

On some variational algebraic problems

Giovanni Molica Bisci and Dušan Repovš

Abstract. In this paper by exploiting critical point theory, the existence of two distinct nontrivial solutions for a nonlinear algebraic system with a parameter is established. Our goal is achieved by requiring an appropriate behavior of the nonlinear term f at zero and at infinity. Some applications to difference equations are also presented.

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1 Introduction

In this paper we deal with the following problem:

$$Au = \lambda f(u), \quad (S_{A,\lambda}^f)$$

where $u = (u_1, \dots, u_n)^t \in \mathbb{R}^n$ is a column vector in \mathbb{R}^n , $A = (a_{ij})_{n \times n}$ is a given positive definite matrix, $f(u) := (f_1(u_1), \dots, f_n(u_n))^t$ with $f_k : \mathbb{R} \rightarrow \mathbb{R}$ a continuous function for every $k \in \mathbb{Z}[1, n] := \{1, \dots, n\}$, and λ is a positive parameter.

Discrete problems involving functions with two or more discrete variables are very relevant and have been deeply investigated. Such great interest is undoubtedly due to the advance of modern digital computing devices.

Indeed, since these relations can be simulated in a relatively easy manner by means of such devices and since such simulations often reveal important information about the behavior of complex systems, a large number of recent investigations related to image processing, population models, neural networks, social behaviors, digital control systems are described in terms of such functional relations.

Moreover, a large number of problems can be formulated as special cases of the nonlinear algebraic system $(S_{A,\lambda}^f)$. For a survey on these topics we cite the recent paper [25]. A similar approach has also been used in others works (see for instance, the papers [21–23] and [24, 26, 27]).

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Here, motivated by the interest on the subject, by using variational methods in finite dimensional setting, we prove the existence of two nontrivial solutions for suitable values of the parameter λ .

More precisely, in Theorem 3.1 we prove the existence of two nontrivial solutions, for every λ sufficiently large, by only requiring sublinear conditions at infinity and an appropriate behaviour of the nonlinear terms at zero.

In Theorem 3.4 we determine an open interval of positive parameters such that problem $(S_{A,\lambda}^f)$ admits at least two nontrivial solutions which are uniformly bounded in norm with respect to the parameter λ .

Our main tool, in this case, is a useful abstract result obtained in [3, Theorem 2.1] which ensures the existence of an open interval $\Lambda \subset (0, +\infty)$ such that for each $\lambda \in \Lambda$ the function J_λ associated to problem $(S_{A,\lambda}^f)$ admits two critical points which are uniformly bounded in norm with respect to λ (see also [5, 6] for related topics).

A direct application of our result to fourth-order difference equations yields the following:

Proposition 1.1. *Assume that*

$$\sup_{t \in \mathbb{R}} \sum_{k=1}^n \int_0^t f_k(s) ds > 0,$$

in addition to

$$\lim_{|s| \rightarrow \infty} \frac{f_k(s)}{s} = \lim_{s \rightarrow 0} \frac{f_k(s)}{s} = 0,$$

for every $k \in \mathbb{Z}[1, n]$. Then there exist a nonempty open interval $\Lambda \subset (0, +\infty)$ and a number $\gamma > 0$ such that for every $\lambda \in \Lambda$, problem

$$\begin{cases} \Delta^4 u_{k-2} = \lambda f_k(u_k), & \forall k \in \mathbb{Z}[1, n], \\ u_{-2} = u_{-1} = u_0 = 0, \\ u_{n+1} = u_{n+2} = u_{n+3} = 0, \end{cases} \quad (D_\lambda^f)$$

has at least two distinct, nontrivial solutions $u_\lambda^1, u_\lambda^2 \in \mathbb{R}^n$, and

$$\|u_\lambda^i\|_2 < \gamma, \quad i \in \{1, 2\}.$$

Further, requiring a suitable growth of the primitive of f , we are able to establish suitable intervals of values of the parameter λ for which the problem $(S_{A,\lambda}^f)$ admits at least three weak solutions. More precisely, the main result ensures the existence of two real intervals of parameters Λ_1, Λ_2 such that, for each $\lambda \in \Lambda_1 \cup \Lambda_2$,

the problem $(S_{A,\lambda}^f)$ admits at least three weak solutions whose norms are uniformly bounded with respect to every $\lambda \in \Lambda_2$ (see Theorem 3.5 and Example 3.7). Our method is mostly based on a useful critical point theorem given in [4, Theorem 3.1].

In conclusion, we also emphasize that if the functions f_k are nonnegative, for every $k \in \mathbb{Z}[1, n]$, our results guarantee two positive solutions (see Remark 3.8 for more details). For a complete and exhaustive overview of variational methods we refer the reader to the monographs [1, 16, 19].

The plan of the paper is as follows. In Section 2 we introduce some basic notations. In Section 3 we obtain our existence results (see Theorems 3.1 and 3.4). Finally, in Section 4, some concrete examples of applications of the obtained results are presented.

2 Preliminaries

As the ambient space X , we consider the n -dimensional Banach space \mathbb{R}^n endowed by the norm

$$\|u\|_2 := \left(\sum_{k=1}^n u_k^2 \right)^{1/2}.$$

More generally, we set

$$\|u\|_r := \left(\sum_{k=1}^n |u_k|^r \right)^{1/r} \quad (r \geq 1)$$

for every $u \in X$.

