Global existence and exponential growth for a viscoelastic wave equation with dynamic boundary conditions

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Abstract. The goal of this work is to study a model of the wave equation with dynamic boundary conditions and a viscoelastic term. First, applying the Faedo–Galerkin method combined with the fixed point theorem, we show the existence and uniqueness of a local in time solution. Second, we show that under some restrictions on the initial data, the solution continues to exist globally in time. On the other hand, if the interior source dominates the boundary damping, then the solution is unbounded and grows as an exponential function. In addition, in the absence of the strong damping, then the solution ceases to exist and blows up in finite time.

Keywords. Damped viscoelastic wave equations, global solutions, exponential growth, blow up in finite time, dynamic boundary conditions.

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1 Introduction

We consider the following problem:

$$
\begin{cases}
u_{tt} - \Delta u - \alpha \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u, & x \in \Omega,\ t > 0, \\[2mm]
u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\[2mm]
u_{tt}(x,t) = -\left[\dfrac{\partial u}{\partial \nu}(x,t) - \int_0^t g(t-s)\dfrac{\partial u}{\partial \nu}(x,s)ds \right. \\[4mm]
\qquad\qquad \left. + \dfrac{\alpha \partial u_t}{\partial \nu}(x,t) + h(u_t) \right], & x \in \Gamma_1,\ t > 0, \\[2mm]
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega,
\end{cases}
\tag{1.1}
$$

where $u = u(x,t), t \geq 0, x \in \Omega$, Δ denotes the Laplacian operator with respect to the x variable, Ω is a regular and bounded domain of $\mathbb{R}^N$, $N \geq 1$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\mathrm{mes}(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \varnothing$ and $\partial/\partial\nu$ denotes the unit outer normal derivative,

α is a positive constant, $p > 2$, h and g are functions whose properties will be discussed in the next section, u_0, u_1 are given functions.

Nowadays the wave equation with dynamic boundary conditions is used in a wide field of applications [2, 3, 25]. See [19, 24] for some applications. Problems similar to (1.1) arise (for example) in the modeling of longitudinal vibrations in a homogeneous bar in which there are viscous effects. The term Δu_t, indicates that the stress is proportional not only to the strain, but also to the strain rate, see [6] fore more details.

In [14] the author introduced the model

$$u_{tt} - u_{xx} - u_{txx} = 0, \quad x \in (0, L), \, t > 0, \tag{1.2}$$

which describes the damped longitudinal vibrations of a homogeneous flexible horizontal rod of length L when the end $x = 0$ is rigidly fixed while the other end $x = L$ is free to move with an attached load. Thus she considered Dirichlet boundary condition at $x = 0$ and dynamic boundary conditions at $x = L$, namely

$$u_{tt}(L, t) = -[u_x + u_{tx}](L, t), \quad t > 0. \tag{1.3}$$

By rewriting the whole system within the framework of the abstract theories of the so-called B-evolution theory, the existence of a unique solution in the strong sense has been shown. An exponential decay result was also proved in [15] for a problem related to (1.2)–(1.3), which describe the weakly damped vibrations of an extensible beam. See [15] for more details.

Subsequently, Zang and Hu [29], considered the problem

$$u_{tt} - p(u_x)_{xt} - q(u_x)_x = 0, \quad x \in (0, 1), \, t > 0,$$

with

$$u(0, t) = 0, \quad p(u_x)_t + q(u_x)(1, t) + k u_{tt}(1, t) = 0, \quad t \geq 0.$$

By using the Nakao inequality, and under appropriate conditions on p and q, they established both exponential and polynomial decay rates for the energy depending on the form of the terms p and q.

Recently, the present authors have considered, in [11] and [12], problem (1.1) with $g = 0$ and a nonlinear boundary damping of the form $h(u_t) = |u_t|^{m-2}u_t$. A local existence result was obtained by combining the Faedo–Galerkin method with the contraction mapping theorem. Concerning the asymptotic behavior, the authors showed that the solution of such a problem is unbounded and grows up exponentially when time goes to infinity provided that the initial data are large enough and the damping term is nonlinear. The blow up result was shown when the damping is linear (i.e. $m = 2$). Also, we proved in [12] that under some re-

strictions on the exponents m and p, we can always find initial data for which the solution is global in time and decays exponentially to zero. These results have recently been generalized for a wide range of nonlinearities in the equation and in the boundary term: the authors proved the local existence and uniqueness by a sophisticated application of the nonlinear semigroup theory, see [13].

In the absence of the strong damping $\alpha \Delta u_t$ and for Dirichlet boundary conditions on the whole boundary $\partial \Omega$, the question of blow up in finite time of problem (1.1) has been investigated by many authors. Messaoudi [21] showed that if the initial energy is negative and if the relaxation function g satisfies the following assumption:

$$\int_0^\infty g(s)ds < \frac{(p/2) - 1}{(p/2) - 1 + (1/2p)}, \tag{1.4}$$

then the solution blows up in finite time. In fact this last condition has been assumed by other researchers. See for instance [16, 17, 20, 23, 26, 28].

The main goal of this paper is to prove the local existence and to study the asymptotic behavior of the solution of problem (1.1).

One of the main questions is to show a blow-up result of the solution. This question is a difficult open problem, since in the presence of the strong damping term, i.e. when $\alpha \neq 0$, the problem has a parabolic structure, which means that the solution gains more regularity. However, in this paper, we give a partial answer to this question and show that for $\alpha \neq 0$ and for large initial data, the solution is unbounded and grows exponentially as t goes to infinity. While for the case $\alpha = 0$, the solution has been shown to blow up in finite time.

The main contribution of this paper in this blow up result is the following: the exponential growth and blow-up results hold without making the assumption (1.4). In fact the only requirement is that the exponent p has to be large enough which is a condition much weaker than condition (1.4). Moreover, unlike in the works of Messaoudi and coworkers, we do not assume any polynomial structure on the damping term $h(u_t)$, to obtain an exponential growth of the solution or a blow up in finite time.

This paper is organized as follows: Firstly, applying the Faedo–Galerkin method combined with the fixed point theorem, we show, in Section 2, the existence and uniqueness of a local in time solution. Secondly, under the smallness assumption on the initial data, we show, in Section 3, that the solution continues to exist globally in time. On the other hand, in Section 4, we prove that under some restrictions on the initial data and if the interior source dominates the boundary damping, then the L^p-norm of the solution grows as an exponential function. Lastly, in Section 5, we investigate the case when $\alpha = 0$ and we prove that the solution ceases to exist and blows up in finite time.

2   Preliminary and local existence

In this section, we introduce some notations used throughout this paper. We also prove a local existence result of the solution of problem (1.1).

We denote

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u_{\Gamma_0} = 0\}.$$

By $(\cdot, \cdot)$ we denote the scalar product in $L^2(\Omega)$, i.e.

$$(u, v)(t) = \int_\Omega u(x, t)v(x, t)dx.$$

Also we mean by $\|\cdot\|_q$ the $L^q(\Omega)$-norm for $1 \leq q \leq \infty$, and by $\|\cdot\|_{q,\Gamma_1}$ the $L^q(\Gamma_1)$-norm.

Let $T > 0$ be a real number and X be a Banach space endowed with norm $\|\cdot\|_X$. We denote by $L^p(0, T; X)$, $1 \leq p < \infty$, the space of functions f which are L^p over $(0, T)$ with values in X, which are measurable and $\|f\|_X \in L^p(0, T)$. This space is a Banach space endowed with the norm

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f\|_X^p dt\right)^{1/p}.$$

By $L^\infty(0, T; X)$ we denote the space of functions $f :]0, T[\to X$ which are measurable and $\|f\|_X \in L^\infty(0, T)$. This space is a Banach space endowed with the norm

$$\|f\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|f\|_X.$$

We recall that if X and Y are two Banach spaces such that $X \hookrightarrow Y$ (continuous embedding), then

$$L^p(0, T; X) \hookrightarrow L^p(0, T; Y), \quad 1 \leq p \leq \infty.$$

We will also use the embedding (see [1, Therorem 5.8])

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^p(\Omega), \quad 2 \leq p \leq \bar{p}, \quad \text{where } \bar{p} = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2, \end{cases}$$

and also

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Gamma_1), \quad 2 \leq q \leq \bar{q}, \quad \text{where } \bar{q} = \begin{cases} \dfrac{2(N-1)}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2. \end{cases}$$

For $2 \leq m \leq \bar{q}$, let us denote $V = H^1_{\Gamma_0}(\Omega) \cap L^m(\Gamma_1)$.

