx \quad \textit{for all } v \in X.$$

*Proof.* Use the previous results and observe that there is some $F^* \in \partial F(Du)$ such that

$$\langle E^*, v\rangle = \langle L^* F^*, v\rangle = \langle F^*, Lv\rangle \quad \text{for all } v \in X. \qquad \square$$

## A.2 Existence of critical points

Many general results about the existence of critical points are based on the category as topological index. However, for the verification of assumptions or for subsequent arguments the genus as topological index appears to be more convenient. Therefore we will provide some critical point result for our analysis based on the genus where we use the notation introduced in Section 7. Let $F : M \to \mathbb{R}$ be a function on the metric space $M$. For a closed set $A \subset M$ we define the *category* $\operatorname{cat}_M A$ of $A$ within $M$ to be the smallest number $k \in \mathbb{N}$ of closed sets $A_1, \ldots, A_k \subset M$ such that all $A_i$ are contractible in $M$ and $A \subset \bigcup_{i=1}^k A_i$. In particular, $\operatorname{cat}_M A = \infty$ if there is no finite cover of that kind and $\operatorname{cat}_M \emptyset = 0$. Let us recall the following general result from Degiovanni & Marzocchi [10, Theorem 3.10].

**Proposition A.4.** *Let $M$ be a weakly locally contractible complete metric space, let $F : M \to \mathbb{R}$ be a continuous function that is bounded from below and that satisfies the Palais–Smale condition at any level $\gamma \in \mathbb{R}$, and assume*

$$\sup\{\operatorname{cat}_M A \mid A \subset M \textit{ compact}\} = \infty.$$

*Then $F$ has a sequence of critical points $\{u_k\}_{k\in\mathbb{N}}$ in $M$ with critical values*

$$\gamma_k = \inf_{A \in \mathcal{A}_k} \max_{u \in A} F(u)$$

*where*

$$\mathcal{A}_k = \{A \subset M \mid A \textit{ compact, } A \neq \emptyset, \operatorname{cat}_M A \geq k\}.$$

*Moreover,* $\gamma_k \to \infty$ *if* $\sup_M F = \infty$.

For analyzing a special case we consider a Banach space $Z$ and a function $E : Z \to \mathbb{R}$ that is continuous and symmetric, i.e. $E(-u) = E(u)$. We are looking for critical points of $E$ on the metric space

$$K := \{u \in Z \mid G(u) = 1\}$$

where $G : Z \to \mathbb{R}$ is also assumed to be continuous and symmetric, i.e. we are looking for $u \in K$ with vanishing weak slope $|dE|_K(u) = 0$. Moreover we assume that there is some $\beta > 0$ with

$$G(tu) = t^\beta G(u) \quad \text{for all } t \geq 0. \tag{A.3}$$

For a nontrivial application of the previous proposition to this situation we have to work in the projective space identifying the antipodal points $\{u, -u\}$. Hence we define

$$A^* := \{\{v, -v\} \mid v \in A\} \quad \text{for any symmetric } A \subset Z \setminus \{0\}$$

(often also denoted as $A/\mathbb{Z}^2$) and we can consider $E, G$ as functions on $(Z \setminus \{0\})^*$ in a natural way. Proposition A.4 now provides critical points $u^*$ of $E$ as function on the space $K^*$ (endowed with the induced metric), i.e. $|dE|_{K^*}(u^*) = 0$. But we are interested in critical points of $E$ on $K$ and we claim to formulate some existence result with conditions in terms of $Z$ and $K$ rather than $K^*$. We first observe that $u$ and $-u$ are critical points of $E$ on $K$ if the corresponding $u^*$ is a critical point of $E$ on $K^*$ (cf. Milbers & Schuricht [20]). Moreover we readily see that $E$ is continuous, bounded from below, and satisfies the Palais–Smale condition at level $\gamma$ as function on $K^*$ if and only if these properties are satisfied for $E$ as function on $K$ and, clearly, $\sup_K E = \sup_{K^*} E$. Let now $\mathcal{A}_k$ denote the sets defined in Proposition A.4 with respect to the metric space $K^*$ and let us consider

$$\mathcal{B}_k := \{A \subset K \mid A \text{ compact, symmetric, } A \neq \emptyset, \operatorname{gen} A \geq k\}.$$

In order to compare $\mathcal{A}_k$ and $\mathcal{B}_k$ we fix $A^* \in \mathcal{A}_k$ and let $A$ be the corresponding set in $K$. A simple normalization argument using (A.3) shows that

$$\operatorname{cat}_{K^*} A^* = \operatorname{cat}_{(Z \setminus \{0\})^*} A^*$$

and from Rabinowitz [21, Theorem 3.7] we obtain that

$$\operatorname{cat}_{(Z \setminus \{0\})^*} A^* = \operatorname{gen} A.$$

But this means that the sets $\mathcal{A}_k$ and $\mathcal{B}_k$ uniquely correspond to each other. Using (A.3) we obtain that $K$ and $K^*$ are always weakly locally contractible (i.e. any point has a contractible neighborhood in $K$ and $K^*$, respectively). Hence Proposition A.4 directly implies the following result.