Let \mathfrak{X}_n denote the class of all symmetric and positive definite matrices of order n . Further, we denote by $\lambda_1, \dots, \lambda_n$ the eigenvalues of A ordered as follows: $0 < \lambda_1 \leq \dots \leq \lambda_n$.

It is well known that if $A \in \mathfrak{X}_n$, for every $u \in X$, then one has

$$\lambda_1 \|u\|_2^2 \leq u^t A u \leq \lambda_n \|u\|_2^2, \quad (2.1)$$

and

$$\|u\|_\infty \leq \frac{1}{\sqrt{\lambda_1}} (u^t A u)^{1/2}, \quad (2.2)$$

where $\|u\|_\infty := \max_{k \in [1, n]} |u_k|$.

From now on we will assume that $A \in \mathfrak{X}_n$. Set

$$\Phi(u) := \frac{u^t A u}{2}, \quad \Psi(u) := \sum_{k=1}^n F_k(u_k) \quad \text{and} \quad J_\lambda(u) := \Phi(u) - \lambda \Psi(u), \quad (2.3)$$

for every $u \in X$, where

$$F_k(t) := \int_0^t f_k(s)ds, \quad \text{for every } (k, t) \in \mathbb{Z}[1, n] \times \mathbb{R}.$$

Standard arguments show that $J_\lambda \in C^1(X, \mathbb{R})$ as well as that the critical points of J_λ are exactly the solutions of problem $(S_{A, \lambda}^f)$.

Indeed, a column vector $\bar{u} = (\bar{u}_1, \dots, \bar{u}_n)^t \in X$ is a critical point of the functional J_λ if the gradient of J_λ at \bar{u} is zero, i.e.,

$$\frac{\partial J_\lambda(u)}{\partial u_1} \Big|_{u=\bar{u}} = 0, \quad \frac{\partial J_\lambda(u)}{\partial u_2} \Big|_{u=\bar{u}} = 0, \quad \dots, \quad \frac{\partial J_\lambda(u)}{\partial u_n} \Big|_{u=\bar{u}} = 0.$$

Moreover, for every $k \in \mathbb{Z}[1, n]$, one has that

$$\frac{\partial u^t Au}{\partial u_k} = 2(Au)_k,$$

where $(Au)_k := \sum_{j=1}^n a_{kj}u_j$. Thus

$$\frac{\partial J_\lambda(u)}{\partial u_k} = (Au)_k - \lambda f_k(u_k), \quad \text{for all } k \in \mathbb{Z}[1, n],$$

which yields our assertion.

3 Main results

Our first result is a multiplicity theorem obtained as a consequence of Tonelli's approach together with a careful analysis of the meaningful Mountain Pass geometry of the functional J_λ . More precisely, we consider the case when the continuous functions $f_k : \mathbb{R} \rightarrow \mathbb{R}$ fulfill the following hypotheses:

(h₁) For every $k \in \mathbb{Z}[1, n]$,

$$\lim_{|s| \rightarrow \infty} \frac{f_k(s)}{s} = 0.$$

(h₂) There exists $\nu_0 > 1$ such that

$$\lim_{|s| \rightarrow 0} \frac{f_k(s)}{|s|^{\nu_0}} = 0,$$

for every $k \in \mathbb{Z}[1, n]$.

Note that a typical example when (h₁) holds is the following:

(h₁^{*}) There exist $q \in (0, 1)$ and $c > 0$ such that $|f_k(s)| \leq c|s|^q$, for every $s \in \mathbb{R}$.

With the above notations, we can prove the following multiplicity result.

Theorem 3.1. *Assume that conditions (h₁) and (h₂) hold in addition to*

$$\sup_{t \in \mathbb{R}} \sum_{k=1}^n F_k(t) > 0.$$

Then:

(i) *There exists a positive parameter λ^* given by*

$$\lambda^* := \left(\frac{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}}{2} \right) \left(\max_{t \neq 0} \frac{\sum_{k=1}^n F_k(t)}{t^2} \right)^{-1},$$

such that, for every $\lambda > \lambda^$, problem $(S_{A,\lambda}^f)$ has at least two distinct, non-trivial solutions $u_\lambda^1, u_\lambda^2 \in \mathbb{R}^n$, where u_λ^1 is the global minimum of the energy functional J_λ associated to $(S_{A,\lambda}^f)$.*

(ii) *If (h₁^{*}) holds, then*

$$\|u_\lambda^1\|_2 = o(\lambda^{1/(1-r)}), \quad \text{for every } r \in (q, 1),$$

but

$$\|u_\lambda^1\|_2 \neq O(\lambda^{1/(1-\mu)}), \quad \text{for every } \mu > 1,$$

as $\lambda \rightarrow \infty$.