We assume that the relaxation function g is of class C^1 on $\mathbb{R}$ and satisfies

$$g(s) \geq 0, \quad \text{for all } s \in \mathbb{R}, \quad \text{and} \quad \left(1 - \int_0^\infty g(s)ds\right) = l > 0. \qquad (2.1)$$

Moreover, we suppose that

$$g'(s) \leq 0, \quad \text{for all } s \geq 0. \qquad (2.2)$$

The hypotheses on the function h are the following:

(H1) h is continuous and strongly monotone, i.e. for $2 \leq m \leq \bar{q}$, there exists a constant $m_0 > 0$ such that

$$(h(s) - h(v))(s - v) \geq m_0|s - v|^m, \qquad (2.3)$$

(H2) there exist two positive constants c_m and C_m such that

$$c_m|s|^m \leq h(s)s \leq C_m|s|^m, \quad \text{for all } s \in \mathbb{R}. \qquad (2.4)$$

For a function $u \in C([0, T], H_{\Gamma_0}^1(\Omega))$, let us introduce the following notation:

$$(g \diamond u)(t) = \int_0^t g(t - s)\|\nabla u(s) - \nabla u(t)\|_2^2 ds.$$

Thus, when $u \in C([0, T], H_{\Gamma_0}^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$ such that

$$u_t \in L^2(0, T; H_{\Gamma_0}^1(\Omega)),$$

we have

$$\begin{aligned}
\frac{d}{dt}(g \diamond u)(t) &= \int_0^t g'(t - s)\|\nabla u(s) - \nabla u(t)\|_2^2 ds \\
&\quad + \frac{d}{dt}\left(\|\nabla u(t)\|_2^2\right) \int_0^t g(s)ds \\
&\quad - 2 \int_\Omega \int_0^t g(t - s)\nabla u(s)\nabla u_t(t)dsdx \\
&= (g' \diamond u)(t) - 2 \int_\Omega \int_0^t g(t - s)\nabla u(s)\nabla u_t(t)dsdx \\
&\quad + \frac{d}{dt}\left\{\|\nabla u(t)\|_2^2 \int_0^t g(s)ds\right\} - g(t)\|\nabla u(t)\|_2^2. \quad (2.5)
\end{aligned}$$

This last identity implies

$$\int_\Omega \int_0^t g(t-s)\nabla u(s)\nabla u_t(t)dsdx$$

$$= \frac{1}{2}(g' \diamond u)(t) + \frac{1}{2}\frac{d}{dt}\left\{\|\nabla u(t)\|_2^2 \int_0^t g(s)ds\right\}$$

$$- \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 - \frac{1}{2}\frac{d}{dt}(g \diamond u)(t). \qquad (2.6)$$

For $u \in C([0,T], H_{\Gamma_0}^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$ such that

$$u_t \in L^2(0,T; H_{\Gamma_0}^1(\Omega)),$$

let us define the modified energy functional E by

$$E(t,u,u_t) = E(t)$$

$$= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_{2,\Gamma_1}^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2$$

$$+ \frac{1}{2}(g \diamond u)(t) - \frac{1}{p}\|u(t)\|_p^p. \qquad (2.7)$$

The following local existence result of the solution of problem (1.1) is closely related to the one we have proved for a slightly different problem in [11, Theorem 2.1], where no memory term was present. Let us state it:

Theorem 2.1. *Assume that* (2.1)*,* (2.2) *and* (2.3) *hold. Let* $2 \le p \le \bar{q}$ *and*

$$\max\left(2, \frac{\bar{q}}{\bar{q}+1-p}\right) \le m \le \bar{q}.$$

Then given $u_0 \in H_{\Gamma_0}^1(\Omega)$ *and* $u_1 \in L^2(\Omega)$*, there exists* $T > 0$ *and a unique solution* u *of problem* (1.1) *on* $(0,T)$ *such that*

$$u \in C([0,T], H_{\Gamma_0}^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$$

and

$$u_t \in L^2(0,T; H_{\Gamma_0}^1(\Omega)) \cap L^m((0,T) \times \Gamma_1).$$

Let us mention that Theorem 2.1 also holds for $\alpha = 0$. The proof of Theorem 2.1 can be done along the same line as in [11, Theorem 2.1]. The main idea of the proof is based on the combination between the Fadeo–Galerkin approximations and the contraction mapping theorem. However, for the convenience of the reader we give only the outline of the proof here.

For $u \in C([0, T], H^1_{\Gamma_0}(\Omega)) \cap C^1([0, T], L^2(\Omega))$ given, let us consider the following problem:

$$\begin{cases} v_{tt} - \Delta v - \alpha \Delta v_t + \displaystyle\int_0^t g(t-s)\Delta v(s)ds = |u|^{p-2}u, & x \in \Omega, \, t > 0, \\[2mm] v(x,t) = 0, & x \in \Gamma_0, \, t > 0, \\[2mm] v_{tt}(x,t) = -\left[\dfrac{\partial v}{\partial \nu}(x,t) - \displaystyle\int_0^t g(t-s)\dfrac{\partial v}{\partial \nu}(x,s)ds \right. & \\[2mm] \qquad\qquad \left. + \dfrac{\alpha \partial v_t}{\partial \nu}(x,t) + h(v_t) \right], & x \in \Gamma_1, \, t > 0, \\[2mm] v(x,0) = u_0(x), \; v_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{2.8}$$

Definition 2.2. A function $v(x,t)$ such that

$$v \in L^\infty(0, T; H^1_{\Gamma_0}(\Omega)),$$
$$v_t \in L^2(0, T; H^1_{\Gamma_0}(\Omega)) \cap L^m((0,T) \times \Gamma_1),$$
$$v_t \in L^\infty(0, T; H^1_{\Gamma_0}(\Omega)) \cap L^\infty(0, T; L^2(\Gamma_1)),$$
$$v_{tt} \in L^\infty(0, T; L^2(\Omega)) \cap L^\infty(0, T; L^2(\Gamma_1))$$

and

$$v(x,0) = u_0(x),$$
$$v_t(x,0) = u_1(x),$$

is a generalized solution to problem (2.8) if for any function

$$\omega \in H^1_{\Gamma_0}(\Omega) \cap L^m(\Gamma_1)$$

and $\varphi \in C^1(0, T)$ with $\varphi(T) = 0$, we have the following identity:

$$\int_0^T (|u|^{p-2}u, w)(t)\, \varphi(t)\, dt$$
$$= \int_0^T \left[(v_{tt}, w)(t) + (\nabla v, \nabla w)(t) - \int_0^t g(t-s)(\nabla v(s), \nabla w(t))ds \right.$$
$$\left. + \alpha(\nabla v_t, \nabla w)(t)\varphi(t)dt \right]$$
$$+ \int_0^T \varphi(t)\left(\int_{\Gamma_1} v_{tt}(t)w\, d\Gamma + h(v_t)d\Gamma \right)dt.$$

Lemma 2.3. *Let* $2 \leq p \leq \bar{q}$ *and* $2 \leq m \leq \bar{q}$. *Let* $u_0 \in H^2(\Omega) \cap V$, $u_1 \in H^2(\Omega)$. *Then for any* $T > 0$, *there exists a unique generalized solution (in the sense of Definition* 2.2*)* $v(t, x)$ *of problem* (2.8).

The proof of Lemma 2.3 is essentially based on the Fadeo–Galerkin approximations combined with the compactness method and can be done along the same lines as in [11, Lemma 2.2]; we omit the details.

In the following lemma we state a local existence result of problem (2.8).

Lemma 2.4. *Let* $2 \leq p \leq \bar{q}$ *and*

$$\max\left(2, \frac{\bar{q}}{\bar{q} + 1 - p}\right) \leq m \leq \bar{q}.$$

Then given $u_0 \in H^1_{\Gamma_0}(\Omega)$ *and* $u_1 \in L^2(\Omega)$ *there exists* $T > 0$ *and a unique solution* v *of problem* (2.8) *on* $(0, T)$ *such that*

$$v \in C([0, T], H^1_{\Gamma_0}(\Omega)) \cap C^1([0, T], L^2(\Omega)),$$
$$v_t \in L^2(0, T; H^1_{\Gamma_0}(\Omega)) \cap L^m((0, T) \times \Gamma_1)$$

and it satisfies the energy inequality

$$\frac{1}{2}\left[\|u_t(t)\|_2^2 + \|u_t(t)\|_{2,\Gamma_1}^2 + \left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + (g \diamond u)(t)\right]_s^t$$
$$+ \alpha \int_s^t \|\nabla v_t(\tau)\|_2^2 d\tau + \int_s^t \int_{\Gamma_1} h(v_t(\sigma, \tau))d\sigma d\tau$$
$$\leq \int_s^t \int_\Omega |u(\tau)|^{p-2}u(\tau)v_t(\tau)d\tau dx$$

for $0 \leq s \leq t \leq T$.