**Proposition A.5.** *Let $Z$ be a real Banach space, let $E, G : Z \to \mathbb{R}$ be continuous and symmetric, let $G$ satisfy* (A.3)*, and let $K$ be the metric space as defined above. Moreover, let $E$ be bounded from below on $K$, let it satisfy the Palais–Smale condition on $K$ at any level $\gamma \in \mathbb{R}$, and let*

$$\sup\{\operatorname{gen} A \mid A \subset K \textit{ compact, symmetric}\} = \infty.$$

*Then $E$ has a sequence of critical points $\{u_k\}$ in $K$ with critical values*

$$\gamma_k = \inf_{A \in \mathcal{B}_k} \max_{u \in A} E(u)$$

*where $\mathcal{B}_k$ is given as above. In addition, $\gamma_k \to \infty$ if $\sup_K E = \infty$.*

## A.3 Capacity

Here we formulate in which way we use the notion of capacity and related notions (cf. Heinonen et al. [15]). For $\Omega \subset \mathbb{R}^n$ open, $K \subset \Omega$ compact, $U \subset \Omega$ open, and $E \subset \Omega$ arbitrary we define the (variational) *capacity*

$$\operatorname{cap}_p(K, \Omega) := \inf\left\{ \int_\Omega |Du|^p \, dx \;\middle|\; \varphi \in C_0^\infty(\Omega),\ \varphi \geq 1 \ \text{ on } \ K \right\},$$

$$\operatorname{cap}_p(U, \Omega), := \sup\{\operatorname{cap}_p(K, \Omega) \mid K \subset \Omega \text{ compact}\},$$

$$\operatorname{cap}_p(E, \Omega), := \inf\{\operatorname{cap}_p(U, \Omega) \mid U \text{ open with } E \subset U \subset \Omega\}.$$

We say that $E \subset \mathbb{R}^n$ has *capacity zero*, written $\operatorname{cap}_p(E) = 0$, if

$$\operatorname{cap}_p(E \cap U, U) = 0 \quad \text{for all } U \subset \mathbb{R}^n \text{ open.}$$

Otherwise we write $\operatorname{cap}_p(E) > 0$. A property holds *quasi-everywhere* (or *q.e.*) if it holds except on a set of capacity zero.

In addition, for $E \subset \mathbb{R}^n$ we consider the *Sobolev-capacity*

$$\operatorname{Cap}_p(E) := \inf\left\{ \int_{\mathbb{R}^n} |u|^p + |Du|^p \, dx \;\middle|\; \begin{array}{l} u \in W^{1,p}(\mathbb{R}^n), \\ u = 1 \text{ on some open } U,\ E \subset U \end{array} \right\}.$$

We have $\operatorname{Cap}_p(E_1) \leq \operatorname{Cap}_p(E_2)$ if $E_1 \subset E_2$ and

$$\operatorname{Cap}_p\Big(\bigcup_{k \in \mathbb{N}} E_k\Big) \leq \sum_{k \in \mathbb{N}} \operatorname{Cap}_p E_k \quad \text{for any } E_1, E_2, \ldots \subset \mathbb{R}^n.$$

Moreover, for $E \subset \mathbb{R}^n$

$$\mathrm{Cap}_p(E) = 0 \quad \text{if and only if } \mathrm{cap}_p(E) = 0$$

(cf. [15, Corollaries 2.37 and 2.39]). A function $u : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *quasi-continuous* if for any $\varepsilon > 0$ there is an open $U$ with $\mathrm{Cap}_p(U) < \varepsilon$ such that $u$ is finite and continuous on $\Omega \setminus U$.

For $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we define the precise representative $u^\star$ by

$$u^\star(x) := \begin{cases} \lim_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)\,dy & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition A.6** (Poincaré inequality). *Let $U \subset \mathbb{R}^n$ be an open and bounded domain with Lipschitz boundary, let $V \subset U$ with $\mathrm{Cap}_p(V) > 0$ and $1 < p < \infty$, and set*

$$Z := \{v \in W^{1,p}(U) \mid v^\star = 0 \text{ on } V\}.$$

*Then there is $c > 0$ such that*

$$\int_U |v|^p\,dx \le c \int_U |Dv|^p\,dx \quad \text{for all } v \in Z\,.$$

*Proof.* For contradiction let us assume that there are $v_k \in Z$ with

$$\|v_k\|_{W^{1,p}} = 1, \quad \int_U |v_k|^p\,dx > k \int_U |Dv_k|^p\,dx \quad \text{for all } k \in \mathbb{N}. \tag{A.4}$$