Proof. Due to conditions (h₁) and (h₁) the term $F_k(t)/t^2$ tends to zero as $|t| \rightarrow \infty$ and $t \rightarrow 0$, respectively. Moreover, since

$$\sup_{t \in \mathbb{R}} \sum_{k=1}^n F_k(t) > 0,$$

there exists $t_0 \in \mathbb{R}$ such that

$$\sum_{k=1}^n F_k(t_0) > 0.$$

Thus, the value λ^* is well-defined. Hence, there exists a number $t_\star \in \mathbb{R} \setminus \{0\}$ such that

$$\frac{\sum_{k=1}^n F_k(t_\star)}{t_\star^2} = \max_{t \neq 0} \frac{\sum_{k=1}^n F_k(t)}{t^2}.$$

So

$$\lambda^* := \left(\frac{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}}{2} \right) \frac{t_\star^2}{\sum_{k=1}^n F_k(t_\star)}.$$

At this point fix $\lambda > \lambda^*$ and let us first consider the vector $u^* \in X$ of components $u_k^* = t_*$, for every $k \in \mathbb{Z}[1, n]$.

One has

$$\begin{aligned} J_\lambda(u^*) &= \Phi(u^*) - \lambda\Psi(u^*) = \left(\frac{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}}{2} \right) t_*^2 - \lambda \sum_{k=1}^n F_k(t_*) \\ &= (\lambda^* - \lambda) \sum_{k=1}^n F_k(t_*) < 0. \end{aligned}$$

Thus $\inf_{u \in X} J_\lambda(u) \leq J_\lambda(u^*) < 0$. Due to (h_1) , for an arbitrarily $\varepsilon < \frac{\lambda_1}{\lambda}$ there exists $c(\varepsilon) > 0$ such that

$$F_k(t) \leq |F_k(t)| \leq \frac{\varepsilon}{2} t^2 + c(\varepsilon) |t|,$$

for every $t \in \mathbb{R}$ and $k \in \mathbb{Z}[1, n]$. Consequently, from the left-hand side of (2.1), we have

$$J_\lambda(u) \geq \left(\frac{\lambda_1 - \lambda\varepsilon}{2} \right) \|u\|_2^2 - \lambda c_1 c(\varepsilon) \|u\|_2,$$

where c_1 is a positive constant such that $\|u\|_1 \leq c_1 \|u\|_2$, for every $u \in X$.

It follows from this that J_λ is bounded from below and coercive. Hence, since our ambient space is finite dimensional, the functional J_λ satisfies the classical compactness (PS)-condition.

Since J_λ verifies the (PS)-condition and it is bounded from below, by [16, Theorem 1.7], one can fix $u_\lambda^1 \in X$ such that $J(u_\lambda^1) = \inf_{u \in X} J_\lambda(u)$. Therefore, $u_\lambda^1 \in X$ is the first solution of $(S_{A,\lambda}^f)$ and $u_\lambda^1 \neq 0$, since $J_\lambda(0_X) = 0$.

Now, we prove that for every $\lambda > \lambda^*$ the functional J_λ has the standard Mountain Pass geometry. Indeed, by (h_1) and (h_2) , one can fix two constants $\nu > 1$ and $C > 0$ such that

$$|F_k(t)| \leq C |t|^{\nu+1},$$

for every $t \in \mathbb{R}$ and $k \in \mathbb{Z}[1, n]$. Moreover, bearing in mind condition (2.1), one has

$$\begin{aligned} J_\lambda(u) &= \Phi(u) - \lambda\Psi(u) \\ &\geq \frac{\lambda_1}{2} \|u\|_2^2 - \lambda C \|u\|_{\nu+1}^{\nu+1} \\ &\geq \frac{\lambda_1}{2} \|u\|_2^2 - \lambda c_{\nu+1}^{\nu+1} C \|u\|_2^{\nu+1}, \end{aligned} \tag{3.1}$$

where $c_{\nu+1}$ is a positive constant such that

$$\|u\|_{\nu+1} \leq c_{\nu+1} \|u\|_2, \quad \text{for all } u \in X.$$

Let us take $\rho_\lambda > 0$ to be so small that

$$\rho_\lambda < \min \left\{ \left(\frac{\lambda_1}{2\lambda c_{v+1}^{v+1} C} \right)^{\frac{1}{v-1}}, \sqrt{n}|t_\star| \right\}.$$

By (3.1), for every $u \in X$ complying with $\|u\|_2 = \rho_\lambda$, we have

$$\begin{aligned} J_\lambda(u) &\geq \left(\frac{\lambda_1}{2} - \lambda c_{v+1}^{v+1} C \|u\|_2^{v-1} \right) \|u\|_2^2 \\ &= \left(\frac{\lambda_1}{2} - \lambda c_{v+1}^{v+1} C \rho_\lambda^{v-1} \right) \rho_\lambda^2 \\ &=: \eta(\rho_\lambda) > 0. \end{aligned}$$

By construction, one has $\|u\|_2 = \sqrt{n}|t_\star| > \rho_\lambda$, and $J(u^\star) < 0 = J_\lambda(0_X)$.

Hence, we can apply the Mountain Pass Theorem (see [16, Theorem 1.13]). Thus, there exists $u_\lambda^2 \in X$ such that $J'(u_\lambda^2) = 0$ and $J_\lambda(u_\lambda^2) \geq \eta(\rho_\lambda) > 0$. Further, $u_\lambda^2 \neq 0_X$ and the vectors u_λ^1 and u_λ^2 are distinct. The proof of point (i) is complete.