Proof. We first approximate $u \in C([0, T], H^1_{\Gamma_0}(\Omega)) \cap C^1([0, T], L^2(\Omega))$ endowed with the standard norm

$$\|u\| = \max_{t \in [0,T]} \|u_t(t)\|_2 + \|u(t)\|_{H^1_{\Gamma_0}(\Omega)}$$

by a sequence

$$(u^k)_{k \in \mathbb{N}} \subset C^\infty([0, T] \times \overline{\Omega})$$

by a standard convolution arguments (see [4]). Next, we approximate the initial data $u_1 \in L^2(\Omega)$ by a sequence

$$(u_1^k)_{k \in \mathbb{N}} \subset C_0^\infty(\Omega).$$

Finally, as the space $H^2(\Omega) \cap V \cap H_{\Gamma_0}^1(\Omega)$ is dense in $H_{\Gamma_0}^1(\Omega)$ for the H^1-norm, we approximate $u_0 \in H_{\Gamma_0}^1(\Omega)$ by a sequence

$$(u_0^k)_{k \in \mathbb{N}} \subset H^2(\Omega) \cap V \cap H_{\Gamma_0}^1(\Omega).$$

We consider now the set of the following problems:

$$
\begin{cases}
v_{tt}^k - \Delta v^k - \alpha \Delta v_t^k + \displaystyle\int_0^t g(t-s) \Delta v^k(s) ds = |u^k|^{p-2} u^k, & x \in \Omega, \, t > 0, \\[2mm]
v^k(x,t) = 0, & x \in \Gamma_0, \, t > 0, \\[2mm]
v_{tt}^k(x,t) = -\left[\dfrac{\partial v^k}{\partial \nu}(x,t) - \displaystyle\int_0^t g(t-s) \dfrac{\partial v^k}{\partial \nu}(x,s) ds \right. \\[4mm]
\qquad\qquad \left. + \dfrac{\alpha \partial v_t^k}{\partial \nu}(x,t) + h(v_t^k) \right], & x \in \Gamma_1, \, t > 0, \\[4mm]
v^k(x,0) = u_0^k, \quad v_t^k(x,0) = u_1^k, & x \in \Omega.
\end{cases}
\tag{2.9}
$$

Since the hypotheses of Lemma 2.3 are verified, we can find a sequence of unique solutions $(v_k)_{k \in \mathbb{N}}$ of problem (2.9). Our goal now is to show that $(v^k, v_t^k)_{k \in \mathbb{N}}$ is a Cauchy sequence in the space

$$
Y_T = \Big\{ (v, v_t) : v \in C([0,T], H_{\Gamma_0}^1(\Omega)) \cap C^1([0,T], L^2(\Omega)),
$$
$$
v_t \in L^2(0,T; H_{\Gamma_0}^1(\Omega)) \cap L^m((0,T) \times \Gamma_1) \Big\}
$$

endowed with the norm

$$
\|(v, v_t)\|_{Y_T}^2 = \max_{0 \le t \le T} \big[\|v_t\|_2^2 + l \|\nabla v\|_2^2 \big] + \|v_t\|_{L^m((0,T) \times \Gamma_1)}^2 + \int_0^t \|\nabla v_t(s)\|_2^2 ds.
\tag{2.10}
$$

For this purpose, we set $U = u^k - u^{k'}$, $V = v^k - v^{k'}$. It is straightforward to see that V satisfies

$$
\begin{cases}
V_{tt} - \Delta V - \alpha \Delta V_t + \displaystyle\int_0^t g(t-s) \Delta V(s) ds \\[2mm]
\qquad = |u^k|^{p-2} u^k - |u^{k'}|^{p-2} u^{k'}, & x \in \Omega, \, t > 0, \\[2mm]
V(x,t) = 0, & x \in \Gamma_0, \, t > 0, \\[2mm]
V_{tt}(x,t) = -\left[\dfrac{\partial V}{\partial \nu}(x,t) - \displaystyle\int_0^t g(t-s) \dfrac{\partial V}{\partial \nu}(x,s) ds \right. \\[4mm]
\qquad\qquad \left. + \dfrac{\alpha \partial V_t}{\partial \nu}(x,t) + h(v_t^k) - h(v_t^{k'}) \right], & x \in \Gamma_1, \, t > 0, \\[4mm]
V(x,0) = u_0^k - u_0^{k'}, \quad V_t(x,0) = u_1^k - u_1^{k'}, & x \in \Omega.
\end{cases}
$$

We multiply the above differential equations by V_t, we integrate over $(0, t) \times \Omega$, we use integration by parts and the identity (2.5) to obtain

$$
\frac{1}{2}\left(\|V_t(t)\|_2^2 + \|V_t(t)\|_{2,\Gamma_1}^2 + \left(1 - \int_0^t g(r)dr\right)\|\nabla V(t)\|_2^2\right) + \alpha \int_0^t \|\nabla V_t\|_2^2 ds
$$

$$
+ \int_0^t \int_{\Gamma_1} \left(h(v_t^k(x,\tau)) - h(v_t^{k'}(x,\tau))\right)\left(v_t^k(x,\tau) - v_t^{k'}(x,\tau)\right)d\Gamma d\tau
$$

$$
- \frac{1}{2}\int_0^t (g' \diamond V)(s)ds + \frac{1}{2}\int_0^t g(s)\|\nabla V(s)\|_2^2 ds
$$

$$
= \frac{1}{2}\left(\|V_t(0)\|_2^2 + \|\nabla V(0)\|_2^2 + \|V_t(0)\|_{2,\Gamma_1}^2\right)
$$

$$
+ \int_0^t \int_\Omega \left(|u^k|^{p-2}u^k - |u^{k'}|^{p-2}u^{k'}\right)\left(v_t^k - v_t^{k'}\right)dxds \qquad (2.11)
$$

for all $t \in (0, T)$.

Consequently, the above inequality together with (2.1), (2.2) and (2.3) gives

$$
\frac{1}{2}\left(\|V_t(t)\|_2^2 + \|V_t(t)\|_{2,\Gamma_1}^2 + l\|\nabla V(t)\|_2^2\right) + \alpha \int_0^t \|\nabla V_t\|_2^2 ds
$$

$$
+ m_0 \int_0^t \|V_t\|_{m,\Gamma_1}^m ds
$$

$$
\leq \frac{1}{2}\left(\|V_t(0)\|_2^2 + \|\nabla V(0)\|_2^2 + \|V_t(0)\|_{2,\Gamma_1}^2\right)
$$

$$
+ \int_0^t \int_\Omega \left(|u^k|^{p-2}u^k - |u^{k'}|^{p-2}u^{k'}\right)\left(v_t^k - v_t^{k'}\right)dxds \qquad (2.12)
$$

for all $t \in (0, T)$.

Following the same method as in [11], we deduce that there exists a $C > 0$ depending only on Ω and p such that

$$
\|V\|_{Y_T} \leq C\left(\|V_t(0)\|_2^2 + \|\nabla V(0)\|_2^2 + \|V_t(0)\|_{2,\Gamma_1}^2\right) + CT\|U\|_{Y_T}.
$$

Since $(u_0^k)_{k \in \mathbb{N}}$ converges in $H_{\Gamma_0}^1(\Omega)$, $(u_1^k)_{k \in \mathbb{N}}$ converges in $L^2(\Omega)$ and $(u^k)_{k \in \mathbb{N}}$ converges in $C([0,T], H_{\Gamma_0}^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$ (so in Y_T also), we conclude that $(v^k, v_t^k)_{k \in \mathbb{N}}$ is a Cauchy sequence in Y_T. Thus (v^k, v_t^k) converges to a limit $(v, v_t) \in Y_T$. Now by the same procedure used by Georgiev and Todorova in [10], we prove that this limit is a weak solution of problem (2.8). This completes the proof of Lemma 2.4.   □

Proof of Theorem 2.1. In order to prove Theorem 2.1, we use the Contraction Mapping Theorem.

Indeed, for $T > 0$, let us define the convex closed subset of Y_T:

$$X_T = \{(v, v_t) \in Y_T : v(0) = u_0, \ v_t(0) = u_1\}.$$

Let us denote

$$B_R(X_T) = \{v \in X_T : \|v\|_{Y_T} \leq R\},$$

the ball of radius R in X_T. Then, Lemma 2.4 implies that for any $u \in X_T$, we may define $v = \Phi(u)$ the unique solution of (2.8) corresponding to u. Our goal now is to show that for a suitable $T > 0$, Φ is a contractive map satisfying

$$\Phi(B_R(X_T)) \subset B_R(X_T).$$

Let $u \in B_R(X_T)$ and $v = \Phi(u)$. Then for all $t \in [0, T]$ we have as in (2.11):

$$\|v_t\|_2^2 + l\|\nabla v\|_2^2 + \|v_t\|_{2,\Gamma_1}^2 + 2\alpha \int_0^t \|\nabla v_t\|_2^2 ds + c \int_0^t \|v_t\|_{m,\Gamma_1}^m ds$$

$$\leq \|u_1\|_2^2 + \|\nabla u_0\|_2^2 + \|u_1\|_{2,\Gamma_1}^2 + 2\int_0^t \int_\Omega |u(\tau)|^{p-2}u(\tau)v_t(\tau)dxd\tau. \quad (2.13)$$

Using Hölder's inequality, we can control the last term on the right hand side of inequality (2.13) as follows:

$$\int_0^t \int_\Omega |u(\tau)|^{p-2}u(\tau)v_t(\tau)dxd\tau$$

$$\leq \int_0^t \|u(\tau)\|_{2N/(N-2)}^{p-1}\|v_t(\tau)\|_{2N/(3N-Np+2(p-1))}d\tau.$$

Since $p \leq \frac{2N}{N-2}$, we have

$$\frac{2N}{3N - Np + 2(p-1)} \leq \frac{2N}{N-2}.$$

Thus, by Young's and Sobolev's inequalities, we get for all $\delta > 0$ that there exists a $C(\delta) > 0$ such that for all $t \in (0, T)$,

$$\int_0^t \int_\Omega |u(\tau)|^{p-2}u(\tau)v_t(\tau)dxd\tau \leq C(\delta)tR^{2(p-1)} + \delta \int_0^t \|\nabla v_t(\tau)\|_2^2 d\tau.$$

Inserting the last estimate in inequality (2.13) we choose δ small enough such that

$$\|v\|_{Y_T}^2 \leq \frac{1}{2}R^2 + CTR^{2(p-1)}.$$

Thus, for T sufficiently small, we have $\|v\|_{Y_T} \leq R$. This shows that $v \in B_R(X_T)$.