Since $v_k$ is bounded in $W^{1,p}(U)$, we have (at least for a subsequence) that

$$v_k \overset{W^{1,p}}{\rightharpoonup} v \quad \text{and} \quad v_k \overset{L^p}{\to} v.$$

By (A.4),

$$\int_U |Dv_k|^p\,dx \le \frac{1}{k} \int_U |v_k|^p\,dx \le \frac{1}{k} \to 0.$$

Hence $v_k \to v$ in $W^{1,p}(U)$ with $Dv = 0$ a.e. on $U$. Thus $v^\star$ is constant on $U$ with $\|v\|_{W^{1,p}} = 1$ and, at least for a subsequence, $v_k \to v^\star$ a.e. on $U$. Clearly, $v^\star$ is quasi-continuous.

Using [15, Theorem 4.3] and $v_k \in Z$, we find smooth $w_k \in W^{1,p}(U) \cap C^\infty(U)$ approximating $v_k$ and open $W_k \subset U$ such that

$$\|v_k - w_k\|_{W^{1,p}} < \frac{1}{k}, \quad \mathrm{cap}_p W_k < \frac{\gamma}{4^k}, \quad |w_k(x)| < \frac{1}{k} \text{ on } V \setminus W_k \quad \text{for all } k \in \mathbb{N}$$

where $\gamma := \mathrm{Cap}_p V > 0$. Consequently, $w_k \to v$ in $W^{1,p}(U)$. Again by [15, Theorem 4.3] there is some quasi-continuous $\tilde{v} \in W^{1,p}(U)$ such that, at least for a subsequence,

$$w_k \to \tilde{v} \text{ q.e. on } U, \quad w_k \to v^\star \text{ a.e. on } U \quad \text{and} \quad w_k \to 0 \text{ on } V \setminus W \tag{A.5}$$

with $W := \bigcup_{k\in\mathbb{N}} W_k$. Obviously $\tilde{v} = v^\star = v$ a.e. on $U$, i.e. $\tilde{v} = v^\star = v$ within $W^{1,p}(U)$. Since $\tilde{v}$ and $v^\star$ are both quasi-continuous, it follows that $\tilde{v} = v^\star$ q.e. on $U$ by [15, Theorem 4.14]. By

$$\gamma = \operatorname{Cap}_p V \le \operatorname{Cap}_p(V \cap W) + \operatorname{Cap}_p(V \setminus W) \le \operatorname{Cap}_p(W) + \operatorname{Cap}_p(V \setminus W)$$

and

$$\operatorname{Cap}_p W \le \sum_{k\in\mathbb{N}} \operatorname{Cap}_p W_k < \frac{\gamma}{2}$$

we obtain $\operatorname{Cap}_p(V \setminus W) > \gamma/2$. By (A.5) we get $\tilde{v} = 0$ q.e. on $V \setminus W$ and, hence, $v^\star = 0$ q.e. on $V \setminus W$. Since $v^\star$ is constant on $U$, we conclude that $v = v^\star$ is zero in $W^{1,p}(U)$. But this contradicts $\|v\|_{W^{1,p}} = 1$, which confirms the assertion. □

Let us finally recall Proposition 2.1 from Kawohl et al. [16] where we include the short proof for the convenience of the reader.

**Proposition A.7.** *Let $u : \Omega \to \mathbb{R}$ be quasi-continuous and let*

$$\mathcal{Z}_t := \{x \in \Omega \mid u(x) = t\}$$

*be a level set. Then there are sets $F_n \subset \mathcal{Z}_t$ closed in $\Omega$ such that*

$$\operatorname{Cap}_p\Big(\mathcal{Z}_t \setminus \bigcup_{n\in\mathbb{N}} F_n\Big) = 0. \tag{A.6}$$

*Proof.* Since $u$ is quasi-continuous, for any $n \in \mathbb{N}$ there is an open $U_n$ such that $\operatorname{Cap}_p U_n \le \frac{1}{n}$ and $u$ continuous on $\Omega \setminus U_n$, which is closed in $\Omega$. Hence the set $F_n := \mathcal{Z}_t \cap (\Omega \setminus U_n)$ is also closed in $\Omega$ and

$$\begin{aligned}\operatorname{Cap}_p\Big(\mathcal{Z}_t \setminus \bigcup_{n\in\mathbb{N}} F_n\Big) &\le \operatorname{Cap}_p(\mathcal{Z}_t \setminus F_n)\\ &= \operatorname{Cap}_p(\mathcal{Z}_t \cap U_n) \le \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.\end{aligned}$$