Now, we assume that (h_1^\star) holds. Since $J_\lambda(u_\lambda^1) < 0$, it follows that

$$\frac{\lambda_1}{2} \|u_\lambda^1\|_2^2 - \frac{\lambda c}{(q+1)} c_{q+1}^{q+1} \|u\|_2^{q+1} \leq J_\lambda(u_\lambda^1) < 0.$$

In particular, $\|u_\lambda^1\|_2 = O(\lambda^{1/(1-q)})$ as $\lambda \rightarrow \infty$. Therefore, for any $r \in (q, 1)$, one has $\|u_\lambda^1\|_2 = o(\lambda^{1/(1-r)})$ as $\lambda \rightarrow \infty$.

Let us assume that $\|u_\lambda^1\|_2 = O(\lambda^{1/(1-\mu)})$ for some $\mu > 1$ as $\lambda \rightarrow \infty$. Then $\|u_\lambda^1\|_2 \rightarrow 0$ as $\lambda \rightarrow \infty$. On the other hand,

$$J(u_\lambda^1) \leq (\lambda^\star - \lambda) \sum_{k=1}^n F_k(t_\star),$$

hence $J_\lambda(u_\lambda^1) \rightarrow -\infty$. Now, since

$$\frac{\lambda_1}{2} \|u_\lambda^1\|_2^2 - \lambda c_{\mu+1}^{\mu+1} C \|u_\lambda^1\|_2^{\mu+1} \leq J_\lambda(u_\lambda^1),$$

one has

$$\left(\frac{\lambda_1}{2} - \lambda c_{\mu+1}^{\mu+1} C \|u_\lambda^1\|_2^{\mu-1} \right) \|u_\lambda^1\|_2^2 \rightarrow -\infty,$$

as $\lambda \rightarrow \infty$. This fact contradicts the initial assumption.

The proof is thus complete. \square

Remark 3.2. We observe that Theorem 3.1 can be checked by a careful analysis of a three critical points theorem contained in [7, Theorem 3.6].

Now, instead of (h_2) we will assume a weaker condition, namely:

$$(h'_2) \quad \lim_{s \rightarrow 0} \frac{f_k(s)}{s} = 0, \text{ for every } k \in \mathbb{Z}[1, n].$$

The next theorem below shows that assumption (h'_2) is still strong enough to prove a similar multiplicity result as Theorem 3.1. In this setting we obtain that the solutions are uniformly bounded in norm with respect to the parameter λ but, unfortunately, we lose the precise location of the eigenvalues. The main tool for our goal is a theoretical result given in [3, Theorem 2.1] (see, for completeness, [16, Theorem 1.13]).

We prove the following preliminary fact.

Proposition 3.3. *Assume that condition (h_1) holds in addition to (h'_2) . Then*

$$\lim_{\varrho \rightarrow 0^+} \frac{\sup_{u \in \Phi^{-1}([- \infty, \varrho])} \Psi(u)}{\varrho} = 0.$$

Proof. Due to (h'_2) , for an arbitrary small $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$|f_k(s)| < \varepsilon |s|,$$

for every $|s| < \delta_\varepsilon$ and $k \in \mathbb{Z}[1, n]$. On the other hand, on account of (h_1) , one can fix $\nu > 1$ and

$$|f_k(s)| < \varepsilon |s|^\nu,$$

for every $|s| \geq \delta_\varepsilon$ and $k \in \mathbb{Z}[1, n]$. Combining these two facts, we obtain

$$F_k(t) \leq \varepsilon \frac{t^2}{2} + \frac{c(\varepsilon)}{(\nu + 1)} |t|^{\nu+1},$$

for every $t \in \mathbb{R}$ and $k \in \mathbb{Z}[1, n]$.

Now, fix $\varrho > 0$. For every $u \in \Phi^{-1}([- \infty, \varrho])$, due to the above estimates, we have

$$\Psi(u) \leq \frac{\varepsilon}{2} \|u\|_2^2 + \frac{c(\varepsilon)}{(\nu + 1)} c_{\nu+1}^{\nu+1} \|u\|_2^{\nu+1} < \frac{\varepsilon \varrho}{\lambda_1} + c(\varepsilon) \frac{c_{\nu+1}^{\nu+1}}{(\nu + 1)} \left(\frac{2\varrho}{\lambda_1} \right)^{\frac{\nu+1}{2}},$$

taking into account that

$$\{u \in X : u^t A u < 2\varrho\} \subset \left\{ u \in X : \|u\|_2 < \left(\frac{2\varrho}{\lambda_1} \right)^{1/2} \right\}.$$

Thus, there exists $\varrho(\varepsilon) > 0$ such that, for every $0 < \varrho < \varrho(\varepsilon)$, we have

$$0 \leq \frac{\sup_{u \in \Phi^{-1}([- \infty, \varrho])} \Psi(u)}{\varrho} < \frac{\varepsilon}{\lambda_1} + c(\varepsilon) \frac{\varrho^{\frac{\nu-1}{2}}}{(\nu + 1)} \left(\frac{2}{\lambda_1} \right)^{\frac{\nu+1}{2}} < \varepsilon,$$

which completes the proof. \square

Our multiplicity result reads as follows.