To prove that Φ is a contraction, we have to follow the same steps (up to minor changes) as in [11]. We omit the details. Thus the proof is finished. $\qquad \square$

Remark 2.5. Let us say that the hypothesis on m, $\max(2, \frac{\bar{q}}{\bar{q}+1-p}) \le m \le \bar{q}$, is made to pass to the limit in the nonlinear term, by the same way we have used in [11, equation (2.28)].

3 Global existence

In this section, we show that, under some restrictions on the initial data, the local solution of problem (1.1) can be continued in time and the lifespan of the solution will be $[0, \infty)$.

Definition 3.1. Let $2 \le p \le \bar{q}$,

$$\max\left(2, \frac{\bar{q}}{\bar{q} + 1 - p}\right) \le m \le \bar{q},$$

$u_0 \in H^1_{\Gamma_0}(\Omega)$ and $u_1 \in L^2(\Omega)$. We denote by u the solution of (1.1). We define

$$T_{\max} = \sup\{T > 0 : u = u(t) \text{ exists on } [0, T]\}.$$

Since the solution $u \in Y_T$ (the solution is "regular enough"), from the definition of the norm given by (2.10), let us recall that if $T_{\max} < \infty$, then

$$\lim_{\substack{t \to T_{\max} \\ t < T_{\max}}} \|\nabla u(t)\|_2 + \|u_t(t)\|_2 = +\infty.$$

- If $T_{\max} < \infty$, we say that the solution of (1.1) blows up and that $T_{\max}$ is the blow up time.

- If $T_{\max} = \infty$, we say that the solution of (1.1) is global.

In order to study the blow up phenomenon or the global existence of the solution of (1.1), for all $0 \le t < T_{\max}$, we define

$$I(t) = I(u(t)) = \left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + (g \diamond u)(t) - \|u\|_p^p, \qquad (3.1)$$

$$J(t) = J(u(t)) = \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \diamond u)(t) - \frac{1}{p}\|u\|_p^p. \quad (3.2)$$

Thus the energy functional defined in (2.7) can be rewritten as

$$E(u(t)) = E(t) = J(t) + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2. \qquad (3.3)$$

As in [9,27], we denote by B the best constant in the Poincaré–Sobolev embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^p(\Omega)$ defined by

$$B^{-1} = \inf\{\|\nabla u\|_2 : u \in H^1_{\Gamma_0}(\Omega), \|u\|_p = 1\}. \qquad (3.4)$$

For $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$, we define

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|u_1\|_{2,\Gamma_1}^2 + \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p}\|u_0\|_p^p.$$

The first goal is to prove that the above energy $E(t)$ defined in (2.7) is a non-increasing function along the trajectories. More precisely, we have the following result:

Lemma 3.2. *Let* $2 \le p \le \bar{q}$ *and*

$$\max\left(2, \frac{\bar{q}}{\bar{q}+1-p}\right) \le m \le \bar{q}$$

and let u be the solution of (1.1). Then, for all $t > 0$, we have

$$\frac{dE(t)}{dt} = \frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 - \alpha\|\nabla u_t\|_2^2 - \int_{\Gamma_1} h(u_t)u_t d\Gamma$$

$$\le \frac{1}{2}(g' \diamond u)(t) - \alpha\|\nabla u_t\|_2^2 - \int_{\Gamma_1} h(u_t)u_t d\Gamma \qquad (3.5)$$

for all $t \in [0, T_{\max})$.

Proof. Multiplying the first equation in (1.1) by u_t, integrating over Ω, using integration by parts, we get

$$\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p\right\}$$

$$- \int_\Omega \int_0^t g(t-s)\nabla u(s)\nabla u_t(t)dsdx$$

$$= -\alpha\|\nabla u_t\|_2^2 - \int_{\Gamma_1} h(u_t)u_t d\Gamma. \qquad (3.6)$$

A simple use of identity (2.6) gives (3.5). This completes the proof. □

Lemma 3.3. *Let* $2 \le p \le \bar{q}$ *and*

$$\max\left(2, \frac{\bar{q}}{\bar{q}+1-p}\right) \le m \le \bar{q}.$$

Assume that (2.1) and (2.2) hold. Then given $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$ satisfying

$$\begin{cases} \beta = \dfrac{B^p}{l}\left(\dfrac{2p}{l(p-2)}E(0)\right)^{(p-2)/2} < 1, \\ I(u_0) > 0, \end{cases} \qquad (3.7)$$

we have

$$I(u(t)) > 0 \quad \text{for all } t \in [0, T_{\max}).$$

Proof. Since $I(u_0) > 0$, by continuity, there exists a $T^* < T_{\max}$ such that

$$I(t) > 0 \quad \text{for all } t \in [0, T^*],$$

which implies that for all $t \in [0, T^*]$,

$$J(t) = \frac{1}{p} I(t) + \frac{p-2}{2p} \left\{ \left(1 - \int_0^t g(s) ds \right) \|\nabla u(t)\|_2^2 + (g \diamond u)(t) \right\}$$

$$\geq \frac{p-2}{2p} \left\{ \left(1 - \int_0^t g(s) ds \right) \|\nabla u(t)\|_2^2 + (g \diamond u)(t) \right\}. \tag{3.8}$$

By using (2.1), (2.2), (3.3) and (3.5), we easily get, for all $t \in [0, T^*]$,

$$l \|\nabla u(t)\|_2^2 \leq \frac{2p}{p-2} J(t),$$

$$\leq \frac{2p}{p-2} E(t) \leq \frac{2p}{p-2} E(0). \tag{3.9}$$

From the definition of the constant B in (3.4), we first get

$$\|u(t)\|_p^p \leq B^p \|\nabla u(t)\|_2^p \quad \text{for all } t \in [0, T^*].$$

Since we have

$$B^p \|\nabla u(t)\|_2^p = \frac{B^p}{l} \|\nabla u(t)\|_2^{p-2} (l \|\nabla u(t)\|_2^2) \quad \text{for all } t \in [0, T^*],$$

by exploiting (3.9) and (3.7), we obtain, for all $t \in [0, T^*]$,

$$\|u(t)\|_p^p \leq \beta l (\|\nabla u(t)\|_2^2)$$

$$\leq \beta \left(1 - \int_0^t g(s) ds \right) \|\nabla u(t)\|_2^2$$

$$< \left(1 - \int_0^t g(s) ds \right) \|\nabla u(t)\|_2^2.$$

Therefore, by using (3.1), we conclude that

$$I(t) > 0 \quad \text{for all } t \in [0, T^*].$$

Using the fact that E is decreasing along the trajectory, we get

$$\frac{B^p}{l} \left(\frac{2p}{l(p-2)} E(t) \right)^{(p-2)/2} \leq \beta < 1 \quad \text{for all } 0 \leq t < T_{\max}.$$

By repeating this procedure, T^* is extended to $T_{\max}$. □

Now, we are able to state the global existence theorem.

Theorem 3.4. *Let* $2 \le p \le \bar{q}$ *and*

$$\max\left(2, \frac{\bar{q}}{\bar{q}+1-p}\right) \le m \le \bar{q}.$$

Assume that (2.1) and (2.2) hold. Then given $u_0 \in H^1_{\Gamma_0}(\Omega)$ *and* $u_1 \in L^2(\Omega)$ *satisfying (3.7), the solution of (1.1) is global and bounded.*

Proof. To prove Theorem 3.4, using the definition of $T_{\max}$, we have just to check that $\|\nabla u(t)\|^2_2 + \|u_t(t)\|^2_2$ is uniformly bounded in time. To achieve this, we use (3.2), (3.3), (3.5) and (3.9) to get

$$E(0) \ge E(t) = J(t) + \frac{1}{2}\|u_t\|^2_2 + \frac{1}{2}\|u_t\|^2_{2,\Gamma_1}$$

$$\ge \frac{p-2}{2p}\|\nabla u(t)\|^2_2 + \frac{1}{2}\|u_t(t)\|^2_2. \tag{3.10}$$

Therefore,

$$\|\nabla u(t)\|^2_2 + \|u_t(t)\|^2_2 \le CE(0),$$

where C is a positive constant, which depends only on p.   □

4   Exponential growth for $\alpha > 0$

In this section we will prove that when the initial data are large enough, the energy of the solution of problem (1.1) defined by (2.7) grows exponentially and thus so the L^p-norm.

In order to state and prove the exponential growth result, we introduce the following constants:

$$B_1 = \frac{B}{l}, \quad \alpha_1 = B_1^{-p/(p-2)}, \quad E_1 = \left(\frac{1}{2}-\frac{1}{p}\right)\alpha_1^2, \quad E_2 = \left(\frac{l}{2}-\frac{1}{p}\right)\alpha_1^2. \tag{4.1}$$

Let us first mention that $E_2 < E_1$.

The following lemma will play an essential role in the proof of the exponential growth result, and it is inspired by the work in [7], where the authors proved a similar lemma for the wave equation.