But this implies (A.6). □

### A.4 Uniform convexity

Let us recall that a Banach space $X$ with norm $\|\cdot\|$ is uniformly convex if for any $\varepsilon > 0$ there is some $\delta > 0$ such that, for $u, v \in X$,

$$\|u\| = \|v\| = 1,\ \|u - v\| \ge \varepsilon \quad \text{implies that} \quad \frac{\|u + v\|}{2} \le 1 - \delta.$$

Note that Sobolev spaces such as $W_0^{1,p}(\Omega)$ and $D_0^{1,p}(\Omega)$ equipped with the standard norm $(\int_\Omega |Du|_p^p\,dx)^{1/p}$ are uniformly convex for $1 < p < \infty$ (cf. Adams [1, p. 7f.]). Clearly

$$\|Du\|_{s,p} := \left(\int_\Omega |Du|_s^p\,dx\right)^{1/p}$$

are equivalent norms on these spaces for $1 \le s \le \infty$. Moreover these norms are also uniformly convex for $1 < s < \infty$ (cf. Fan & Guan [14, Theorem 2.4] with $A(\xi) = \varphi(|\xi|_s)$ and $\varphi(t) = |t|^p$).

We now provide some result for the construction of further uniformly convex norms (cf. also Beauzamy [6] where this is stated as Exercise 3.II.1 without proof). This will allow the construction of nonsmooth examples for $A$ satisfying condition (A3) (cf. Remark 7.6 (2)).

**Proposition A.8.** *Let $X$ be a Banach space that is uniformly convex with norm $\|\cdot\|$ and let $\|\cdot\|_*$ be a further norm on $X$ such that*

$$\|u\|_* \le c\|u\| \quad \textit{for all } u \in X$$

*with some $c > 0$. Then*

$$\|u\|_0 := \left(\|u\|^p + \|u\|_*^p\right)^{1/p} \quad \textit{with } 1 < p < \infty$$

*is an equivalent norm on $X$ that is also uniformly convex.*

*Proof.* Let us fix $p \in (1,\infty)$. Then a norm $\|\cdot\|_\sim$ is uniformly convex on $X$ if and only if for any $\varepsilon > 0$ there is some $\delta > 0$ such that any $u, v \in X$ with

$$\|u\|_\sim \le 1, \quad \|v\|_\sim \le 1, \quad \|u - v\|_\sim \ge \varepsilon$$

satisfy

$$\left\|\frac{u+v}{2}\right\|_\sim \le (1-\delta)\frac{\|u\|_\sim^p + \|v\|_\sim^p}{2}$$

(cf. Beauzamy [6]).

For verifying this condition we consider $u, v \in X$ satisfying

$$\|u\|_0 \le 1, \quad \|v\|_0 \le 1, \quad \|u - v\|_0 \ge \varepsilon$$

and we fix $\varepsilon > 0$. Then

$$\varepsilon^p \le \|u - v\|_*^p + \|u - v\|^p \le (c^p + 1)\|u - v\|^p$$

Now we take $\tilde{\varepsilon} > 0$ such that $\tilde{\varepsilon}^p := \frac{\varepsilon^p}{c^p+1}$ and, for the norm $\|\cdot\|_\sim = \|\cdot\|$, we choose the corresponding $\delta(\tilde{\varepsilon})$ according to the condition for uniform convexity stated above. Then, also using convexity of $\|\cdot\|_*$, we obtain

$$
\begin{aligned}
\left\|\frac{u+v}{2}\right\|_0^p &= \left\|\frac{u+v}{2}\right\|_*^p + \left\|\frac{u+v}{2}\right\|^p \\
&\le \left(\frac{\|u\|_* + \|v\|_*}{2}\right)^p + (1-\delta(\tilde{\varepsilon}))\frac{\|u\|^p + \|v\|^p}{2} \\
&\le \frac{\|u\|_*^p + \|v\|_*^p}{2} + \frac{\|u\|^p + \|v\|^p}{2} \\
&\qquad - \delta(\tilde{\varepsilon})\left(\frac{\|u\|^p + \|v\|^p}{4} + \frac{\|u\|^p + \|v\|^p}{4}\right) \\
&\le \frac{\|u\|_0^p + \|v\|_0^p}{2} - \frac{\delta(\tilde{\varepsilon})}{2c}\frac{\|u\|_*^p + \|v\|_*^p}{2} - \frac{\delta(\tilde{\varepsilon})}{2}\frac{\|u\|^p + \|v\|^p}{2} \\
&\le (1-\delta_0(\varepsilon))\frac{\|u\|_0^p + \|v\|_0^p}{2}
\end{aligned}
\tag{A.7}
$$

where

$$
\delta_0(\varepsilon) := \min\left\{\frac{\delta(\tilde{\varepsilon})}{2c}, \frac{\delta(\tilde{\varepsilon})}{2}\right\}.
$$

But this implies the uniform convexity of $\|\cdot\|_0$. The equivalence of $\|\cdot\|_0$ to $\|\cdot\|$ follows easily. □

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