Theorem 3.4. *Assume that conditions (h_1) and (h'_2) hold. Then there exist a non-empty open interval $\Lambda \subset (0, +\infty)$ and a number $\gamma > 0$ such that for every $\lambda \in \Lambda$, problem $(S_{A,\lambda}^f)$ has at least two distinct, nontrivial solutions $u_\lambda^1, u_\lambda^2 \in X$, and $\|u_\lambda^i\|_2 < \gamma$, $i \in \{1, 2\}$.*

Proof. Let $X := \mathbb{R}^n$, and consider the functionals Φ and Ψ defined in (2.3). Note that $J_\lambda := \Phi - \lambda\Psi$. We already know that for every positive parameter λ the functional J_λ is coercive and consequently satisfies the Palais–Smale condition, because X is finite dimensional.

Due to the fact that the functions f_k are sublinear at infinity and superlinear at zero, the terms $F_k(t)/t^2 \rightarrow 0$ as $|t| \rightarrow \infty$ and $t \rightarrow 0$, respectively.

Since $\sup_{t \in \mathbb{R}} \sum_{k=1}^n F_k(t) > 0$, there exists $t_0 \in \mathbb{R}$ such that $\sum_{k=1}^n F_k(t_0) > 0$, and we may fix a number $t_\star \in \mathbb{R} \setminus \{0\}$ such that

$$\frac{\sum_{k=1}^n F_k(t_\star)}{t_\star^2} = \max_{t \neq 0} \frac{\sum_{k=1}^n F_k(t)}{t^2}.$$

Therefore the number

$$\lambda^\star := \left(\frac{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}}{2} \right) \frac{t_\star^2}{\sum_{k=1}^n F_k(t_\star)},$$

is well-defined.

Now, let us choose $u^0 = 0_X$ and $u^1 \in X$ such that

$$u_k^1 = t_\star, \quad \text{for every } k \in \mathbb{Z}[1, n].$$

Fixing $\varepsilon \in (0, 1)$, due to Proposition 3.3, one can choose $\varrho > 0$ such that

$$\frac{\sup_{u \in \Phi^{-1}([-\infty, \varrho])} \Psi(u)}{\varrho} < \frac{\varepsilon}{\lambda^\star} \quad \text{and} \quad \varrho < \left(\frac{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}}{2} \right) t_\star^2.$$

Note that

$$\frac{\varepsilon}{\lambda^\star} < \frac{1}{\lambda^\star} = \frac{\Psi(u^1)}{\Phi(u^1)} \quad \text{and} \quad \Phi(u^1) = \left(\frac{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}}{2} \right) t_\star^2.$$

Therefore, by choosing

$$\bar{a} := \frac{1 + \varepsilon}{\frac{\Psi(u^1)}{\Phi(u^1)} - \frac{\sup_{u \in \Phi^{-1}([-\infty, \varrho])} \Psi(u)}{\varrho}},$$

all the assumptions of [3, Theorem 2.1] can be verified.

Hence, there exist a non-empty open interval $\Lambda \subset [0, \bar{a}]$ and a positive real γ such that for every $\lambda \in \Lambda$, the functional J_λ admits at least three distinct critical points in X having $\|\cdot\|_2$ -norm less than γ . The proof is complete. \square

As a direct application of [4, Theorem 3.1] we give the following multiplicity property.

Theorem 3.5. *Let $f_k : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function for every $k \in \mathbb{Z}[1, n]$. Assume that there exist positive constants γ and δ such that*

$$(g_1) \quad \delta > \left(\frac{\lambda_1}{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}} \right)^{1/2} \gamma.$$

(g₂) *The following inequality holds:*

$$\sum_{k=1}^n \max_{|\xi| \leq \gamma} F_k(\xi) < \eta(\gamma, \delta) \left(\sum_{k=1}^n F_k(\delta) \right),$$

where

$$\eta(\gamma, \delta) := \frac{\lambda_1 \gamma^2}{\lambda_1 \gamma^2 + (\text{Tr}(A) + 2 \sum_{i < j} a_{ij}) \delta^2}.$$

Further, require that

$$(g_3) \quad \limsup_{|\xi| \rightarrow \infty} \frac{F_k(\xi)}{\xi^2} < \frac{\lambda_1}{2}, \text{ for all } k \in \mathbb{Z}[1, n].$$

Then, for each

$$\lambda \in \Lambda_1 :=]\lambda_1^*, \lambda_2^*[,$$

where

$$\lambda_1^* := \frac{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}}{2 \left(\sum_{k=1}^n F_k(\delta) - \sum_{k=1}^n \max_{|\xi| \leq \gamma} F_k(\xi) \right)},$$

and

$$\lambda_2^* := \frac{\lambda_1 \gamma^2}{2 \left(\sum_{k=1}^n \max_{|\xi| \leq \gamma} F_k(\xi) \right)},$$

problem $(S_{A, \lambda}^f)$ has at least three distinct solutions and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2 \subset [0, \lambda_{3,h}^*],$$

where

$$\lambda_{3,h}^* := \frac{\lambda_1 h \gamma^2}{2 \left(\lambda_1 \gamma^2 \left(\frac{\sum_{k=1}^n F_k(\delta)}{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}} \right) - \sum_{k=1}^n \max_{|\xi| \leq \gamma} F_k(\xi) \right)},$$

and a positive real number $\sigma > 0$ such that, for each $\lambda \in \Lambda_2$, problem $(S_{A, \lambda}^f)$ has at least three solutions whose norms are less than σ .