First, we define the function

$$\gamma(t) := \left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2_2 + (g \diamond u)(t). \tag{4.2}$$

Let us rewrite the energy functional E defined by (2.7) as

$$E(t) = \frac{1}{2}\|u_t(t)\|^2_2 + \frac{1}{2}\|u_t(t)\|^2_{2,\Gamma_1} + \frac{1}{2}\gamma(t) - \frac{1}{p}\|u(t)\|^p_p. \tag{4.3}$$

Lemma 4.1. *Let $2 \le p \le \bar{q}$ and*

$$\max\left(2, \frac{\bar{q}}{\bar{q} + 1 - p}\right) \le m \le \bar{q}.$$

Let u be the solution of (1.1). Assume that

$$E(0) < E_1 \quad and \quad \|\nabla u_0\|_2 \ge \alpha_1. \tag{4.4}$$

Then there exists a constant $\alpha_2 > \alpha_1$ such that

$$(\gamma(t))^{1/2} \ge \alpha_2 \quad for \; all \; t \in [0, T_{\max}) \tag{4.5}$$

and

$$\|u(t)\|_p \ge B_1 \alpha_2 \quad for \; all \; t \in [0, T_{\max}). \tag{4.6}$$

Proof. We first note that, by (4.3), we have

$$
\begin{aligned}
E(t) &\ge \frac{1}{2}\gamma(t) - \frac{1}{p}\|u(t)\|_p^p \\
&\ge \frac{1}{2}\gamma(t) - \frac{B_1^p}{p}(l\|\nabla u(t)\|_2)^p \\
&\ge \frac{1}{2}\gamma(t) - \frac{B_1^p}{p}(\gamma(t))^{p/2} \\
&= \frac{1}{2}\alpha^2 - \frac{B_1^p}{p}\alpha^p = F(\alpha),
\end{aligned} \tag{4.7}
$$

where $\alpha = (\gamma(t))^{1/2}$. It is easy to verify that F is increasing for $0 < \alpha < \alpha_1$, decreasing for $\alpha > \alpha_1$, $F(\alpha) \to -\infty$ as $\alpha \to +\infty$, and

$$F(\alpha_1) = E_1,$$

where α_1 is given in (4.1). Therefore, since $E(0) < E_1$, there exists an $\alpha_2 > \alpha_1$ such that $F(\alpha_2) = E(0)$.

If we set $\alpha_0 = (\gamma(0))^{1/2}$, then by (4.7) we have

$$F(\alpha_0) \le E(0) = F(\alpha_2),$$

which implies that $\alpha_0 \ge \alpha_2$.

Now to establish (4.5), we suppose by contradiction that

$$(\gamma(t_0))^{1/2} < \alpha_2,$$

for some $t_0 > 0$ and by the continuity of $\gamma(\cdot)$, we may choose t_0 such that

$$(\gamma(t_0))^{1/2} > \alpha_1.$$

Using again (4.7) leads to

$$E(t_0) \geq F(\gamma(t_0)^{1/2}) > F(\alpha_2) = E(0).$$

But this is impossible since for all $t > 0$, $E(t) \leq E(0)$. Hence (4.5) is established.
To prove (4.6), we use (4.3) to get

$$\frac{1}{2}\gamma(t) \leq E(0) + \frac{1}{p}\|u(t)\|_p^p.$$

Consequently, using (4.5) leads to:

$$\frac{1}{p}\|u(t)\|_p^p \geq \frac{1}{2}\gamma(t) - E(0)$$

$$\geq \frac{1}{2}\alpha_2^2 - E(0).$$

But we have

$$\frac{1}{2}\alpha_2^2 - E(0) = \frac{1}{2}\alpha_2^2 - F(\alpha_2) = \frac{B_1^p}{p}\alpha_2^p.$$

Therefore (4.6) holds. This finishes the proof of Lemma 4.1.   □

The exponential growth result reads as follows:

Theorem 4.2. *Suppose that* (2.1), (2.2), (2.4) *hold. Assume that*

$$2 \leq m \quad and \quad \max\left(m, \frac{2}{l}\right) < p \leq \overline{p}.$$

Then, the solution of (1.1) *satisfying*

$$E(0) < E_2 \quad and \quad \|\nabla u_0\|_2 \geq \alpha_1, \tag{4.8}$$

grows exponentially in the L^p-norm.

Remark 4.3. It is obvious that for $g = 0$, we have $E_1 = E_2$, and Theorem 4.2 reduces to [11, Theorem 3.1].

Remark 4.4. In Theorem 4.2, the condition

$$\int_0^\infty g(s)ds < \frac{(p/2) - 1}{(p/2) - 1 + (1/2p)}$$

used in [16, 17, 20, 21, 23, 26, 28] is unnecessary and our result holds without it.

Remark 4.5. Let us denote $c_1 = (l - \frac{2}{p}) - 2E_2(B_1\alpha_2)^{-p}$. Since we have seen that $\alpha_2 > \alpha_1$, using the definition of E_2, we easily get $c_1 > 0$. This constant will play an important role in the proof of Theorem 4.2.

Proof of Theorem 4.2. We implement the so-called Georgiev–Todorova method
(see [10, 20] and also [22]). So, we suppose that the solution exists for all time
and we will prove an exponential growth. For this purpose, we set

$$\mathcal{H}(t) = E_2 - E(t). \tag{4.9}$$

Of course by (4.4) and (3.5) and since $E_2 < E_1$, we deduce that $\mathcal{H}$ is a non-
decreasing function. So, by using (4.3) and, (4.9) we get successively

$$0 < \mathcal{H}(0) \le \mathcal{H}(t) \le E_2 - E(t) \le E_1 - \frac{1}{2}\gamma(t) + \frac{1}{p}\|u(t)\|_p^p.$$

On one hand as $F(\alpha_1) = E_1$ and $\gamma(t) \ge \alpha_2^2 > \alpha_1^2$ for all $t > 0$, we obtain

$$E_1 - \frac{1}{2}\gamma(t) < F(\alpha_1) - \frac{1}{2}\alpha_1^2.$$

On the other hand, since

$$F(\alpha_1) - \frac{1}{2}\alpha_1^2 = -\frac{B_1^p}{p}\alpha_1^p,$$

we obtain the following inequality:

$$0 < \mathcal{H}(0) \le \mathcal{H}(t) \le \frac{1}{p}\|u(t)\|_p^p \quad \text{for all } t \ge 0. \tag{4.10}$$

For ε small to be chosen later, and inspired by the ideas of the authors in [11], we
then define the auxiliary function

$$\mathcal{L}(t) = \mathcal{H}(t) + \varepsilon \int_\Omega u_t u \, dx + \varepsilon \int_{\Gamma_1} u_t u \, d\Gamma + \frac{\varepsilon\alpha}{2}\|\nabla u\|_2^2. \tag{4.11}$$

Let us remark that $\mathcal{L}$ is a small perturbation of the energy. By taking the time
derivative of (4.11), using problem (1.1), we obtain

$$\frac{d\mathcal{L}(t)}{dt} = \alpha\|\nabla u_t\|_2^2 + \int_{\Gamma_1} h(u_t)u_t \, d\Gamma + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2$$

$$+ \varepsilon\|u\|_p^p + \varepsilon\|u_t\|_{2,\Gamma_1}^2 - \varepsilon \int_{\Gamma_1} h(u_t)u(x,t) \, d\sigma$$

$$+ \int_\Omega \nabla u(t) \int_0^t g(t-s)\nabla u(s) \, ds \, dx. \tag{4.12}$$

By making use of (2.4) and the following Young's inequality,

$$XY \le \frac{\lambda^\mu X^\mu}{\mu} + \frac{\lambda^{-\nu}Y^\nu}{\nu}, \tag{4.13}$$

where $X, Y \geq 0$, $\lambda > 0$, $\mu, \nu \in \mathbb{R}^+$ such that $1/\mu + 1/\nu = 1$, we get

$$\int_{\Gamma_1} h(u_t) u \, d\Gamma \leq C_m \int_{\Gamma_1} |u_t|^{m-2} u_t u \, d\Gamma$$

$$\leq C_m \frac{\lambda^m}{m} \|u\|_{m,\Gamma_1}^m + C_m \frac{m-1}{m} \lambda^{-m/(m-1)} \|u_t\|_{m,\Gamma_1}^m. \quad (4.14)$$

Now, the term involving g on the right-hand side of (4.12) can be written as

$$\int_{\Omega} \nabla u(t,x) \int_0^t g(t-s) \nabla u(s,x) \, ds \, dx$$

$$= \|\nabla u(t)\|_2^2 \left(\int_0^t g(s) \, ds \right)$$

$$+ \int_{\Omega} \nabla u(t,x) \int_0^t g(t-s) (\nabla u(s,x) - \nabla u(t,x)) \, ds \, dx. \quad (4.15)$$

On the other hand, by using Hölder's and Young's inequalities, we infer that for all $\mu > 0$ we get

$$\int_{\Omega} \nabla u(t,x) \int_0^t g(t-s)(\nabla u(s,x) - \nabla u(t,x)) \, ds \, dx$$

$$\leq \int_0^t g(t-s) \|\nabla u(t)\|_2 \|\nabla u(s) - \nabla u(t)\|_2 \, ds$$

$$\leq \mu(g \diamond u)(t) + \frac{1}{4\mu} \left(\int_0^t g(s) \, ds \right) \|\nabla u(t)\|_2^2. \quad (4.16)$$

Inserting the estimates (4.14) and (4.15) into (4.12), taking into account the inequality (4.16) and making use of (2.4), we obtain by choosing $\mu = 1/2$ and multiplying by l

$$l\mathcal{L}'(t) \geq \alpha l \|\nabla u_t\|_2^2 + l \left(c_m - C_m \varepsilon \frac{m-1}{m} \lambda^{-m/(m-1)} \right) \|u_t\|_{m,\Gamma_1}^m + \varepsilon l \|u_t\|_2^2$$

$$+ \varepsilon l \|u\|_p^p + \varepsilon l \|u_t\|_{2,\Gamma_1}^2 - C_m \varepsilon l \frac{\lambda^m}{m} \|u\|_{m,\Gamma_1}^m$$

$$- \frac{\varepsilon l}{2} (g \diamond u)(t) - \varepsilon l \|\nabla u(t)\|_2^2. \quad (4.17)$$

We want now to estimate the term involving $\|u\|_{m,\Gamma_1}^m$ in (4.17). We proceed as in [11]. Then, we have

$$\|u\|_{m,\Gamma_1} \leq C \|u\|_{H^s(\Omega)},$$

which holds for

$$m \geq 1 \quad \text{and} \quad 0 < s < 1, \ s \geq \frac{N}{2} - \frac{N-1}{m} > 0,$$

where C here and in the sequel denotes a generic positive constant which may change from line to line.