Proof. We use the notations adopted above. Our aim is to apply [4, Theorem 3.1]. First of all let us verify that J_λ is a coercive functional for every positive parameter λ . By (g₃) there are constants $\epsilon \in]0, \lambda_1/2[$ and $\sigma > 0$ such that

$$\frac{1}{\xi^2} \int_0^\xi f_k(s) ds < \frac{\lambda_1}{2} - \epsilon, \quad (3.2)$$

for every $|\xi| \geq \sigma$ and $k \in \mathbb{Z}[1, n]$. Let us put

$$M_1 := \max_{(k, \xi) \in \mathbb{Z}[1, n] \times [-\sigma, \sigma]} \int_0^\xi f_k(s) ds. \quad (3.3)$$

At this point note that, for every $\xi \in \mathbb{R}$ and $k \in \mathbb{Z}[1, n]$, one has

$$\int_0^\xi f_k(s) ds \leq M_1 + M_2 \xi^2,$$

where

$$M_2 := \frac{\lambda_1}{2} - \epsilon.$$

Moreover, the following inequality holds:

$$J_\lambda(u) \geq \frac{u^t Au}{2} - \sum_{k=1}^n [M_1 + M_2 u_k^2], \quad \text{for all } u \in X.$$

Hence,

$$J_\lambda(u) \geq \frac{u^t Au}{2} - M_2 \|u\|_2^2 - nM_1, \quad \text{for all } u \in X,$$

and by relation (2.1), one has

$$J_\lambda(u) \geq \epsilon \|u\|_2^2 - nM_1, \quad \text{for all } u \in X, \quad (3.4)$$

which clearly shows that

$$\lim_{\|u\|_2 \rightarrow \infty} J_\lambda(u) = +\infty. \quad (3.5)$$

Hence J_λ is coercive for every positive parameter $\lambda > 0$.

Next, consider the vector $u^\star \in X$ of components $u_k^\star = \delta$, for every $k \in \mathbb{Z}[1, n]$. Thus

$$\Phi(u^\star) = \left(\frac{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}}{2} \right) \delta^2. \quad (3.6)$$

Put

$$r := \frac{\lambda_1}{2} \gamma^2.$$

It follows now from (g_1) that $\Phi(u^*) > r$. Further, we explicitly observe that, in view of (2.2), one has

$$\Phi^{-1}(]-\infty, r]) \subset \{u \in X : \|u\|_\infty \leq \gamma\}. \quad (3.7)$$

Moreover, taking (3.7) into account, a direct computation ensures that

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) \leq \sum_{k=1}^n \max_{|\xi| \leq \gamma} F_k(\xi). \quad (3.8)$$

At this point, by definition of u^* , we can clearly write

$$\Psi(u^*) = \sum_{k=1}^n F_k(u^*) = \sum_{k=1}^n F_k(\delta). \quad (3.9)$$

Moreover, by using hypothesis (g_2) from (3.8) and (3.9), we have

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) < \frac{r}{r + \Phi(u^*)} \Psi(u^*),$$

taking into account that

$$\frac{r}{r + \Phi(u^*)} = \eta(\gamma, \delta).$$

Thus, we can apply [4, Theorem 3.1], bearing in mind that

$$\frac{\Phi(u^*)}{\Psi(u^*) - \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \leq \lambda_1,$$

and

$$\frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \geq \lambda_2$$

as well as

$$\frac{hr}{r \frac{\Psi(u^*)}{\Phi(u^*)} - \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \leq \lambda_{3,h}^*.$$

The proof is complete. \square

Remark 3.6. As observed in [4, Remark 2.1], the real intervals Λ_1 and Λ_2 in Theorem 3.5 are such that either

$$\Lambda_1 \cap \Lambda_2 = \emptyset,$$

or

$$\Lambda_1 \cap \Lambda_2 \neq \emptyset.$$

In the first case, we actually obtain two distinct open intervals of positive real parameters for which problem $(S_{A,\lambda}^f)$ admits two nontrivial solutions; otherwise,

we obtain only one interval of positive real parameters, precisely $\Lambda_1 \cup \Lambda_2$, for which problem $(S_{A,\lambda}^f)$ admits three solutions and in addition, the subinterval Λ_2 for which the solutions are uniformly bounded.

The following is a simple application of Theorem 3.5.

Example 3.7. Let $g_k : \mathbb{R} \rightarrow \mathbb{R}$ be as follows:

$$g_k(s) := \begin{cases} 0 & \text{if } s < 2, \\ k\sqrt{s-2} & \text{if } s \geq 2, \end{cases}$$

whose potentials are given by

$$G_k(t) := \int_0^t g_k(s)ds = \begin{cases} 0 & \text{if } t < 2, \\ \frac{2k}{3}(t-2)^{3/2} & \text{if } t \geq 2, \end{cases}$$

for every $k \in \mathbb{Z}[1, n]$. Consider the algebraic nonlinear system

$$Au = \lambda g(u), \tag{S_{A,\lambda}^g}$$

where $A \in \mathfrak{X}_n$ and $g(u) := (g_1(u_1), \dots, g_n(u_n))^t$.

We observe that there exist two positive constants $\gamma = 2$ and

$$\delta > 2 \max \left\{ 1, \left(\frac{\lambda_1}{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}} \right)^{1/2} \right\},$$

such that all the conditions of Theorem 3.5 hold. Then for each

$$\lambda \in \Lambda'_1 :=]\lambda_1^*, +\infty[,$$

where

$$\lambda_1^* := \frac{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}}{2(\sum_{k=1}^n G_k(\delta))},$$

problem $(S_{A,\lambda}^g)$ has at least three distinct solutions (two nontrivial) and moreover, for each $h > 1$, there exist an open interval

$$\Lambda'_2 \subset [0, \lambda_{3,h}^*],$$

where

$$\lambda_{3,h}^* := h \frac{\text{Tr}(A) + 2 \sum_{i < j} a_{ij}}{2(\sum_{k=1}^n G_k(\delta))} = h\lambda_1^*,$$

and a positive real number $\sigma > 0$ such that for each $\lambda \in \Lambda'_2$, problem $(S_{A,\lambda}^g)$ has at least three solutions whose norms are less than σ .

Remark 3.8. A vector $\bar{u} := (\bar{u}_1, \dots, \bar{u}_n)^t \in \mathbb{R}^n$ is said to be *positive* (nonnegative) if $\bar{u}_k > 0$ ($\bar{u}_k \geq 0$) for every $k \in \mathbb{Z}[1, n]$. Now, let $A \in \mathfrak{X}_n$ and consider the following conditions:

(A₁) If $i \neq j$, then $a_{ij} \leq 0$.

(A₂) For every $i \in \mathbb{Z}[2, n]$, there exists $j_i < i$ such that $a_{ij_i} < 0$.

Assume that (A₁) holds. Then, if $\bar{u} := (\bar{u}_1, \dots, \bar{u}_n)^t \in X$ is a solution of

$$\sum_{j=1}^n a_{ij} u_j \geq 0, \quad \text{for all } i \in \mathbb{Z}[1, n], \quad (S_A^*)$$

then $\bar{u}_i \geq 0$, for every $i \in \mathbb{Z}[1, n]$ (see [11, 28] and [9, Proposition 2.1]). If, in addition to (A₁), condition (A₂) holds, then any solution of (S_A^*) is trivial or otherwise is positive (see [9, Proposition 2.2]). Hence, if f_k are nonnegative, for every $k \in \mathbb{Z}[1, n]$, our results guarantee the existence of two nonnegative solutions if A satisfies hypothesis (A₁). Finally, if (A₂) holds together with (A₁), then the obtained solutions are positive.

4 Applications

In this section we present some direct applications to discrete equations.

4.1 Tridiagonal matrices

Let $n > 1$ and $(a, b) \in \mathbb{R}^- \times \mathbb{R}^+$ be such that

$$\cos\left(\frac{\pi}{n+1}\right) < -\frac{b}{2a}.$$

Set

$$\text{Trid}_n(a, b, a) = \begin{pmatrix} b & a & 0 & \dots & 0 \\ a & b & a & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & a & b & a \\ 0 & \dots & 0 & a & b \end{pmatrix}_{n \times n}.$$

Note that $\text{Trid}_n(a, b, a)$ is a symmetric and positive definite matrix whose first eigenvalue is given by

$$\lambda_1 = b + 2a \cos\left(\frac{\pi}{n+1}\right),$$

see, for instance, [20, Example 9, page 179]. In this setting an important case is given by the following matrix:

$$\text{Trid}_n(-1, 2, -1) := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}_{n \times n} \in \mathfrak{X}_n,$$

which is associated to the second-order discrete boundary value problem

$$\begin{cases} -\Delta^2 u_{k-1} = \lambda f_k(u_k), & \forall k \in \mathbb{Z}[1, n], \\ u_0 = u_{n+1} = 0, \end{cases} \quad (S_\lambda^j)$$

where $\Delta^2 u_{k-1} := \Delta(\Delta u_{k-1})$, and, as usual, $\Delta u_{k-1} := u_k - u_{k-1}$ denotes the forward difference operator. We point out that the matrix $\text{Trid}_n(-1, 2, -1)$ was considered in order to study the existence of nontrivial solutions of nonlinear second-order difference equations [8, 14, 15, 17]. For completeness, we just mention here that there is a vast literature on nonlinear difference equations based on fixed point and upper and lower solution methods (see, for instance, the papers [2, 12]).

Example 4.1. By Theorem 3.4, there are a non-empty open interval $\Lambda \subset (0, +\infty)$ and a number $\gamma > 0$ such that for every $\lambda \in \Lambda$, the following problem,

$$\text{Trid}_n(a, b, a)u = \lambda g(u), \quad (T_\lambda^g)$$

where $g(u) := (g_1(u_1), \dots, g_n(u_n))^t$, in which

$$g_i(u_i) := \begin{cases} -iu_i^2 & \text{if } u_i \leq 0, \\ \frac{iu_i}{\log u_i} & \text{if } 0 < u_i \leq e^{-1}, \\ -\frac{i}{e} & \text{if } u_i > e^{-1}, \end{cases}$$

has at least two distinct nontrivial solutions $u_\lambda^1, u_\lambda^2 \in \mathbb{R}^n$, and

$$\|u_\lambda^i\|_2 < \gamma, \quad i \in \{1, 2\}.$$

Note that g in Example 4.1 satisfies (h'_2) but not (h_2) for any constant $v_0 > 1$. Therefore, one can apply Theorem 3.4 but not Theorem 3.1.