Recalling the interpolation and Poincaré's inequalities (see [18])

$$\|u\|_{H^s(\Omega)} \leq C \|u\|_2^{1-s} \|\nabla u\|_2^s,$$
$$\leq C \|u\|_p^{1-s} \|\nabla u\|_2^s,$$

we finally have the following inequality:

$$\|u\|_{m,\Gamma_1} \leq C \|u\|_p^{1-s} \|\nabla u\|_2^s. \tag{4.18}$$

If $s < 2/m$, using again Young's inequality, we get

$$\|u\|_{m,\Gamma_1}^m \leq C \left[\left(\|u\|_p^p \right)^{\frac{m(1-s)\mu}{p}} + \left(\|\nabla u\|_2^2 \right)^{\frac{ms\theta}{2}} \right] \tag{4.19}$$

for $1/\mu + 1/\theta = 1$. Here we choose $\theta = 2/ms$ to get $\mu = 2/(2-ms)$. Therefore the previous inequality becomes

$$\|u\|_{m,\Gamma_1}^m \leq C \left[\left(\|u\|_p^p \right)^{\frac{m(1-s)2}{(2-ms)p}} + \|\nabla u\|_2^2 \right]. \tag{4.20}$$

Now, choosing s such that

$$0 < s \leq \frac{2(p-m)}{m(p-2)},$$

we get

$$\frac{2m(1-s)}{(2-ms)p} \leq 1. \tag{4.21}$$

Once inequality (4.21) is satisfied, we use the classical algebraic inequality

$$z^\nu \leq (z+1) \leq \left(1 + \frac{1}{\omega} \right)(z+\omega) \quad \text{for all } z \geq 0,\ 0 < \nu \leq 1,\ \omega \geq 0, \tag{4.22}$$

to obtain the following estimate:

$$\left(\|u\|_p^p \right)^{\frac{m(1-s)2}{(2-ms)p}} \leq d \left(\|u\|_p^p + \mathscr{H}(0) \right)$$
$$\leq d \left(\|u\|_p^p + \mathscr{H}(t) \right) \quad \text{for all } t \geq 0, \tag{4.23}$$

where we have set $d = 1 + 1/\mathscr{H}(0)$. Inserting the estimate (4.23) into (4.19), we obtain the following important inequality:

$$\|u\|_{m,\Gamma_1}^m \leq C \left[\|u\|_p^p + l \|\nabla u\|_2^2 + \mathscr{H}(t) \right]. \tag{4.24}$$

Keeping in mind that

$$l = 1 - \int_0^\infty g(s)ds,$$

in order to control the term $\|\nabla u\|_2^2$ in equation (4.17), we preferably use (as we have $\mathcal{H}(t) > 0$), the following estimate:

$$\|u\|_{m,\Gamma_1}^m \leq C\left[\|u\|_p^p + l\|\nabla u\|_2^2 + 2\mathcal{H}(t)\right],$$

which gives finally

$$\|u\|_{m,\Gamma_1}^m \leq C\left[2E_2 + \left(1 + \frac{2}{p}\right)\|u\|_p^p - \|u_t\|_2^2 - \|u_t\|_{2,\Gamma_1}^2\right.$$
$$\left. + \left(l - \left(1 - \int_0^t g(s)ds\right)\right)\|\nabla u\|_2^2 - (g \diamond u)(t)\right]. \quad (4.25)$$

Since

$$1 - \int_0^t g(s)ds \geq l,$$

we obtain from above

$$\|u\|_{m,\Gamma_1}^m \leq C\left[2E_2 + \left(1 + \frac{2}{p}\right)\|u\|_p^p - \|u_t\|_2^2 - \|u_t\|_{2,\Gamma_1}^2 - (g \diamond u)(t)\right]. \quad (4.26)$$

Now, inserting (4.26) into (4.17), then we infer that

$$l\mathcal{L}'(t) \geq \alpha l\|\nabla u_t\|_2^2 + l\left(c_m - C_m\varepsilon\frac{m-1}{m}\lambda^{-m/(m-1)}\right)\|u_t\|_{m,\Gamma_1}^m$$
$$+ \varepsilon\left(l + lC_m\frac{\lambda^m}{m}C\right)\|u_t\|_2^2 + \varepsilon\left(l + lC_m\frac{\lambda^m}{m}C\right)\|u_t\|_{2,\Gamma_1}^2$$
$$+ \varepsilon\left\{l - C_m l\frac{\lambda^m}{m}C\left(1 + \frac{2}{p}\right)\right\}\|u\|_p^p$$
$$+ \varepsilon\left(C_m l\frac{\lambda^m}{m}C - \frac{l}{2}\right)(g \diamond u)(t)$$
$$- \varepsilon l\|\nabla u(t)\|_2^2 - 2C_m\varepsilon l\frac{\lambda^m}{m}CE_2. \quad (4.27)$$

From (4.9), we get

$$\mathcal{H}(t) \leq E_2 - \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 - \frac{1}{2}(g \diamond u)(t) + \frac{1}{p}\|u(t)\|_p^p$$
$$\leq E_2 - \frac{l}{2}\|\nabla u(t)\|_2^2 - \frac{1}{2}(g \diamond u)(t) + \frac{1}{p}\|u(t)\|_p^p.$$

This last inequality gives

$$-l\|\nabla u(t)\|_2^2 \geq 2\left(\mathcal{H}(t) - E_2 + \frac{1}{2}(g \diamond u)(t) - \frac{1}{p}\|u(t)\|_p^p\right). \qquad (4.28)$$

Consequently, (4.27) takes the form

$$l\mathcal{L}'(t) \geq \alpha l\|\nabla u_t\|_2^2 + l\left(c_m - C_m\varepsilon\frac{m-1}{m}\lambda^{-m/(m-1)}\right)\|u_t\|_{m,\Gamma_1}^m$$

$$+ \varepsilon\left(l + lC_m\frac{\lambda^m}{m}C\right)\|u_t\|_2^2 + \varepsilon\left(l + lC_m\frac{\lambda^m}{m}C\right)\|u_t\|_{2,\Gamma_1}^2$$

$$+ \varepsilon\left\{l - \frac{2}{p} - C_m l\frac{\lambda^m}{m}C\left(1 + \frac{2}{p}\right)\right\}\|u\|_p^p - 2\varepsilon E_2$$

$$+ \varepsilon\left(C_m l\frac{\lambda^m}{m}C - \frac{l}{2} + 1\right)(g \diamond u)(t)$$

$$+ 2\varepsilon\mathcal{H}(t) - 2C_m\varepsilon l\frac{\lambda^m}{m}CE_2. \qquad (4.29)$$

Now to estimate the terms involving $\|u\|_p^p$ and E_2 in (4.29), we simply write

$$\left(l - \frac{2}{p}\right)\|u\|_p^p - 2E_2 = \left(l - \frac{2}{p}\right)\|u\|_p^p - 2E_2\frac{\|u\|_p^p}{\|u\|_p^p}.$$

Then by using (4.6), we get

$$\left(l - \frac{2}{p}\right)\|u\|_p^p - 2E_2 \geq c_1\|u\|_p^p,$$

where $c_1 > 0$ is defined in Remark 4.5. Thus, (4.29) becomes

$$l\mathcal{L}'(t) \geq \alpha l\|\nabla u_t\|_2^2 + l\left(c_m - C_m\varepsilon\frac{m-1}{m}\lambda^{-m/(m-1)}\right)\|u_t\|_{m,\Gamma_1}^m$$

$$+ \varepsilon\left(l + lC_m\frac{\lambda^m}{m}C\right)\|u_t\|_2^2 + \varepsilon\left(l + lC_m\frac{\lambda^m}{m}C\right)\|u_t\|_{2,\Gamma_1}^2$$

$$+ \varepsilon\left\{c_1 - C_m l\frac{\lambda^m}{m}C(1 + \frac{2}{p}) - 4C_m\varepsilon l\frac{\lambda^m}{m}CE_2(B_1\alpha_2)^{-p}\right\}\|u\|_p^p$$

$$+ \varepsilon\left(C_m l\frac{\lambda^m}{m}C - \frac{l}{2} + 1\right)(g \diamond u)(t)$$

$$+ 2\varepsilon\left(\mathcal{H}(t) + C_m l\frac{\lambda^m}{m}CE_2\right). \qquad (4.30)$$