4.2 Fourth-order difference equations

As it is well known, boundary value problems involving fourth-order difference equations such as

$$\begin{cases} \Delta^4 u_{k-2} = \lambda f_k(u_k), & \forall k \in \mathbb{Z}[1, n], \\ u_{-2} = u_{-1} = u_0 = 0, \\ u_{n+1} = u_{n+2} = u_{n+3} = 0, \end{cases} \quad (D_\lambda^f)$$

can also be expressed as the problem $(S_{A,\lambda}^f)$, where A is the real symmetric and positive definite matrix of the form

$$A := \begin{pmatrix} 6 & -4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & \dots & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & \dots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \dots & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -4 & 6 \end{pmatrix} \in \mathcal{X}_n.$$

Hence, Proposition 1.1 is a direct consequence of Theorem 3.4.

4.3 Partial difference equations

A *lattice point* $z := (i, j)$ in the plane is a point with integer coordinates. Two lattice points are said to be *neighbors* if their Euclidean distance is one. An edge is a set $\{z, z^*\}$ consisting of two neighboring points, whereas a directed edge is an ordered pair (z, z^*) of neighboring points. A *path* between two lattice points z and z^* is a sequence $z = z_0, \dots, z_s = z^*$ of lattice points such that z_i and z_{i+1} are neighbors for $0 \leq i \leq s-1$. A set S of lattice points is said to be *connected* if there is a path contained in S between any two points of S . A finite and connected set of lattice points is called a *net*. An *exterior boundary* point of a net S is a point outside S but has a neighbor in S . The set of all exterior boundary points is denoted by ∂S . The set of all edges of S is denoted by $\Gamma(S)$ and the set of all directed edges of a net S by $E(S)$. The pair $(S, \Gamma(S))$ is a planar graph and the pair $(S, E(S))$ is a planar directed graph. With the above notations we consider

the problem, namely (E_λ^f) , given by

$$\begin{cases} Du(z) + \lambda f(z, u(z)) = 0, & z \in S, \\ u(z) = 0, & z \in \partial S, \end{cases}$$

where

$$Du(z) := [u(i+1, j) - 2u(i, j) + u(i-1, j)] + [u(i, j+1) - 2u(i, j) + u(i, j-1)]$$

is the well-known discrete Laplacian acting on a function $u : S \cup \partial S \rightarrow \mathbb{R}$. Then problem (E_λ^f) can be written as a nonlinear algebraic system (see, for more details, the monograph of Cheng [10]). We also cite the paper [13] in which the existence of infinitely many solutions for problem (E_λ^f) has been investigated.

Example 4.2. For each

$$\lambda > \frac{1}{0.3787311542} \simeq 2.6,$$

the following problem,

$$\begin{aligned} &[u(i+1, j) - 2u(i, j) + u(i-1, j)] + [u(i, j+1) - 2u(i, j) + u(i, j-1)] \\ &+ \lambda a(u(i, j)) = 0, \quad \forall (i, j) \in \mathbb{Z}[1, 2] \times \mathbb{Z}[1, 2], \end{aligned}$$

with boundary conditions

$$\begin{aligned} u(i, 0) &= u(i, 3) = 0, \quad \forall i \in \mathbb{Z}[1, 2], \\ u(0, j) &= u(3, j) = 0, \quad \forall j \in \mathbb{Z}[1, 2], \end{aligned}$$

where $a(s) := \log(1 + s^2)$ for every $s > 0$ and zero otherwise, admits two non-trivial (positive) solutions.

Indeed, let $h : \mathbb{Z}[1, 2] \times \mathbb{Z}[1, 2] \rightarrow \mathbb{Z}[1, 4]$ be the bijection defined by

$$h(i, j) := i + 2(j - 1), \quad \text{for every } (i, j) \in \mathbb{Z}[1, 2] \times \mathbb{Z}[1, 2].$$

Next, define

$$w_k := u(h^{-1}(k)),$$

and

$$g_k(w_k) = g_k(u(h^{-1}(k))) := a(w_k),$$

for every $k \in \mathbb{Z}[1, 4]$. The above problem can then be written as

$$Bw = \lambda g(w),$$

where

$$B := \begin{pmatrix} 4 & -1 & \vdots & -1 & 0 \\ -1 & 4 & \vdots & 0 & -1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & \vdots & 4 & -1 \\ 0 & -1 & \vdots & -1 & 4 \end{pmatrix},$$

$w := (w_1, \dots, w_k)$ and $g(w) := (g_1(w_1), \dots, g_4(w_4))^t$. Our assertion now immediately follows from Theorem 3.1 and Remark 3.8.

Some recent results about the discontinuous case are obtained in [18].

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Author information

Giovanni Molica Bisci, Dipartimento MECMAT, University of Reggio Calabria,
Via Graziella, Feo di Vito, 89124 Reggio Calabria, Italy.
E-mail: gmolica@unirc.it

Dušan Repovš, Faculty of Education, and Faculty of Mathematics and Physics,
University of Ljubljana, POB 2964, Ljubljana, Slovenia 1001.
E-mail: dusan.repovs@guest.arnes.si