Notice that since $l < 1$, we first have

$$C_m l \frac{\lambda^m}{m} C - \frac{l}{2} + 1 > 0 \quad \text{for all } \lambda > 0.$$

At this point, we pick λ small enough such that

$$c_1 - C_m l \frac{\lambda^m}{m} C \left(1 + \frac{2}{p} \right) - 4 C_m \varepsilon l \frac{\lambda^m}{m} C E_2 (B_1 \alpha_2)^{-p} > 0.$$

Once λ is fixed, we may choose ε small enough such that

$$\begin{cases} c_m - C_m \varepsilon \dfrac{m-1}{m} \lambda^{-m/(m-1)} > 0, \\ \mathscr{L}(0) > 0. \end{cases}$$

Consequently, we end up with the estimate

$$\mathscr{L}'(t) \geq \eta_1 \left(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p + \mathscr{H}(t) + E_2 \right) \quad \text{for all } t \geq 0. \quad (4.31)$$

Next, it is clear that, by Young's and Poincaré's inequalities, we have

$$\mathscr{L}(t) \leq \gamma \left[\mathscr{H}(t) + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|\nabla u\|_2^2 \right] \quad \text{for some } \gamma > 0. \quad (4.32)$$

Since $\mathscr{H}(t) > 0$ for all $t \geq 0$, we have

$$\frac{l}{2} \|\nabla u\|_2^2 \leq \frac{1}{p} \|u\|_p^p + E_2. \quad (4.33)$$

Thus, inequality (4.32) becomes

$$\mathscr{L}(t) \leq \zeta \left[\mathscr{H}(t) + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p + E_2 \right] \quad \text{for some } \zeta > 0. \quad (4.34)$$

From the two inequalities (4.31) and (4.34), we finally obtain the differential inequality

$$\frac{d\mathscr{L}(t)}{dt} \geq \mu \mathscr{L}(t) \quad \text{for some } \mu > 0. \quad (4.35)$$

An integration of the previous differential inequality (4.35) between 0 and t gives the following estimate for the function $\mathscr{L}$:

$$\mathscr{L}(t) \geq \mathscr{L}(0) e^{\mu t}. \quad (4.36)$$

On the other hand, from the definition of the function $\mathscr{L}$, from inequality (4.10) and for small values of the parameter ε, it follows that

$$\mathscr{L}(t) \leq \frac{1}{p} \|u\|_p^p. \quad (4.37)$$

From the two inequalities (4.36) and (4.37) we conclude the exponential growth of the solution in the L^p-norm. $\qquad \square$

5 Blow up in finite time for $\alpha = 0$

In this section, we prove that in the absence of the strong damping $-\Delta u_t$ (i.e. we have $\alpha = 0$), the solution of problem (1.1) blows up in finite time, that is, there exists a $0 < T^* < \infty$ such that $\|u(t)\|_p \to \infty$ as $t \to T^*$.

The blow up result reads as follows:

Theorem 5.1. *Suppose that* (2.1)*,* (2.2) *and* (2.4) *hold. Assume that*

$$2 < m \quad and \quad \max\left(m, \frac{2}{l}\right) < p \le \overline{p}.$$

Then, the solution of (1.1) *satisfying*

$$E(0) < E_2, \quad \|\nabla u_0\|_2 \ge \alpha_1 \tag{5.1}$$

blows up in finite time, that is, $\|u(t)\|_p \to \infty$ *as* $t \to T^*$ *for some* $0 < T^* < \infty$.

Remark 5.2. The requirement $m > 2$ in Theorem 5.1 is technical, but it seems necessary in our proof. The case $m = 2$ cannot be handled with the method we use here. But the same result can be shown for $m = 2$ by using the concavity method. See [12] for more details.

Proof of Theorem 5.1. To prove Theorem 5.1, we suppose that the solution exists for all time and we reach to a contradiction. Following the idea introduced in [10] and developed in [22] and [27], we will define a function $\hat{L}$ which is a perturbation of the total energy of the system and which will satisfy the differential inequality

$$\frac{d\hat{L}(t)}{dt} \ge \xi \hat{L}^{1+\nu}(t), \tag{5.2}$$

where $\nu > 0$. Inequality (5.2) leads to a blow up of the solution in finite time $T^* \ge \hat{L}(0)^{-\nu}\xi^{-1}\nu^{-1}$, provided that $\hat{L}(0) > 0$.

To do so, we define the functional $\hat{L}$ as follows:

$$\hat{L}(t) = \mathcal{H}^{1-\sigma}(t) + \varepsilon \int_\Omega u_t u \, dx + \varepsilon \int_{\Gamma_1} u_t u \, d\Gamma, \tag{5.3}$$

where the functional $\mathcal{H}$ is defined in (4.9), σ is satisfying

$$0 < \sigma \le \min\left(\frac{p-m}{p(m-1)}, \frac{p-2}{2p}, \frac{m-2}{2m}, \hat{\sigma}\right), \tag{5.4}$$

where $\hat{\sigma}$ is defined later in (5.7) and ε is a small positive constant to be chosen later. Taking the time derivative of $\hat{L}(t)$ and following the same steps as in the

proof of Theorem 4.2, we get (instead of inequality (4.17)), for all $\lambda > 0$,

$$l\hat{L}'(t) \geq lc_m(1-\sigma)\mathscr{H}^{-\sigma}(t)\|u_t\|^m_{m,\Gamma_1} - C_m\varepsilon\frac{m-1}{m}\lambda^{-m/(m-1)}\|u_t\|^m_{m,\Gamma_1}$$

$$+ \varepsilon l\|u_t\|^2_2 + \varepsilon l\|u\|^p_p + \varepsilon l\|u_t\|^2_{2,\Gamma_1} - C_m\varepsilon l\frac{\lambda^m}{m}\|u\|^m_{m,\Gamma_1}$$

$$- \frac{\varepsilon l}{2}(g \diamond u)(t) - \varepsilon l\|\nabla u(t)\|^2_2. \tag{5.5}$$

Next, for large positive M, we select $\lambda^{-m/(m-1)} = M\mathscr{H}^{-\sigma}(t)$. Then the estimate (5.5) takes the form

$$l\hat{L}'(t) \geq \left(lc_m(1-\sigma) - MC_m\varepsilon\frac{m-1}{m}\right)\mathscr{H}^{-\sigma}(t)\|u_t\|^m_{m,\Gamma_1} + \varepsilon l\|u_t\|^2_2$$

$$+ \varepsilon l\|u\|^p_p + \varepsilon l\|u_t\|^2_{2,\Gamma_1} - C_m\varepsilon l\frac{M^{-(m-1)}}{m}H^{\sigma(m-1)}\|u\|^m_{m,\Gamma_1}$$

$$- \frac{\varepsilon l}{2}(g \diamond u)(t) - \varepsilon l\|\nabla u(t)\|^2_2. \tag{5.6}$$

Exploiting (4.10 and 4.18), we get

$$\mathscr{H}^{\sigma(m-1)}\|u\|^m_{m,\Gamma_1} \leq C\|u\|^{(1-s)m+\sigma p(m-1)}_p \|\nabla u\|^{sm}_2 .$$

Thus, as in Section 4, we have

$$\|u\|^{(1-s)m+\sigma p(m-1)}_p\|\nabla u\|^{sm}_2 \leq C\left[\left(\|u\|^p_p\right)^{\left(\frac{m(1-s)}{p}+\sigma(m-1)\right)\mu} + \left(\|\nabla u\|^2_2\right)^{\frac{ms\theta}{2}}\right].$$

Choosing μ, θ and s exactly as in Section 4 (with strict inequalities), we choose σ that verifies

$$\sigma \leq \frac{2-ms}{2(m-1)}\left(1 - \frac{2m(1-s)}{(2-ms)p}\right) = \hat{\sigma}. \tag{5.7}$$

The hypotheses on m and p ensure to have $0 < \sigma < 1$.

Consequently, we get from above

$$\mathscr{H}^{\sigma(m-1)}\|u\|^m_{m,\Gamma_1} \leq C\left[\left(\|u\|^p_p\right)^{\left(\frac{m(1-s)}{p}+\sigma(m-1)\right)\mu} + \|\nabla u\|^2_2\right]. \tag{5.8}$$

Since

$$\left(\frac{m(1-s)}{p} + \sigma(m-1)\right)\frac{2}{2-ms} \leq 1,$$

applying the algebraic inequality (4.22), we get

$$\left(\|u\|^p_p\right)^{\left(\frac{m(1-s)}{p}+\sigma(m-1)\right)\frac{2}{2-ms}} \leq d\left(\|u\|^p_p + \mathscr{H}(0)\right)$$

$$\leq d\left(\|u\|^p_p + \mathscr{H}(t)\right) \quad \text{for all } t \geq 0. \tag{5.9}$$

Thus, (5.9) together with (5.8) leads to (see (4.26))

$$\mathscr{H}^{\sigma(m-1)}\|u\|_{m,\Gamma_1}^m \le Cd\left[\|u\|_p^p + l\|\nabla u\|_2^2 + \mathscr{H}(t)\right]$$

$$\le Cd\left[2E_2 + \left(1 + \frac{2}{p}\right)\|u\|_p^p - \|u_t\|_2^2\right.$$

$$\left. - \|u_t\|_{2,\Gamma_1}^2 - (g \diamond u)(t)\right]. \qquad (5.10)$$

Inserting (5.10) into (5.6) and using (4.28), we obtain

$$l\hat{L}'(t) \ge \left(lc_m(1-\sigma) - MC_m\varepsilon\frac{m-1}{m}\right)\mathscr{H}^{-\sigma}(t)\|u_t\|_{m,\Gamma_1}^m$$

$$+ \varepsilon l\left(1 + C_m\varepsilon\frac{M^{-(m-1)}}{m}Cd\right)\{\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2\}$$

$$+ 2\varepsilon\mathscr{H}(t) - 2\varepsilon E_2$$

$$+ \varepsilon\left\{l - \frac{2}{p} - C_m l\frac{M^{-(m-1)}}{m}Cd\left(1 + \frac{2}{p}\right)\right\}\|u\|_p^p$$

$$+ C_m\varepsilon l\frac{M^{-(m-1)}}{m}Cd(g \diamond u)(t)$$

$$- 2\varepsilon C_m l\frac{M^{-(m-1)}}{m}CdE_2 + \varepsilon\left(1 - \frac{l}{2}\right)(g \diamond u)(t). \qquad (5.11)$$

Writing again $E_2 = E_2\|u\|_p^p/\|u\|_p^p$ and using again (4.6), we deduce that

$$l\hat{L}'(t) \ge \left(lc_m(1-\sigma) - MC_m\varepsilon\frac{m-1}{m}\right)\mathscr{H}^{-\sigma}(t)\|u_t\|_{m,\Gamma_1}^m$$

$$+ \varepsilon l\left(1 + C_m\varepsilon\frac{M^{-(m-1)}}{m}Cd\right)\{\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2\}$$

$$+ 2\varepsilon\mathscr{H}(t) + 2C_m\varepsilon l\frac{M^{-(m-1)}}{m}CdE_2$$

$$+ \varepsilon\left\{l - \frac{2}{p} - 2E_2(B_1\alpha_2)^{-p} - C_m l\frac{M^{-(m-1)}}{m}Cd\left(1 + \frac{2}{p}\right)\right.$$

$$\left. - 4C_m l\frac{M^{-(m-1)}}{m}CdE_2(B_1\alpha_2)^{-p}\right\}\|u\|_p^p$$

$$+ \varepsilon C_m l\frac{M^{-(m-1)}}{m}Cd(g \diamond u)(t).$$

Thus, using the definition of c_1 in Remark 4.5, we get

$$l\hat{L}'(t) \geq \left(lc_m(1-\sigma) - MC_m\varepsilon\frac{m-1}{m}\right)\mathcal{H}^{-\sigma}(t)\|u_t\|_{m,\Gamma_1}^m$$

$$+ \varepsilon l\left(1 + C_m\varepsilon\frac{M^{-(m-1)}}{m}Cd\right)\{\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2\}$$

$$+ 2\varepsilon\mathcal{H}(t) + 2C_m\varepsilon l\frac{M^{-(m-1)}}{m}CdE_2$$

$$+ \varepsilon\left\{c_1 - C_m l\frac{M^{-(m-1)}}{m}Cd\left(1 + \frac{2}{p}\right)\right.$$

$$\left. - 4C_m l\frac{M^{-(m-1)}}{m}CdE_2(B_1\alpha_2)^{-p}\right\}\|u\|_p^p$$

$$+ \varepsilon C_m l\frac{M^{-(m-1)}}{m}Cd(g \diamond u)(t).$$

Since $c_1 > 0$, we choose M large enough such that

$$c_1 - C_m l\frac{M^{-(m-1)}}{m}Cd\left(1 + \frac{2}{p}\right) - 4C_m l\frac{M^{-(m-1)}}{m}CdE_2(B_1\alpha_2)^{-p} > 0.$$

Once M is fixed, we pick ε small enough such that

$$lc_m(1-\sigma) - MC_m\varepsilon\frac{m-1}{m} > 0$$

and $\hat{L}(0) > 0$. This leads to

$$\hat{L}'(t) \geq \hat{\eta}\left(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \mathcal{H}(t) + \|u\|_p^p + E_2\right) \tag{5.12}$$

for some $\hat{\eta} > 0$.

On the other hand, it is clear from the definition (5.3) that

$$\hat{L}^{\frac{1}{1-\sigma}}(t) \leq C(\varepsilon,\sigma)\left[\mathcal{H}(t) + \left(\int_\Omega u_t u\,dx\right)^{\frac{1}{1-\sigma}} + \left(\int_{\Gamma_1} u_t u\,d\Gamma\right)^{\frac{1}{1-\sigma}}\right]. \tag{5.13}$$

By the Cauchy–Schwarz inequality and Hölder's inequality, we have

$$\int_\Omega u_t u\,dx \leq \left(\int_\Omega u_t^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega u^2\,dx\right)^{\frac{1}{2}}$$

$$\leq C\left(\int_\Omega u_t^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}},$$

where C is the positive constant which comes from the embedding

$$L^p(\Omega) \hookrightarrow L^2(\Omega).$$

This inequality implies that there exists a positive constant $C_1 > 0$ such that

$$\left(\int_\Omega u_t u \, dx \right)^{\frac{1}{1-\sigma}} \leq C_1 \left[\left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{(1-\sigma)p}} \left(\int_\Omega u_t^2 \, dx \right)^{\frac{1}{2(1-\sigma)}} \right].$$

Applying Young's inequality to the right hand-side of the preceding inequality, there exists a positive constant also denoted by C such that

$$\left(\int_\Omega u_t u \, dx \right)^{\frac{1}{1-\sigma}} \leq C \left[\left(\int_\Omega |u|^p \, dx \right)^{\frac{\tau}{(1-\sigma)p}} + \left(\int_\Omega u_t^2 \, dx \right)^{\frac{\theta}{2(1-\sigma)}} \right] \quad (5.14)$$

for $1/\tau + 1/\theta = 1$. We take $\theta = 2(1-\sigma)$, hence $\tau = 2(1-\sigma)/(1-2\sigma)$, to get

$$\left(\int_\Omega u_t u \, dx \right)^{\frac{1}{1-\sigma}} \leq C \left[\left(\int_\Omega |u|^p \, dx \right)^{\frac{2}{(1-2\sigma)p}} + \int_\Omega u_t^2 \, dx \right].$$

Using the algebraic inequality (4.22) with

$$z = \|u\|_p^p, \quad d = 1 + \frac{1}{\mathcal{H}(0)}, \quad \omega = \mathcal{H}(0) \quad \text{and} \quad \nu = \frac{2}{p(1-2\sigma)}$$

(the condition (5.4) on σ ensuring that $0 < \nu \leq 1$), we get

$$z^\nu \leq d(z + \mathcal{H}(0)) \leq d(z + \mathcal{H}(t)).$$

Therefore, there exists a positive constant denoted by C_2 such that for all $t \geq 0$,

$$\left(\int_\Omega u_t u \, dx \right)^{\frac{1}{1-\sigma}} \leq C_2 \left[\mathcal{H}(t) + \|u(t)\|_p^p + \|u_t(t)\|_2^2 \right]. \quad (5.15)$$

Following the same method as above, we can show that there exists a $C_3 > 0$ such that

$$\left(\int_{\Gamma_1} u_t u \, d\Gamma \right)^{\frac{1}{1-\sigma}} \leq C_3 \left[\mathcal{H}(t) + \|u(t)\|_{m,\Gamma_1}^m + \|u_t(t)\|_{2,\Gamma_1}^2 \right].$$

Applying inequality (4.24), we get

$$\left(\int_{\Gamma_1} u_t u \, d\Gamma \right)^{\frac{1}{1-\sigma}} \leq C_4 \left[\mathcal{H}(t) + \|u(t)\|_p^p + l \|\nabla u(t)\|_2^2 + \|u_t(t)\|_{2,\Gamma_1}^2 \right].$$

Furthermore, inequality (4.33) leads to

$$\left(\int_{\Gamma_1} u_t u \, d\Gamma \right)^{\frac{1}{1-\sigma}} \leq C_5 \left[\mathcal{H}(t) + \|u(t)\|_p^p + \|u_t(t)\|_{2,\Gamma_1}^2 + E_2 \right]. \quad (5.16)$$

Collecting (5.13), (5.15) and (5.16), we obtain, for all $t \geq 0$,

$$\hat{L}^{\frac{1}{1-\sigma}}(t) \leq \hat{\eta}_1 \{ \|u_t(t)\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \mathscr{H}(t) + \|u(t)\|_p^p + E_2 \} \qquad (5.17)$$

for some $\hat{\eta}_1 > 0$.

Combining (5.12) and (5.17), we conclude that there exists a positive constant $\xi > 0$, as small as ε, such that for all $t \geq 0$,

$$\hat{L}'(t) \geq \xi \hat{L}^{\frac{1}{1-\sigma}}(t). \qquad (5.18)$$

Thus, inequality (5.2) holds. Therefore, $\hat{L}(t)$ blows up in a finite time T^*.

On the other hand, from the definition of the function $\hat{L}(t)$ and using inequality (4.10), for small values of the parameter ε, it follows that

$$\hat{L}(t) \leq \kappa \left(\|u(t)\|_p^p \right)^{1-\sigma}, \qquad (5.19)$$

where κ is a positive constant. Consequently, from inequality (5.19) we conclude that the norm $\|u(t)\|_p$ of the solution u blows up in the finite time T^*, which implies the desired result. This completes the proof of Theorem 5.1.   □

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